COMPLEX MANIFOLDS
EXAMPLE SHEET 4

The two questions marked $(*)$ can be handed in to be marked. For this, please leave the work in my DPMMS pigeon hole (labelled "Dervan") by 14:00 on March 10^{th}.

(1) Let E be a holomorphic vector bundle. Show that a connection D is compatible with the holomorphic structure if and only if for every holomorphic frame $(e_1, \ldots, e_r)$, the connection matrix (Θ_{jl}) is a matrix of $(1,0)$-forms.

(2) Let (E, h) be a hermitian vector bundle. Show that a connection D is compatible with h if and only if for every unitary frame $(e_1, \ldots, e_r)$, the connection matrix (Θ_{jl}) is skew-hermitian.

(3) Show that $K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$ (here for X a complex manifold, $K_X = \det T^* X^{1,0}$).

(4) (*) (a) Let (E, h) be a hermitian holomorphic vector bundle. Define the dual connection D^* on E^* by specifying that for local sections σ of E^* and s of E, we have

$$(D^*\sigma)(s) = d(\sigma(s)) - \sigma(Ds).$$

Check that D^* is a connection.

(b) Let h^* be the dual metric, defined such that the dual of a unitary frame is unitary. If D is the Chern connection for E, show that D^* is the Chern connection for E^*.

(c) Check that the Chern connection on (E_1, h_1) and (E_2, h_2) naturally induce the Chern connection on $E_1 \oplus E_2$ with respect to the product metric. What is the Chern connection on $EndE$?

(5) Suppose Y is a smooth hypersurface of X. The normal bundle of Y in X is the holomorphic vector bundle $N_{Y/X}$ on Y which is the cokernel of the inclusion $TY^{(1,0)} \hookrightarrow TX^{(1,0)}|_Y$. Show that

$$N_{Y/X} \cong \mathcal{O}(Y)|_Y.$$

(6) The aim of this question is to show that the image of $c_1 : Pic(X) \to H^2(X, \mathbb{Z})$ lies in $H^{1,1}_{\bar{\partial}}(X) \cap H^2(X, \mathbb{Z}) := H^{1,1}(X, \mathbb{Z})$, where one considers $H^{1,1}_{\bar{\partial}}(X) \subset H^2(X, \mathbb{C})$ via the Hodge decomposition.

(a) Consider the maps

$$f_1 : H^2(X, \mathbb{C}) \to H^2(X, \mathcal{O}) \cong H^{0,2}_{\bar{\partial}}(X)$$

induced from the inclusion $\mathbb{C} \hookrightarrow \mathcal{O}$ and

$$f_2 : H^2(X, \mathbb{C}) \to H^{0,2}_{\bar{\partial}}(X)$$

induced from the Hodge decomposition. Show that $f_1 = f_2$.

(b) Let $\alpha = c_1(L) \in H^2(X, \mathbb{Z})$. Show that under the Hodge decomposition $\alpha = \alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}$, the term $\alpha^{2,0}$ vanishes. Conclude that the image of c_1 lies in $H^{1,1}(X, \mathbb{Z})$.

(7) (*) (a) Let $D = Z(s)$ be a smooth hypersurface on X defined by a section $s \in H^0(X, L)$. Let $\mathcal{I}_D$ denote the ideal sheaf of D, namely the subsheaf of $\mathcal{O}_X$ consisting of functions vanishing along Y. Show that $\mathcal{I}_D$ can be identified with L^*.

(b) Show that there is a short exact sequence

$$0 \to L^{\otimes k-1} \to L^{\otimes k} \to L^{\otimes k}|_D \to 0,$$

where if $\iota : D \hookrightarrow X$ denotes the inclusion, and $\iota_* \mathcal{F}(U) = \mathcal{F}(\iota^{-1}U)$ for $\mathcal{F}$ a sheaf, then $L^{\otimes k}|_D$ means the sheaf $\iota_*(L^{\otimes k}|_D)$, so that

$$\iota_*(L^{\otimes k}|_D)(U) = \mathcal{O}((L^{\otimes k})|_D)(U \cap D).$$

(c) Suppose now that L is positive. Assuming Bertini's Theorem that (the divisor associated to) a general section of a basepoint free line bundle is smooth, and the general result (which we did not prove) that $H^1(X, L^{\otimes k}) = 0$ for $k \gg 0$ when L is ample, show that $\dim H^0(X, L^{\otimes k})$ is a polynomial of degree $\dim X$ for $k \gg 0$.

[One can show in addition that if $\alpha \in c_1(\mathcal{O}(D))$ and $\omega \in c_1(L)$ are closed then $\int_X \alpha \wedge \omega^{n-1} = \int_D \omega^{n-1}$. Using this one can show by induction that the leading order term of the polynomial constructed is $\int_X \frac{\omega^n}{n!}$.]

(8) If h is a hermitian metric on a holomorphic line bundle L with curvature F_D, show that $\frac{i}{2\pi} F_D \in c_1(L)$.

[One can show that given $\omega \in c_1(L)$, there exists a hermitian metric h on L with $\frac{i}{2\pi} F_D = \omega$. If ω is Kähler, it follows that there is a hermitian metric h on L whose curvature is ω. Thus L is positive if and only if $c_1(L)$ is a Kähler class]

(9) Prove the Bianchi identity that $D(F_D) = 0$.

(10) Let E be a holomorphic vector bundle and let $X^* \subset X$ be a complex submanifold with $X \backslash X^*$ of codimension at least 2. Suppose s is a holomorphic section of E over X^*. Show that E extends to a holomorphic section of E.

(11) Let ω be a Kähler metric on a complex manifold X. Explain how ω^n can be seen as a hermitian metric on K_X^*. The Ricci curvature $\operatorname{Ric} \omega$ is the curvature of this hermitian metric

$$\operatorname{Ric} \omega = -\frac{i}{2\pi} \partial \bar{\partial} \log \omega^n$$

(this agrees with the Ricci curvature in Riemannian geometry). Show that if ω and ω' are two (not necessarily cohomologous) Kähler metrics, then $[\operatorname{Ric} \omega] = [\operatorname{Ric} \omega']$ (this can be done in two ways: using $\frac{i}{2\pi} F_D \in c_1(L)$, or directly from the definition of the Ricci curvature).

[This class is denoted $c_1(X)$ *and equals* $c_1(K_X^*)$ *through the interpretation of* ω^n*. Yau's solution of the Calabi conjecture states that for any Kähler class* $\kappa \in H^2(X, \mathbb{R})$*, given any closed* $\alpha \in c_1(X)$*, there exists a unique* $\omega \in \kappa$ *with* $\mathrm{Ric}\,\omega = \alpha$*.* X *is said to be Calabi-Yau if* $c_1(X) = 0$*, and hence taking* $\alpha = 0$ *there is a Ricci flat metric in each Kähler class on a Calabi-Yau manifold.]*

(12) Let L be a line bundle on a compact complex manifold X. Show that if $c_1(L)$ admits a Kähler metric, then L admits a hermitian metric h with curvature $\frac{i}{2\pi}F_h \in c_1(L)$ which is itself Kähler.