COMPLEX MANIFOLDS
EXAMPLE SHEET 3

The two questions marked $(*)$ can be handed in to be marked. For this, please leave the work in my DPMMS pigeon hole (labelled "Dervan") by 14:00 on February 25^{th}.

(1) Suppose $\mathcal{U} = \{U_\alpha)$ is an open cover of a topological space X, and that on each U_α we have a sheaf $\mathcal{F}_\alpha$. If on each $U_\alpha \cap U_\beta$ we have an isomorphism of sheaves

$$\varphi_{\alpha\beta} : \mathcal{F}_\beta|_{U_\alpha \cap U_\beta} \to \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta},$$

such that $\varphi_{\alpha\alpha} = Id$ and

$$\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} = Id$$

where defined, show that there exists a sheaf $\mathcal{F}$ on X with $\mathcal{F}|_{U_\alpha} \cong \mathcal{F}_\alpha$ for all α.

(2) Suppose (U_α, ϕ_α) is a trivialisation for a vector bundle $\pi : E \to X$ and let $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ be a collection of maps such that $s_\alpha = \phi_{\alpha\beta} s_\beta$. Show that the s_α glue to a global section $s : X \to L$. Conversely show that any global section satisfies this property.

(3) *(For those who know some commutative algebra)* Let $\mathcal{O}_n$ denote the local ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$. If $w_1, \dots, w_n$ are the holomorphic coordinate functions on $\mathbb{C}^n$, let $\frac{\partial}{\partial w_i}$ denote the map $\mathcal{O}_n \to \mathbb{C}$ given by $f \to \frac{\partial f}{\partial w_i}(0)$. Show that the $\frac{\partial}{\partial w_i}$ are derivations in the sense of ring theory and that they form a basis for the holomorphic vector bundle $(T\mathbb{C}^n)^{(1,0)}$ at 0.

(4) (*) (i) Let X be a complex manifold with almost complex structure J. Explain how multiplication by i gives and $\mathbb{R}$-linear map

$$\tilde{J} : T^*X^{(1,0)} \to T^*X^{(1,0)}$$

such that $\tilde{J}^2 = -Id$. Prove that there is a natural identification

$$T^*X \cong (T^*X)^{(1,0)}$$

taking J to $\tilde{J}$.

(ii) Suppose now that g is a Riemannian metric on X compatible with J and let $g_{\mathbb{C}}$ be the extension to $TX_{\mathbb{C}}$ given by

$$g_{\mathbb{C}}(\lambda v, \mu w) = \lambda \bar{\mu} g(v, w) \quad \lambda, \mu \in \mathbb{C}, v, w \in T_x X.$$

Show that under the isomorphism in part (i) we can identify $g_{\mathbb{C}}$ with $g - i\omega$ (up to a factor of 2) on $T^*X^{(1,0)}$.

(iii) Now suppose we have local coordinates $z_1, \dots z_n$ on X so that $dz_1, \dots dz_n$ are a frame for $T^*X^{(1,0)}$. Show that if $h_{ij} = 2g_{\mathbb{C}}(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$, then (h_{ij}) is a hermitian matrix and the associated fundamental form is

$$\omega = \frac{i}{2}\sum_{i,j} h_{ij} dz_i \wedge \bar{d}z_j.$$

(5) Let $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ be the standard linear inclusion. Show that the restriction of the Fubini-Study metric on $\mathbb{P}^n$ gives the Fubini-Study metric on $\mathbb{P}^{n-1}$.

(6) The k^{th} Betti number of a smooth manifold X is $b_k = \dim_{\mathbb{R}} H^k(X, \mathbb{R})$. Show that the odd Betti numbers b_{2k+1} of a compact Kähler manifold are even.

(7) Show that the Hopf surface from Example Sheet 1 does not admit a Kähler form (this will require some topology). This gives an example of a compact complex manifold which is not Kähler, and hence not projective.

(8) (*) Let X be a compact complex manifold with Kähler form ω arising from a Riemannian metric g and suppose $\mathcal{H}^{0,1}(X, g) = 0$. Suppose that $\alpha \in \mathcal{A}^{0,1}(X)$ has $\bar{\partial}\alpha = 0$. Show that $\alpha = \bar{\partial}\beta$ for some β (a) directly using the Hodge decomposition and (b) using the identification between harmonic forms and Dolbeault cohomology.

(9) Show that a Kähler form is harmonic.

(10) Show how Poincaré duality $H^k(X, \mathbb{C}) \cong H^{2n-k}(X, \mathbb{C})$ can be deduced from Serre duality on a compact Kähler manifold.