$\bar{\partial}$-POINCARÉ LEMMA

PART III COMPLEX MANIFOLDS 2019

We prove the $\bar{\partial}$-Poincaré Lemma in one variable, correcting a couple of typos in the lectures, and mostly following Huybrechts' book.

We begin with the generalised Cauchy integral formula.

Proposition 0.1. *Let $D = D(a, r)$ be a disc in $\mathbb{C}$ with $r < \infty$, $z \in D$ and $f \in C^\infty(\bar{D})$. Then*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i}\int_D \frac{\partial f}{\partial \bar{w}}(w)\frac{dw \wedge d\bar{w}}{w-z}.$$

Proof. Let $D_\epsilon = D(z, \epsilon)$ and denote

$$\eta = \frac{1}{2\pi i}\frac{f(w)}{w-z} dw \in \mathcal{A}^1_{\mathbb{C}}(D \backslash D_\epsilon).$$

Then $d\eta = \bar{\partial}\eta$ since $dw \wedge dw = 0$ and so

$$d\eta = -\frac{1}{2\pi i}\frac{\partial f(w)}{\partial \bar{w}}\frac{dw \wedge d\bar{w}}{w-z}.$$

Stokes' Theorem gives

$$\frac{1}{2\pi i}\int_{\partial D_\epsilon} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i}\int_{D\backslash D_\epsilon} \frac{\partial f}{\partial \bar{w}}(w)\frac{dw \wedge d\bar{w}}{w-z}.$$

We first show

$$\frac{1}{2\pi i}\int_{\partial D_\epsilon} \frac{f(w)}{w-z} dw \to f(z) \text{ as } z \to 0.$$

Indeed, changing variables by $w - z = re^{i\theta}$ gives

$$\frac{1}{2\pi i}\int_{\partial D_\epsilon} \frac{f(w)}{w-z} dw = \frac{1}{2\pi}\int_0^{2\pi} f(z + \epsilon e^{i\theta}) d\theta.$$

The latter integral converges to $f(z)$ as $z \to 0$ as f is smooth.

As $dw \wedge d\bar{w} = -2ir dr \wedge d\theta$ we have

$$\left|\frac{\partial f}{\partial \bar{w}}\frac{dw \wedge d\bar{w}}{w-z}\right| = 2\left|\frac{\partial f}{\partial \bar{w}} dr \wedge d\theta\right| \leq C|dr \wedge d\theta|,$$

so by absolute integrability of $|\frac{\partial f}{\partial \bar{w}}\frac{dw \wedge d\bar{w}}{w-z}|$ over D we see

$$\int_{D_\epsilon} \frac{\partial f}{\partial \bar{w}}(w)\frac{dw \wedge d\bar{w}}{w-z} \to 0 \text{ as } \epsilon \to 0.$$

Thus taking the limit as $\epsilon \to 0$ gives

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i}\int_D \frac{\partial f}{\partial \bar{w}}(w)\frac{dw \wedge d\bar{w}}{w-z},$$

as required.

□

Using this we can prove the $\bar{\partial}$-Poincaré Lemma in one variable.

Theorem 0.2. *Let $D = D(a, r)$ be a disc in $\mathbb{C}$ with $r < \infty$, $z \in D$ and $g \in C^\infty(\bar{D})$. Then*

$$f(z) = \frac{1}{2\pi i}\int_D \frac{g(w)}{w-z} dw \wedge d\bar{w} \in C^\infty(D)$$

is smooth and on D satisfies

$$\frac{\partial f(z)}{\partial \bar{z}} = g(z).$$

Proof. We first reduce to the case that g has compact support.

Pick $z_0 \in D$ and choose $\epsilon > 0$ such that

$$D_{2\epsilon} = D(z_0, 2\epsilon) \subsetneq D.$$

Using a partition of unity for the cover $\{D \backslash D_\epsilon, D_{2\epsilon}\}$, write

$$g(z) = g_1(z) + g_2(z)$$

with $g_1(z), g_2(z) \in g \in C^\infty(\bar{D})$, such that g_1 vanishes outside of $D_{2\epsilon}$ and g_2 vanishes in D_ϵ.

Define

$$f_2(z) = \frac{1}{2\pi i}\int_D \frac{g_2(w)}{w-z} dw \wedge d\bar{w}.$$

Then $f_2 \in C^\infty(D_\epsilon)$ as g_2 vanishes near z when $z \in D_\epsilon$. Differentiating under the integral sign (permissible as f_2 is smooth) gives

$$\frac{\partial f_2}{\partial \bar{z}}(z) = \frac{1}{\pi i}\int_D \frac{\partial}{\partial \bar{z}}\left(\frac{g_2(w)}{w-z}\right) dw \wedge d\bar{w} = 0$$

as $g_2(w)$ is independent of $\bar{z}$.

As $g_1(z)$ has compact support, we can write

$$\begin{aligned}
\frac{1}{2\pi i}\int_D \frac{g_1(w)}{w-z} dw \wedge d\bar{w} &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(w)}{w-z} dw \wedge d\bar{w}, \\
&= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(u+z)}{u} du \wedge d\bar{u}, \\
&= -\frac{1}{\pi}\int_{\mathbb{C}} g_1(z + re^{i\theta}) e^{-i\theta} dr \wedge d\theta.
\end{aligned}$$

We then set

$$f_1(z) = \frac{1}{2\pi i}\int_D \frac{g_1(w)}{w-z} dw \wedge d\bar{w} \in C^\infty(D),$$

where the smoothness follows from the last representation of the integral. Again by smoothness differentiating under the integral sign and using the chain rule in two variables

$$\begin{aligned}
\frac{\partial f_1}{\partial \bar{z}}(z) &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial g_1(z + re^{i\theta})}{\partial \bar{z}} e^{-i\theta} dr \wedge d\theta, \\
&= -\frac{1}{\pi}\int_{\mathbb{C}} \left(\frac{\partial g_1}{\partial \bar{w}}\frac{\partial(\bar{z} + re^{-i\theta})}{\partial \bar{z}} + \frac{\partial g_1}{\partial w}\frac{\partial(z + re^{-i\theta})}{\partial \bar{z}}\right) e^{-i\theta} dr \wedge d\theta, \\
&= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial g_1(z + re^{i\theta})}{\partial \bar{w}} e^{-i\theta} dr \wedge d\theta \\
&= \int_D \frac{\partial g_1(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z}.
\end{aligned}$$

By the generalised Cauchy integral formula we obtain

$$g_1(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{g_1(w)}{w-z}dw + \frac{1}{2\pi i}\int_D \frac{\partial g_1}{\partial \bar{w}}(w)\frac{dw \wedge d\bar{w}}{w-z} = \frac{\partial f_1}{\partial \bar{z}}(z),$$

since g_1 vanishes on ∂D.

Setting $f = f_1 + f_2$, we have $f \in C^\infty(D_\epsilon)$ and for $z \in D_\epsilon$ we have

$$g(z) = g_1(z) = \frac{\partial f_1}{\partial \bar{z}}(z) = \frac{\partial f}{\partial \bar{z}}(z).$$

To finish the proof, note that z_0 was arbitrary and the functions constructed on $D(z_0, \epsilon)$ and some other $D(\hat{z}_0, \hat{\epsilon})$ agree when the discs overlap.

□