SIGNS FOR THE HODGE STAR OPERATOR

PART III COMPLEX MANIFOLDS 2019

1. Manifolds

For manifolds we defined the Hodge star operator in such a way that

$$(\alpha, \beta)_h \, \mathrm{dVol} = \alpha \wedge \star\beta.$$

(The point is that this is slightly different to Huybrechts' definition). We then defined adjoints

$$\partial^* = - \star \partial \star$$

and

$$\bar{\partial}^* = - \star \bar{\partial} \star .$$

Then we proved:

Lemma 1.1. *∂^* is the L^2-adjoint of ∂. That is, for $\alpha \in \mathcal{A}_{\mathbb{C}}^{p,q}(X)$ and $\beta \in \mathcal{A}_{\mathbb{C}}^{p+1,q}(X)$, we have*

$$\langle \partial\alpha, \beta \rangle_{L^2} = \langle \alpha, \partial^* \beta \rangle_{L^2}.$$

Proof. The L^2-inner product is

$$\begin{aligned} \langle \partial\alpha, \beta \rangle_{L^2} &= \int_X (\partial\alpha, \beta)_h \, \mathrm{dVol}, \\ &= \int_X \partial\alpha \wedge \star\beta. \end{aligned}$$

By Stokes' Theorem

$$0 = \int_X d(\alpha \wedge \star\beta) = \int_X \partial(\alpha \wedge \star\beta),$$

using that $\alpha \wedge \star\beta$ is an $(n-1, n)$-form. Next

$$\int_X \partial(\alpha \wedge \star\beta) = \int_X (\partial\alpha \wedge \star\beta + (-1)^{p+q} \alpha \wedge \partial \star \beta).$$

Thus

$$\begin{aligned} \langle \partial\alpha, \beta \rangle_{L^2} &= \int_X \partial\alpha \wedge \star\beta, \\ &= (-1)^{p+q+1} \int_X \alpha \wedge \partial \star \beta, \\ &= (-1)^{p+q+1+(p+q)(2n-(p+q))} \int_X \alpha \wedge \star(\star\partial \star \beta), \\ &= \langle \alpha, \partial^* \beta \rangle \end{aligned}$$

using

$$\star\star = (-1)^{(p+q)(2n-p-q)}.$$

$\square$

2. Bundles

Let h be a Hermitian metric on E, viewed as giving a map $h : E \to E^*$. The consistent way to define the Hodge star operator

$$\star_E : \wedge^{p,q} T^*X \otimes E \to \wedge^{p,q} T^*X \otimes E^*$$

is to define

$$\star_E(\varphi \otimes s) = (\star\varphi) \otimes h(s)$$

for $\varphi \in \mathcal{A}^{p,q}_{\mathbb{C}}(U)$ and $s \in C^\infty(E)(U)$ and extend linearly. The Hodge star operator then satisfies

$$(\alpha, \beta)\,\mathrm{dVol} = \alpha \wedge \star_E \beta$$

using the natural inner product on the left, for $\alpha, \beta \in \mathcal{A}^{p,q}_{\mathbb{C}}(E)(U)$ as in the lectures. Note also that this operator satisfies $\star_E \star_{E^*} = (-1)^{(p+q)(2n-p-q)}$ as before, where $\star_{E^*}$ is the Hodge star using the dual Hermitian metric h^*, which satisfies $h^*(h(s)) = s$. Here we remark again that "$\alpha \wedge \star\beta$" means wedge on the form part and evaluation $E \otimes E^* \to \mathbb{C}$ on the bundle part. One then defines an adjoint

$$\bar\partial^*_E = - \star_{E^*} \bar\partial_{E^*} \star_E .$$

To check this is a reasonable, using the usual L^2-inner product we explicitly prove:

Lemma 2.1. *$\bar\partial^*$ is the L^2-adjoint of $\bar\partial$. That is, for $\alpha \in \mathcal{A}^{p,q}_{\mathbb{C}}(E)$ and $\beta \in \mathcal{A}^{p,q+1}_{\mathbb{C}}(E)$, we have*

$$\langle \bar\partial\alpha, \beta\rangle_{L^2} = \langle \alpha, \bar\partial^*\beta\rangle_{L^2}.$$

Proof. The proof is almost verbatim the one given above. The L^2-inner product is

$$\begin{aligned}\langle \bar\partial\alpha, \beta\rangle_{L^2} &= \int_X (\bar\partial\alpha, \beta)\,\mathrm{dVol}, \\ &= \int_X \bar\partial\alpha \wedge \star_E \beta.\end{aligned}$$

By Stokes' Theorem

$$0 = \int_X d(\alpha \wedge \star_E \beta) = \int_X \bar\partial(\alpha \wedge \star_E \beta),$$

using that $\alpha \wedge \star_E \beta$ is an $(n, n-1)$-form. Next one checks that the definitions of $\bar\partial_E$ and $\bar\partial_{E^*}$ provide

$$\int_X \bar\partial(\alpha \wedge \star_E \beta) = \int_X (\bar\partial_E \alpha \wedge \star_E \beta + (-1)^{p+q} \alpha \wedge \bar\partial_{E^*} \star_E \beta).$$

Thus

$$\begin{aligned}\langle \bar\partial\alpha, \beta\rangle_{L^2} &= \int_X \bar\partial_E \alpha \wedge \star_E \beta, \\ &= (-1)^{p+q+1} \int_X \alpha \wedge \bar\partial_{E^*} \star_E \beta, \\ &= (-1)^{p+q+1+(p+q)(2n-(p+q))} \int_X \alpha \wedge \star_E (\star_{E^*} \bar\partial \star_E \beta), \\ &= \langle \alpha, \bar\partial^*_E \beta\rangle_{L^2}\end{aligned}$$

using

$$\star_E \star_{E^*} = (-1)^{(p+q)(2n-p-q)}.$$

□

Lemma 2.2 (Nakano Identity). *Defining* $(D^{1,0})^* = - \star_{E^*} D^{1,0}_{E^*} \star_E$*, we have*
$$[\Lambda, \bar{\partial}_E] = -i(D^{1,0})^*.$$

Proof. This now follows from the argument in lectures using the normal frame. $\square$