CORRIGENDUM TO "MODULI OF POLARISED MANIFOLDS VIA CANONICAL KÄHLER METRICS"

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M. Hattori has kindly pointed out an error in the construction of the CM line bundle in [1, Theorem 1.3], whom we greatfully thank, and which we take the opportunity to correct. The moduli space of cscK manifolds constructed in [1, Theorem 1] parametrises compact Kähler manifolds (X, α), with α a Kähler class on X, such that α admits a cscK metric. Through the construction, two families $(\mathcal{X}, \alpha_{\mathcal{X}}) \to B$ and $(\mathcal{X}', \alpha_{\mathcal{X}'}) \to B$, with $\alpha_{\mathcal{X}}$ and $\alpha_{\mathcal{X}'}$ relative Kähler classes, are considered isomorphic if there is a morphism $\varphi : \mathcal{X} \to \mathcal{X}'$, commuting with the projections to B, such that $\alpha_{\mathcal{X}} = \varphi^* \alpha_{\mathcal{X}'}$ (up to pullback from B). This corresponds to what Hattori calls a moduli space of *numerical equivalence classes* [2], and leads to a moduli space $\mathcal{M}_{\mathrm{num}}$ with a natural Weil–Petersson Kähler metric $\omega_{\mathcal{M}_{\mathrm{num}}}$.

It is further claimed in [1, Theorem 1.3] that—in the projective setting, so that all Kähler classes are integral—the space $\omega_{\mathcal{M}_{\mathrm{num}}}$ admits a natural CM line bundle $\mathcal{L}$ with $\omega_{\mathcal{M}_{\mathrm{num}}} \in c_1(\mathcal{L})$. As explained by Hattori [2, Remark 1.3], the moduli space $\mathcal{M}_{\mathrm{num}}$ is not the appropriate one to admit a genuine ample line bundle: the moduli space of numerical equivalence classes merely admits the numerical equivalence class of an ample line bundle (in particular, a Kähler class). The issue in the proof of [1, Theorem 1.3] arises in [1, Section 4.2], where line bundles are constructed in local charts on the moduli space (which are GIT quotients of Kuranishi spaces), with the line bundle being glued across charts. The gluing argument requires the isomorphisms of the line bundles to be constructed in a canonical way, which requires more structure than is provided by the approach of [1, Section 4.2].

To construct the relevant moduli space which does admit a CM line bundle, one follows an identical strategy, with a slightly different basic setup. We recall that the moduli space $\omega_{\mathcal{M}_{\mathrm{num}}}$ is constructed by taking a symplectic quotient of $\mathcal{J}^{\mathrm{Int}}(M, \omega)$ of the space of integrable almost complex structures on a fixed compact symplectic manifold, where the moment map is the scalar curvature and the group $\mathcal{G}$ is the group of exact symplectomorphisms. The proof then proceeds by producing a complex structure on this space.

In the projective case, it is more natural to incorporate line bundles from the beginning of the setup. Our basic data is now a collection $(M, \omega, L, h, \nabla)$, where (M, ω) is compact symplectic, L is a Hermitian line bundle with a unitary connection connection, and the curvature of ∇ is $-i\omega$. If J is an integrable almost complex structure on L, then the condition that the curvature of ∇ is $-i\omega$ implies that the $(0, 2)$-component of the curvature of ∇ vanishes, giving L the structure of a holomorphic line bundle. Letting $\mathcal{J}_{\mathrm{lin}}$ denote the space of almost complex structures compatible with (M, ω), we proceed as before, but with a different gauge group: we let $\mathcal{G}_{\mathrm{lin}}$ be the collection of fibrewise linear maps $\varphi : L \to L$ which cover

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a diffeomorphism $\varphi_M : M \to M$, with φ and φ_M preserving the various structures (so that, in particular, φ_M is a symplectomorphism of (M, ω)). There is a short exact sequence

$$1 \to S^1 \to \mathcal{G}_{\text{lin}} \to \mathcal{G} \to 1,$$

and thus the Lie algebra of $\mathcal{G}_{\text{lin}}$ is isomorphic to $C^\infty(M, \mathbb{R})$.

Starting from this setup, the proofs are completely identical to [1], hence we give only a very brief outline of the difference in outcome. The scalar curvature remains a moment map, and we may define $\mathcal{M}_{\text{lin}}$ as the symplectic quotient of (the integrable locus of) $\mathcal{J}_{\text{lin}}$. We may endow this space with a complex structure, again in an identical manner to [1], producing a moduli space $\mathcal{M}_{\text{lin}}$. The main difference is that the universal family over $\mathcal{J}_{\text{lin}}$ now admits a universal Hermitian line bundle with connection, and when one constructs the Kuranishi spaces as subspaces of $\mathcal{J}_{\text{lin}}$, these Kuranishi spaces admit natural holomorphic line bundles (by pullback from the Kuranishi space, or by direct construction). Importantly, the choice of gauge group means that polystable orbits in the Kuranishi space correspond to isomorphism classes of (X, L_X), where $c_1(L_X)$ admits a cscK metric, and where isomorphism means that the line bundles (and not just their first Chern classes) are identified. This allows the construction of the CM line bundle to proceed as claimed in [1, Section 4.2].

Note that the resulting moduli space $\mathcal{M}_{\text{lin}}$ corresponds to the *moduli space of linear equivalence classes* in [2]: families $(\mathcal{X}, \mathcal{L}_\mathcal{X}) \to B$ and $(\mathcal{X}, \mathcal{L}_\mathcal{X}) \to B$ are identified if there is an isomorphism $\varphi : \mathcal{X} \to \mathcal{X}'$, commuting with the projections to B, with $\varphi^* \mathcal{L}_{\mathcal{X}'} = \mathcal{L}_\mathcal{X}$ (up to pullback from B).

References

[1] R. Dervan and P. Naumann, *Moduli of polarised manifolds via canonical Kähler metrics*. To appear in Ann. Inst. Fourier, arXiv:1810.02576.

[2] M. Hattori, *On positivity of CM line bundles on the moduli space of klt good minimal models with $\kappa = 1$*. To apper on the arXiv.

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