1 (a) Let $J_{st} : T\mathbb{C}^n \to T\mathbb{C}^n$ be the standard almost complex structure.

Let $\varphi : U \to \varphi(U) \subset \mathbb{C}^n$ be a chart, $U \subset X$

Define $J : TU \to TU$ by

$$J = d\varphi^{-1} \circ J_{st} \circ d\varphi : TU \to TU$$

Want to show that this is independent of chart. Let $\tilde{\varphi} : V \to \tilde{\varphi}(V) \subset \mathbb{C}^n$ be another.

We assume $U = V$.

We need

$$d\varphi^{-1} \circ J_{st} \circ d\varphi = d\tilde{\varphi}^{-1} \circ J_{st} \circ d\tilde{\varphi}$$

where $\tilde{\varphi} \circ \varphi^{-1} : \varphi(U) \to \tilde{\varphi}(U)$

<u>Lemma</u> Suppose $F : U^{\subseteq \mathbb{C}^m} \to V^{\subseteq \mathbb{C}^n}$ is smooth. Then F is holomorphic if and only if $dF \circ J_{st} = J_{st} \circ dF$

<u>Proof</u> F is holomorphic if and only if dF is $\mathbb{C}$-linear (by the Cauchy-Riemann equations)

$$z_k = x_k + i y_k$$
$$F_j = u_j + i v_j$$
$$\frac{\partial u_j}{\partial x_k} = \frac{\partial v_j}{\partial y_k}, \quad \frac{\partial u_j}{\partial y_k} = -\frac{\partial v_j}{\partial x_k}.$$

$$dF = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial x_2} & \cdots \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \cdots & \\ \frac{\partial u_2}{\partial x_1} & & \ddots & \\ \vdots & & & \ddots \end{pmatrix}$$

Then $dF \circ J_{st} = J_{st} \circ dF$

is equivalent to the CR equations.

$$J_{st} = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}$$

$\blacksquare$

(b) $\varphi : X \to Y$ smooth, complex manifolds

$J_X \quad J_Y$ almost complex structures

Want φ is holomorphic if and only if

$$d\varphi \circ J_X = J_Y \circ d\varphi.$$

We use (a)

$\varphi : \underset{U^c}{X} \to \underset{V^c}{Y}$ is holomorphic

if and only if

if and only if

$$\left(\begin{array}{l} \psi_x : U \to \psi_x(U) \subset \mathbb{C}^n \\ \psi_y : V \to \psi_y(V) \subset \mathbb{C}^m \end{array}\right)$$

$$\psi_y \circ \varphi \circ (\psi_x)^{-1} : \psi_x(U) \to \psi_y(U)$$

is holomorphic

if and only if (by (a))

$$\underline{d}(\psi_y \circ \varphi \circ \psi_x^{-1}) \circ J_S = J_{St} \circ d(\psi_y \circ \varphi \circ \psi_x^{-1})$$

$$d\varphi \circ (d\psi_x)^{-1} \circ J_{St} = \overbrace{d\psi_y^{-1} \circ J_{St} \circ d\psi_y} \circ d\varphi \circ d\psi_x^{-1}$$

$$= J_y \circ d\varphi \circ d\psi_x^{-1}$$

$$d\varphi \circ \underbrace{(d\psi_x^{-1}) \circ J_{St} \circ d\psi_x}_{= J_x} = J_y \circ d\varphi$$

$$d\varphi \circ J_x = J_y \circ d\varphi.$$

(c) $\varphi_1 : \mathbb{P}^1 \to \mathbb{P}^1$ smooth

$\varphi_2 : \mathbb{P}^1 \to \mathbb{P}^1$

φ_2 : " "

φ_1 holomorphic

φ not holomorphic.

$\varphi_1 : \mathbb{P}^1 \to \mathbb{P}^1$ identity map

$\varphi_1(z) = z$ for all z

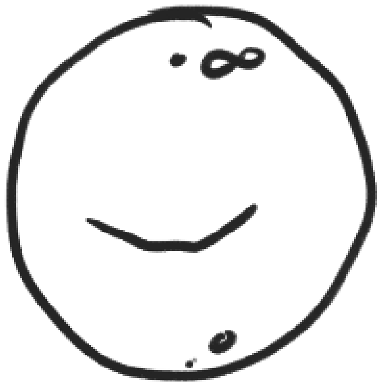
φ_2

$[1:0]$ 0

$[0:1]$ ∞

$\mathbb{P}^1$

$\mathbb{P}^1 = \mathbb{C} \cup \infty$



$\varphi_2(z) = \bar{z}$

$z \in \mathbb{C}$

$\varphi_2(\infty) = \infty$

φ_2 smooth: true locally

$$\mathbb{P}^1 = \underset{\mathbb{P}^1 \setminus \infty}{\underset{\shortparallel}{\mathbb{C}}} \cup \underset{\mathbb{P}^1 \setminus 0}{\underset{\shortparallel}{\mathbb{C}}}$$

Then $\varphi_1 : \mathbb{P}^1 \setminus \infty \to \mathbb{P}^1 \setminus \infty$

Then φ_2. ...

$$\varphi_2(z) = \bar{z}$$

$\varphi_2(z)$ holom

if and only if

$$\frac{\partial \varphi_2(z)}{\partial \bar{z}} = 0$$

but $\frac{\partial \bar{z}}{\partial \bar{z}} = 1$

__Q__ Can one always take an open cover $\mathcal{U}$ of X such that Čech cohomology agrees with sheaf cohomology wrt $\mathcal{U}$?

__Theorem__ (see Voisin's book)
To calculate Čech cohomology it's enough to use an open cover $\mathcal{U}$ of X such that

$$H^p(U_j, F) = 0 \quad \forall\ p > 0$$

(F corresponding to vector bundle)

F coherent sheaf

So e.g. if U_j is affine

or if X is non-algebraic each U_j is Stein.

<u>Q</u> If

$$0 \to E \to F \to G \to 0$$

is a ses

then is there a $\mathcal{U}$

s.t.

$$0 \to E(U) \to F(U) \to G(U) \to 0$$

is exact $\forall\ U \in \mathcal{U}$?

This is true for everything in the course and everything you're likely to see.

Q Do we need to learn about divisors? In Q Sheet 4 they were mentioned

A. They only come up in Q7. (c), where you should replace divisor with "codim one complex submanifold"

If L is very ample then if $s \in H^0(X, L)$ is general then

$D = Z(s)$ is smooth

i.e. a complex submanifold.

Step 1 Calculate Betti numbers

$Z(f_1, f_2, f_3) \subset \mathbb{P}^n$

Step 2 Use symmetries of Hodge diamond

Step 3
Use results like

$$H^{q,p}_{\bar\partial}(X) \cong H^p(X, \Omega^q)$$

Ω^q sheaf of holomorphic q-forms

much more geometric!

Def X is Calabi-Yau & compact if $c_1(\Lambda^n T^* X) = 0$

X Hopf surface

Often one can actually explicitly write down (p,q)-forms that are closed but not exact.

$$0 \to TY^{(1,0)} \to TX^{(1,0)} \to N_{Y/X} \to 0$$

$$N_{Y/X} \cong \mathcal{O}(Y)|_Y$$

1 (d) Unseen

Take determinant of (c)

$$0 \to E \to F \to G \to 0$$

then $\det F \cong \det E \otimes \det G$.
(using transition functions)

2 (b)

(i)
$p_{jl} \in \mathcal{U}_{jl}$

$p_j - p_l |_{U_{jl}}$

$\delta(p_{jl}) = 0$

(ii) $H^1(S, \mathcal{O}) = 0$

$\exists$ G with

$\delta G = (p_{jl})$

Set $F|_{U_j} = G|_{U_j} - p_j$

and check well defined.

(c) Exercise!

Lectures:

$\left(H^q(X, \mathcal{O}) \cong H^{0,q}_{\bar{\partial}}(X) \right)$

$$H^q(X, \Omega^p) \cong H^{p,q}_{\bar\partial}(X).$$

<u>3</u> Not relevant for this year.

L on X

$$L = H_1 \otimes H_2^{-1}$$

H_1, H_2 very ample

$$c_1(L) = c_1(H_1) - c_1(H_2)$$

<u>Kähler</u> classes

$\alpha \in c_1(L)$

$$\alpha = \omega_1 - \omega_2$$

Kähler

$$F_{D_1} = (D_2 + a) \circ (D_2 + a)$$

$$D_1 = D_2 + a$$

(d), (e) unseen

(e) $$\underbrace{F_{D_1}}_{(1,1)} - \underbrace{F_{D_2}}_{(1,1)} = D_2(a) + \overbrace{a\wedge a}^{=0}$$

a $(1,0)$-form

$a\wedge a$ $(2,0)$-form

$$D_2(a) = \bar\partial a$$

5

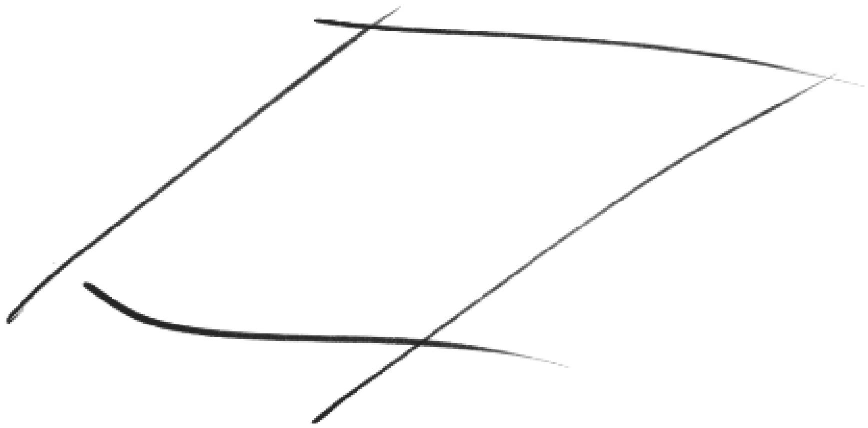
(d) unseen

$\partial\bar\partial$-Lemma

or Huybrechts

Blowups happen in dim $\underline{\underline{n \geq 2}}$

$\mathbb{C}^2$



$\mathbb{P}^2$

$z \in \mathbb{P}^2$ or $\mathbb{C}^2$

or a curve

$C \subset \mathbb{P}^2$

Blowups pass from high codim situations (e.g. a point) to codim $\underline{\underline{1}}$.

$$\mathrm{Bl}_z \mathbb{P}^2 \xrightarrow{\pi} \mathbb{P}^2$$

$$\pi^{-1}(z) = E \cong \mathbb{P}^1$$

curve (codim 1)

$$\pi| \ \mathrm{Bl}_z\mathbb{P}^2 \setminus E \to \mathbb{P}^2 \setminus z$$

biholom

<u>Points</u> E.g.

$$H^p(X, \underline{E} \otimes L^k) = 0$$

$\forall \ p > 0$

E holom bundle

$\underline{E}$ ample line bundle

L <u>ample</u>

$k \gg 1.$

$s \in H^0(X, L^k)$

$\rightsquigarrow Z(s) = D \subset X$

$c_1(X) \overset{def}{:=} c_1(\Lambda^n TX^{(1,0)})$

Kähler

$\Lambda^n TX^{(1,0)} = -K_X$

$K_X = -(-K_X)$

Suppose X 3-dimensional

<u>and</u> $c_1(X)$ Kähler

(X is Fano)

These are completely classified

(up to deformation, there are 105 classes)

Open question

Are there finitely many types of Calabi-Yau 3-folds? ($c_1(X) = 0$.)

Open problem!

(conjecture) Minimal model program

X up to birational transform

is "built" from

$\{X_j'\}$ with $\begin{cases} \underline{c_1(X_j') > 0} \\ c_1(X_j') < 0 \quad \text{general type} \\ \underline{c_1(X_j') = 0} \end{cases}$