12 Convection in rotating spherical fluid shells and its dynamo states

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The problem of convection in rotating spherical fluid shells is reviewed with emphasis on recently obtained results. The generation of magnetic fields by convection shows a strong dependence on the Prandtl number $P$ of the fluid. Results for the computationally accessible regime of convection driven dynamos in the parameter space are given and the validity of the magnetostrophic approximation is discussed. Of particular interest are various types of dipole oscillations, reversals and torsional oscillations.

1. Introduction

The exploration of the solar system by space probes and the expanding observations of stellar magnetic fields have led to a growing interest in the problem of the generation of magnetic fields by convection flow in rotating fluid spheres. In addition to the classical examples of geomagnetism and the solar magnetic field a large variety of magnetic fields generated in celestial bodies have become known and the conditions favoring different types of dynamos are receiving increasing attention. The facts that all planetary and stellar dynamos are highly turbulent and that they usually operate in the deep interiors of these celestial bodies pose severe challenges for the understanding of their nature. Nevertheless it seems desirable to explore the large parameter space of convection driven dynamos in rotating spherical fluid shells as far as it is accessible to computational simulations.

In this paper we shall review recent results on convection in rotating spheres and its dynamo action and add some new results on
its Prandtl number dependence. While most simulations of spherical
dynamos assume a Prandtl number \( P = 1 \), this parameter is likely to
differ from unity even in highly turbulent systems where the molecular
diffusivities are replaced by eddy diffusivities. High Rayleigh number
convection experiments (Ahlers and Xu, 2001) and theoretical consid-
erations (Eschrich and Rüdiger, 1983) suggest that the properties of
fully turbulent convection still depend on the Prandtl number. Low
Prandtl numbers are of special interest for applications because of the
metallic constituents of planetary cores and because the radiative heat
transport in stars contributes to a reduction of the effective Prandtl
number.

In the following we first describe the mathematical formulation of
the model in Section 2 and then turn to the problem of convection in
the absence of a magnetic field. After a short introduction to the lin-
ear problem of the onset of convection in Section 3 we focus on the low
Prandtl number regime which has received less attention in the past.
While Section 4 deals with columnar convection at Prandtl number of
the order \( P = 0.1 \), equatorially attached convection which becomes pre-
dominant at lower values of \( P \) is described in Section 5. The Prandtl
number \( P_c \) separating the two types of convection decreases with in-
creasing the Coriolis number \( \tau \), which is the dimensionless measure of
the rotation rate of the system. Convection-driven dynamos with \( P = 1 \)
are briefly reviewed in Section 6 before we turn to more recent results
obtained for higher Prandtl numbers. Dynamos in this regime appear
to require magnetic Prandtl numbers \( P_m \) which increase with \( P \). The
same trend continues to hold for Prandtl numbers less than unity. This
regime thus seems to be best suited for attaining the goal of minimal
values of \( P_m \). In Section 7 we shall report on typical properties of low
Prandtl number dynamos. Torsional oscillations are of special interest
because of their connection with jerks of the geomagnetic field and will
be considered in Section 8. Various types of oscillatory dipolar dynamos
are discussed in Section 9. The paper closes with a discussion and an
outlook on future research in Section 10.

2. Mathematical formulation of the problem

For the description of finite amplitude convection in rotating spherical
shells and its dynamo action we follow the standard formulation used in
earlier work by the authors (Zhang and Busse, 1989; Busse et al., 1998;
Grote et al., 1999, 2000). But we assume that a more general static
state exists with the temperature distribution $T_S = T_0 - \beta d^2 r^2/2 + \Delta T \eta r^{-1}(1-\eta)^{-2}$ where $\eta$ denotes the ratio of inner to outer radius of the shell and $d$ is its thickness. $\Delta T$ is the temperature difference between the boundaries in the special case $\beta = 0$. The gravity field is given by $g = -\gamma dr$ where $r$ is the position vector with respect to the center of the sphere and $r$ is its length measured in units of $d$. In addition to $d$, the time $d^2/\nu$, the temperature $\nu^2/\gamma \alpha d^4$ and the magnetic flux density $\nu(\mu \rho)^{1/2}/d$ are used as scales for the dimensionless description of the problem where $\nu$ denotes the kinematic viscosity of the fluid, $\kappa$ its thermal diffusivity, $\rho$ its density and $\mu$ is its magnetic permeability. We use the Boussinesq approximation in that we assume $\rho$ to be constant except in the gravity term where its temperature dependence given by $\alpha \equiv (d \rho/dT)/\rho =$ const. is taken into account. Since the velocity field $u$ as well as the magnetic flux density $B$ are solenoidal vector fields, the general representation in terms of poloidal and toroidal components can be used,

$$u = \nabla \times (\nabla u \times r) + \nabla w \times r,$$  

$$B = \nabla \times (\nabla h \times r) + \nabla g \times r.$$  

By multiplying the $(\text{curl})^2$ and the curl of the Navier-Stokes equations in the rotating system by $r$ we obtain two equations for $v$ and $w$

$$[(\nabla^2 - \partial_t) L_2 + \tau \partial_\phi] \nabla^2 v + \tau Qw - L_2 \Theta = -r \cdot \nabla \times [\nabla \times (u \cdot \nabla u - B \cdot \nabla B)],$$

$$[(\nabla^2 - \partial_t) L_2 + \tau \partial_\phi]w - \tau Qv = r \cdot \nabla \times (u \cdot \nabla u - B \cdot \nabla B),$$

where $\partial_t$ and $\partial_\phi$ denote the partial derivatives with respect to time $t$ and with respect to the angle $\phi$ of a spherical system of coordinates $r, \theta, \phi$ and where the operators $L_2$ and $Q$ are defined by

$$L_2 \equiv -r^2 \nabla^2 + \partial_r (r^2 \partial_r),$$

$$Q \equiv r \cos \theta \nabla^2 - (L_2 + r \partial_r)(\cos \theta \partial_r - r^{-1} \sin \theta \partial_\theta).$$

The heat equation for the dimensionless deviation $\Theta$ from the static temperature distribution can be written in the form

$$\nabla^2 \Theta + \left[R_i + R_c \eta r^{-3}(1-\eta)^{-2}\right] L_2 \Theta = P(\partial_t + u \cdot \nabla) \Theta.$$
and the equations for $h$ and $g$ are obtained through the multiplication of the equation of magnetic induction and of its curl by $\mathbf{r}$,

\begin{align}
\nabla^2 L_2 h &= P_m [\partial_t L_2 h - \mathbf{r} \cdot \nabla \times (\mathbf{u} \times \mathbf{B})], \\
\nabla^2 L_2 g &= P_m [\partial_t L_2 g - \mathbf{r} \cdot \nabla \times (\nabla \times (\mathbf{u} \times \mathbf{B}))].
\end{align}

(2d), (2e)

The Rayleigh numbers $R_i$ and $R_e$, the Coriolis parameter $r$, the Prandtl number $P$ and the magnetic Prandtl number $P_m$ are defined by

\begin{align}
R_i &= \frac{\alpha \gamma \beta d^6}{\nu \kappa}, \quad R_e = \frac{\alpha \gamma \Delta T d^4}{\nu \kappa}, \quad r = \frac{2 \Omega d^2}{v}, \quad P = \frac{\nu}{\kappa}, \quad P_m = \frac{\nu}{\lambda},
\end{align}

(3)

where $\lambda$ is the magnetic diffusivity. We assume stress-free boundaries with fixed temperatures:

\begin{align}
v &= \partial_r^2 v = \partial_r (w/r) = \Theta = 0, \quad \text{at} \quad r = r_i \equiv \eta/(1 - \eta) \\
\text{and at} \quad r = r_o = (1 - \eta)^{-1}
\end{align}

(4a)

Throughout this chapter the case $\eta = 0.4$ will be considered unless indicated otherwise. For the magnetic field electrically insulating boundaries are used such that the poloidal function $h$ must be matched to the function $h^{\text{ei}}$ which describes the potential fields outside the fluid shell

\begin{align}
g = h - h^{\text{ei}} = \partial_r (h - h^{\text{ei}}) = 0, \quad \text{at} \quad r = r_i \text{ and } r = r_o.
\end{align}

(4b)

But computations for the case of an inner boundary with no-slip conditions and an electrical conductivity equal to that of the fluid have also been done. The numerical integration of equations (2) together with boundary conditions (4) proceeds with the pseudo-spectral method as described by Tilgner and Busse (1997) which is based on an expansion of all dependent variables in spherical harmonics for the $\theta$, $\phi$-dependences, i.e.,

\begin{align}
v = \sum_{l,m} V_l^n(r, t) P_l^n(\cos \theta) \exp(i m \phi)
\end{align}

(4c)

and analogous expressions for the other variables, $w$, $\Theta$, $h$ and $g$. $P_l^n$ denotes the associated Legendre functions. For the $r$-dependence expansions in Chebychev polynomials are used. (For further details see also Busse et al., 1998, or Grote et al., 1999).

For the computations to be reported in the following a minimum of 33 collocation points in the radial direction and spherical harmonics up to the order 64 have been used. But in many cases the resolution was increased to 49 collocation points and spherical harmonics up to the order 96 or 128.
Figure 1. Banana cells in a thin rotating spherical fluid shell cooled from within. The motion is made visible by a suspension of flakes which become aligned with the shear.

3. Convection in rotating spherical shells

For an introduction to the problem of convection in spherical shells we refer to the recent review of Busse (2002a). The model of the rotating cylindrical annulus has been especially useful for the understanding of convection in rotating spheres. A rough idea of the dependence of the critical Rayleigh number $R_{ic}$ for the onset of convection on the parameters of the problem in the case $R_c = 0$ can be gained from the expressions derived from the annulus model (Busse, 1970):

$$R_{ic} = 3 \left( \frac{P \tau}{1 + P} \right)^{4/3} \frac{(\tan \theta)^{8/3}}{2^{2/3} \bar{r}_m^{13}},$$  \hspace{1cm} (6a)

$$m_c = \left( \frac{P \tau}{1 + P} \right)^{1/3} \frac{(r_m \tan \theta)^{3/2}}{2^{1/6}}$$  \hspace{1cm} (6b)

$$\omega_c = \left( \frac{\tau^2}{(1 + P)^2 P} \right)^{1/3} \frac{(\tan \theta)^{4/3}}{2^{5/6} \bar{r}_m^{2/3}},$$  \hspace{1cm} (6c)

where $r_m$ refers to the mean radius of the fluid shell, $r_m = (r_i + r_o)/2$, and $\theta_m$ to the corresponding colatitude, $\theta_m = \arcsin \left( r_m (1 - \eta) \right)$. The
azimuthal wavenumber of the preferred mode is denoted by $m_c$ and the corresponding angular velocity of the drift of the convection columns in the prograde direction is given by $\omega_c / m_c$. In Fig. 2 the expressions (6a,b) are compared with accurate numerical values which indicate that the general trend is well represented by expressions (6a,b). The same property holds for $m_c$. For a rigorous asymptotic analysis including the radial dependence we refer to Jones et al. (2000). Since we shall continue to restrict the attention to the case $Re = 0$, unless indicated otherwise, we shall drop the subscript $i$ of $R_i$.

There is a second mode of convection which becomes preferred at onset for sufficiently low Prandtl numbers. It is characterized by convection cells attached to the equatorial part of the outer boundary not unlike the “banana cells” seen in the narrow gap experiment of Fig. 1. The equatorially attached mode actually represents an inertial wave modified by the effects of viscous dissipation and thermal buoyancy. An analytical description of this type of convection can thus be attained through the introduction of viscous friction and buoyancy as perturbations as has been done by Zhang (1994) for the case of stress-free as well as for no-slip boundaries (Zhang, 1995). According to Ardes et al. (1997) equatorially attached convection is preferred at onset for $\tau < \tau_l$ where $\tau_l$ increases in proportion to $P^{-1/2}$.
4. Evolution of convection columns at small Prandtl numbers

In general the onset of convection in rotating fluid spheres occurs supercritically. As long as the convection assumes the form of shape-preserving traveling thermal Rossby waves as described by linear theory, its azimuthally averaged properties are time independent. In fact, as seen from a frame of reference drifting together with the convection columns the entire pattern is steady. A differential rotation is generated through the action of the Reynolds stress. The latter is caused by the spiraling cross section of the columns which persists as a dominant feature at moderate Prandtl numbers far into the turbulent regime. The plots of the streamlines \( r \partial v / \partial \phi = \text{const.} \) in the equatorial plane shown in Fig. 3 give a good impression of the spiraling nature of the columns.

A true time dependence of convection develops in the form of vacillations after a subsequent bifurcation. First the transition to amplitude vacillations occurs in which case just the amplitude of convection varies periodically in time as exhibited in the left plot of Fig. 3. At a somewhat higher Rayleigh number shape vacillations become noticeable which are characterized by periodic changes in the structure of the columns as shown in the right plot of Fig. 3. The outer part of the columns is stretched out, breaks off and decays. The tendency toward breakup is caused by the fact that the local frequency of propagation varies with distance from the axis according to expression (6c) after \( \theta_m \) has been replaced by the local colatitude \( \theta \).

![Figure 3](image-url)

**Figure 3.** Lines of constant \( r \partial v / \partial \phi \) in the equatorial plane in the case \( \tau = 3 \times 10^4 \), \( P = 0.1 \) for (a) \( R = 3 \times 10^5 \) and (b) \( R = 3.5 \times 10^5 \). The four sections are a quarter of the vacillation period, \( t_p = 0.0864 \) (left) and \( t_p = 0.0124 \) (right), apart with time progressing in the clockwise sense.
The two types of vacillations also differ significantly in their frequencies of oscillation. This is evident from the time records of the energy densities of convection which have been plotted in Fig. 4, namely

\[ E_p^m = \frac{1}{2} \langle |\nabla \times (\nabla \bar{v} \times \mathbf{r})|^2 \rangle, \quad E_t^m = \frac{1}{2} \langle |\nabla \bar{w} \times \mathbf{r}|^2 \rangle \]  

\[ E_p^f = \frac{1}{2} \langle |\nabla \times (\nabla \tilde{v} \times \mathbf{r})|^2 \rangle, \quad E_t^f = \frac{1}{2} \langle |\nabla \tilde{w} \times \mathbf{r}|^2 \rangle \]  

where \( \bar{v} \) refers to the azimuthally averaged component of \( v \) and \( \tilde{v} \) is defined by \( \tilde{v} = v - \bar{v} \). As the Rayleigh number is increased further a fairly sudden transition into a chaotic regime occurs where convection has become strongly inhomogeneous in space and in time. A typical sequence of plots is shown in Fig. 5 which covers about one period of the relaxation cycles seen in the time record for \( R = 3.8 \times 10^5 \) in Fig. 4. In contrast to the more common relaxation oscillations encountered at higher Rayleigh numbers — see for example the time record for \( R = 5 \times 10^5 \) in Fig. 4 — convection does not die off entirely at any time during the cycle. But the interaction between convection and differential rotation appears to be similar. As the amplitude of convection as measured by \( E_p^f \) and \( E_t^f \) grows, the differential rotation generated by the Reynolds stress grows as well with just a small delay in time. When the differential rotation reaches a critical level the convection columns become disrupted and their amplitude decays. Subsequently the differential rotation decays as well on the timescale of viscous diffusion. It is typical for this type of relaxation cycles that the viscous decay is shorter than the growth time of the differential rotation in contrast to the relaxation oscillations at higher values of \( R \). The regime of relaxation oscillations is interrupted once in a while by a return to the more regular regime of shape vacillations as shown in the inserted enlargement of the time record for \( R = 3.8 \times 10^5 \) in Fig. 4. But in contrast to the vacillation of the right plot Fig. 3 the pattern is now strongly modulated. The component with the azimuthal wavenumber \( m = 1 \) plays a dominant role in this modulation.

As \( R \) is further increased the spatio-temporal structure of convection becomes more irregular as can be seen in the time series for \( R = 4.5 \times 10^5 \) in Fig. 4. But at \( R = 5 \times 10^5 \) the relaxation oscillations with only intermittent convection become firmly established and they continue to persist up to Rayleigh numbers of the order of \( 10^6 \). They are basically the same phenomenon as found by Grote and Busse (2001) at \( \theta = 1 \) and even the period is the same, about 0.1, which corresponds to the viscous decay time of the differential rotation.
Figure 4. Time series of energy densities of convection in the case $\tau = 3 \cdot 10^4$, $P = 0.1$ for $R = 3 \times 10^5$, $3.3 \times 10^5$, $3.5 \times 10^5$, $3.8 \times 10^5$, $4 \times 10^5$, $4.5 \times 10^5$, $5 \times 10^5$ (from bottom to top). Solid, dot-dashed and dashed lines indicate $E_m^t$, $E_f^t$ and $E_p^t$ respectively. The critical Rayleigh number for onset of convection is $R_c = 222518$. $E_p^t$ is smaller by more than an order of magnitude than the other energy densities and has not been plotted for this reason.

The main difference between convection at $P = 0.1$ and Prandtl numbers of the order unity occurs at the transition between regular and irregular patterns. A typical scenario for $P = 0.5$ is shown in Fig. 6. After convection has set in in the form of eight drifting column pairs, the usual amplitude vacillations occur as the Rayleigh number
Figure 5. Sequence of plots equidistant in time (from top to bottom with $\Delta t = 0.05$) for $R = 3.8 \times 10^5$, $\tau = 3 \times 10^4$, $P = 0.1$. The left column shows streamlines, $r \partial v / \partial \phi = \text{const.}$, in the equatorial plane and in the middle column lines of constant radial velocity $u_r$ on the mid surface, $r = r_i + 0.5$, are shown. The left halves of the circles of the right column show lines of constant $u_\phi$, which is the azimuthally averaged azimuthal component of the velocity field. The right halves show streamlines of the axisymmetric meridional circulation.
is increased. The shape vacillations, however, exhibit a modulation with wavenumber \( m = 4 \) as shown in Fig. 7. Only every second pair of columns gets stretched until the outer part separates, a little earlier for the cyclonic column than for the anticyclonic one. Then the same process is repeated for the other pairs of columns such that the sequence shown in Fig. 7 exhibits only half a period of the oscillation. In addition, of course, the column pattern drifts in the prograde direction. With increasing Rayleigh number the stretching process gets out of phase and an \( m = 1 \)-modulation becomes noticeable. The time dependence is still periodic as indicated in the record for \( R = 3.45 \times 10^5 \) in Fig. 6. But the oscillations have become more complex in that some of the separated outer parts become attached to the preceding column pair. By the time when \( R \) has reached \( 3.5 \times 10^5 \) the modulated vacillations

**Figure 6.** Time series of energy densities of convection in the case \( \tau = 1.5 \times 10^4, P = 0.5 \) for (a) \( R = 3 \times 10^5, 3.2 \times 10^5, 3.45 \times 10^5, 3.5 \times 10^5 \) (from bottom to top) and (b) \( R = 5 \times 10^5, 7 \times 10^5, 8 \times 10^5, 10^6 \) (from bottom to top). Solid, dot-dashed and dashed lines indicate \( E_i^m, E_f^i \) and \( E_f^p \), respectively. The critical Rayleigh number for onset is \( R_c = 215142 \).
Figure 7. Modulated shape vacillations of convection for $R = 3.2 \times 10^5$, $\tau = 1.5 \times 10^4$, $P = 0.5$. The sequence of plots equidistant in time ($\Delta t = 0.005$), starting at the upper left and continuing clockwise, shows streamlines, $r\partial u/\partial\phi = \text{const}$, in the equatorial plane. Since the modulation period is about 0.025, the last plot resembles the first plot except for a shift in azimuth.

have become aperiodic and with increasing Rayleigh number convection becomes more and more chaotic. Regularity reappears only in the form of relaxation oscillations as shown in the section for $R = 7 \times 10^5$ of Fig. 6. Remainders of the vacillations can still be seen in this time record. Around $R = 8 \times 10^5$ the relaxation oscillations occur in combination with the phenomenon of localized convection (Grote and Busse, 2001) in that the convection is confined to a meridional sector with an extent of about 90° in longitude as shown in Fig. 8. At $R = 10^6$, however, the relaxation oscillations have become fully established with a sharp rise and a slower decay of the differential rotation and with convection occurring only in an intermittent fashion, but nearly homogeneously in azimuth.

5. **Convection at very small Prandtl numbers**

Since stellar interiors as well as metallic planetary cores are characterized by rather small Prandtl numbers the onset of convection in rotating fluid spheres in the limit of vanishing $P$ has received special attention.
Zhang and Busse (1987) found the equatorially attached mode which is quite distinct from the columnar mode discussed in the preceding section. The new mode represents an inertial oscillation which becomes excited when viscous dissipation is exceeded by the energy provided per unit time by thermal buoyancy. The fact that both energies can be regarded as small perturbations has led Zhang (1994, 1995) to solve the problem of onset of convection by an asymptotic analysis. An detailed numerical study together with some analytical approximations can be found in the paper of Ardes et al. (1997). The results of these various efforts have turned out to be rather complex since the preferred inertial modes travel in the prograde as well in the retrograde directions depending on the parameters of the problem. Moreover, the azimuthal wavenumber $m$ of convection does not increase monotonically with the Coriolis parameter $\tau$ as is usually found for the columnar mode at values of $P$ of the order unity or higher. In addition to the simple “single cell” inertial modes of Zhang and Busse (1987) multicellular modes have been found in the study of Ardes et al. (1997) which appear to be closely related to the multicellular modes described by the Airy function in the

![Figure 8. Streamlines, $r\partial v/\partial \phi = \text{const.}$, in the equatorial plane (upper row) and lines of constant $u_\phi$ (left halves of circles in lower row) and streamlines of the axisymmetric meridional circulation (right halves) are shown for the case $P = 0.5$, $R = 8 \times 10^5$, $\tau = 1.5 \times 10^4$ at the times $t = 0.1 + n0.05$, $n = 1, 2, 3$ (left to right) of the corresponding time series in Fig. 6.](image)
Figure 9. In the middle part: the Rayleigh numbers $R$ of the competing prograde (dotted line, empty circles) and retrograde mode (dashed line, filled circles) as well as the actual critical value (thick solid line) and the corresponding frequencies as a function of $\tau$ in the case $P = 0.001, \eta = 0.2, m = 6$. On the left and right: Contours of constant radial velocity $u_r$ (lower plots) and the streamlines, $w = \text{const.}$ on the spherical surface $r = 0.9$ (upper plots) for $\tau = 5\times 10^4$ and $6\times 10^4$.

analysis by Yano (1992) of the analogous problem of convection in the rotating cylindrical annulus. The numerical solutions for this problem obtained by Pino et al. (2000) also indicate the onset of multicellular convection in parts of the parameter space.

Here we shall present only some examples of the variety of patterns encountered in a rotating sphere of a low Prandtl number fluid. More details can be found in the recent paper of Simitev and Busse (2003).

In Fig. 9 a typical change from the retrograde to the prograde modes as a function of $\tau$ has been indicated and the corresponding changes in the patterns are shown. The retrograde and prograde modes differ little in their form. The opposite phase between toroidal and poloidal components of motion is the most characteristic difference as indicated in Fig. 9. The sense of outward spiraling is also opposite for prograde and retrograde modes as is evident from Fig. 10. But in the limit of vanishing Prandtl number or infinite $\tau$ the sense of spiraling disappears because the phase of the inertial modes does not vary with distance from the axis.

Whenever $\tau$ is sufficiently large the frequency $\omega$ is closely approximated by the frequency of the corresponding inertial modes which is given by the analytical expression (Zhang, 1994; Ardes et al., 1997),

$$\omega = \frac{\tau}{m + 2} \left( 1 \pm \left[ 1 + m(m + 2)(2m + 3)^{-1} \right]^{1/2} \right).$$  

(8)
Figure 10. Lines of constant $r \partial \nu / \partial \phi$ in the equatorial plane corresponding to the values of $R = 15293.7, 17190.8, 31963.2, 763401$, for $\tau = 950, 1500, 8 \times 10^6, 3.5 \times 10^7$, respectively, (left to right, first upper row then lower row) in the case $P = 10^{-4}$.

The negative (positive) sign applies for modes drifting in the pro-grade (retrograde) direction.

6. Convection driven dynamos at moderate Prandtl numbers

Numerous reviews of convection driven dynamos in rotation spherical shells have appeared in recent years (Busse, 2000; Busse et al., 2003; Dormy et al., 2000; Kono and Roberts, 2002 and others) which usually have focused on the case of $P = 1$, although some simulations have been carried out for Prandtl numbers as high as 900 (Glatzmaier and Roberts, 1995). Here we do not wish to provide still another general survey, but instead focus on the Prandtl number dependence and draw attention to a few questions that have not been answered satisfactorily. Most prominent among these questions is the dependence of the average magnetic density energy $M$ on the parameters of the problem. As a convenient definition for $M$ we shall use the sum $M = M_p^m + M_p^f + M_t^f$ where $M_p^m$ etc. are defined in direct analogy to the definitions (7)
for the kinetic energy densities. The most popular rules of thumb are the criterion that the Elsasser number

$$\Lambda = 2MP_m/\tau$$

(9)

assumes a value of the order unity and the equipartition criterion that magnetic and kinetic energies are in approximate balance. This latter criterion is satisfied, for example, when the dynamics of a turbulent plasma are dominated by Alfvén waves. But this situation is not realized in the case of planetary dynamos or in the case of the Sun. The condition $\Lambda \approx 1$ determines minimum Rayleigh number for the onset of convection in a horizontal fluid layer heated from below and rotating about a vertical axis when a homogeneous magnetic field is applied parallel to that axis (Chandrasekhar, 1961). Similar conditions apply in the case of a rotating sphere when an azimuthal magnetic field is applied (Fearn, 1979) or in the corresponding annulus model (Busse, 1983) and in some other configurations involving rotation and a homogeneous magnetic field (Eltayeb, 1972). There is a little evidence that the dynamo generated magnetic field plays a dynamical role similar to that of an nearly homogeneous imposed field. It has not been possible so far to find convection at Rayleigh numbers below the critical value in the presence of dynamo action as must be expected if the main function of the Lorentz force is to counteract the Coriolis force. Instead it has been found – at least for $P$ of the order unity or less – that the generated magnetic field inhibits the differential rotation and thereby increases the amplitude of convection and its heat transport (Grote and Busse, 2001; Busse et al., 2003).

Kinetic and magnetic energies have been plotted as a function of $R$ in Fig. 11 in the case $P = P_m = 1$, $\tau = 5 \times 10^5$. The Elsasser number $\Lambda$ approaches unity for the maximum value of the magnetic energy which is achieved at Rayleigh numbers in the neighborhood of $1.4 \times 10^5$. Flux expulsion and an increasingly filamentary structure of the magnetic field cause a decrease of the energy for higher values of $R$, while the Ohmic dissipation continues to increase. For the same reason axisymmetric components of the fields decay faster with increasing $R$ than the fluctuating nonaxisymmetric components. Since the critical value $R_c$ is only about $8.03 \times 10^4$, convection in the polar region has set in about $R \approx 5 \times 10^5$ and dipolar and quadrupolar components of the magnetic field contribute about equal amounts to the energy for $R \geq 10^6$.

That convection amplitudes in an intermediate range are most suitable for the generation of a global magnetic field appears to be a general
Figure 11. Kinetic (upper left) and magnetic (upper right) energy densities and viscous (lower left) and Ohmic (lower right) dissipation are plotted as a function of $R_i$ for convection driven dynamos in the case $\tau = 5 \times 10^3, P = P_m = 1$. Filled (open) symbols indicate toroidal (poloidal) components of the energies and dissipations, circles (squares) indicate axisymmetric (non-axisymmetric) components. The values of $R_i$ at the abscissa should be multiplied by $10^5$. The scales of the ordinates in the two lower plots must also be multiplied by the factor $10^5$.

The structure of dynamos corresponds to the expectation that quadrupolar dynamos are found for lower values of $P_m$ while dipolar dynamos predominate at higher values of $P_m$ and also at higher Rayleigh numbers. Here, however, a clear distinction can often no longer be made in that components of both symmetries contribute about equally to the magnetic energy. There exists a critical value $P_{mc}$ of the magnetic Prandtl number for dynamo action which appears to increase monotonically with $P$. At Prandtl numbers of the order unity $P_{mc}$ increases more strongly with $P$ because of the concurrent decrease of the differential rotation. The $\omega$--effect provided by the latter certainly enhances the dynamo action. As can be seen from the logarithmic plot of Fig. 14
Figure 12. Dynamo solutions indicated by red (dipolar), blue (quadrupolar), green (hemispherical) and yellow (mixed symmetry) balls in the $R - P - P_m$ parameter space. No dynamo solution could be obtained for values of $P, P_m$ in the shaded region.

Figure 13. Total magnetic energies in the cases $P = P_m = 10$, $\tau = 5 \times 10^5$ and $R = 3 \times 10^5$ (dotted line), $R = 5 \times 10^5$ (solid line), $R = 6 \times 10^5$ (dashed line), $R = 7 \times 10^5$ (dot-dashed line). The abscissa of the case $R = 3 \times 10^5$ is multiplied by the factor 0.65 and in the case $R = 7 \times 10^5$ by 0.286.
the energy $E_i^m$ quickly decreases in comparison to the energies $E_i^f$ and $E_p^f$ when $P$ is increased.

It is of interest to compare the major sources of the energies of the various components of the magnetic field. To do this the equations (2d,e) have been multiplied by $\nabla^2 h$, $\nabla^2 \hat{h}$ and by $-\bar{g}$ and $-\hat{g}$, respectively, and averaged over the spherical shell. Some typical time series are shown in Fig. 15 for the case $P = 1$. Only the major contributors have been included. As expected, the shearing of the mean poloidal field by the differential rotation is the main generator of the azimuthal field. The mean poloidal field is generated primarily by the interaction between fluctuating velocity and magnetic field components. The generation of the latter component also occurs mainly through the interaction of fluctuating components. An $\alpha$-effect in the traditional sense does not operate. This pattern of the dynamo process does not change much as the Prandtl number is increased. Even though the differential rotation decreases, it is still the major energy source for the mean azimuthal field.
Figure 15. The terms \( \langle \delta g h \rangle \) (solid line), \( \langle \dot{\delta} h \dot{h} \rangle \) (dotted line) and \( \langle \ddot{\delta} h \dot{g} \dot{h} \rangle \) (dashed line) in the upper plot and the terms \( \langle \dot{\theta} g h \rangle \) (solid), \( \langle \ddot{\theta} g h \rangle \) (dashed) and \( \langle \dddot{\theta} g \dot{h} \rangle \) (dotted) in the middle plot and the terms \( \langle \dot{\theta} g \dot{h} \rangle \) (solid), \( \langle \ddot{\theta} g \dot{h} \rangle \) (dashed) and \( \langle \dddot{\theta} g \dot{h} \rangle \) (dotted) in the lower plot are displayed as a function of time in the case \( \tau = 5 \times 10^3, R = 14 \times 10^5, P = P_m = 1 \). The notation \( \langle \delta g h \rangle \) indicates the contribution made by terms involving \( \delta \) and \( g \) when the equation (2d) is multiplied by \( \nabla^2 h \) and averaged over the fluid shell.

But the latter weakens significantly in comparison to the mean poloidal field with increasing \( P \) as is also evident from Fig. 16.

The fact that convection driven dynamos in rotating systems depend on a rather large number of parameters has led to numerous attempts to eliminate one or more parameters through reductions of the basic equations. The most important among these reductions is the magnetostrophic approximation in which the acceleration of fluid particles is neglected in comparison to the Coriolis force and the Lorentz force. This approximation is most easily obtained when the thermal timescale \( d^2/\kappa \) is used instead of the viscous timescale and \( \sqrt{\rho \mu \kappa / d} \) is used as scale of the magnetic field. The basic dimensionless equations of motion, equation of induction and the heat equation can then be written in the form

\[
P^{-1}(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \lambda \mathbf{k} \times \mathbf{u} = -\nabla \pi + \Theta \mathbf{r} + \nabla^2 \mathbf{u} + (\nabla \times \mathbf{B}) \times \mathbf{B},
\]

\[
\frac{K}{\lambda} (\partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{u}) = \nabla^2 \mathbf{B},
\]

\[
\partial_t \Theta + \mathbf{u} \cdot \nabla \Theta = R \mathbf{u} \cdot \mathbf{r} + \nabla^2 \Theta,
\]

where \( \mathbf{k} \) is the unit vector parallel to the axis of rotation. From the form of (10) it is clear that the magnetostrophic approximation should certainly be valid in the limit \( P \to \infty \). For values of \( P \) of the order unity or less the question of the validity of the magnetostrophic approximation is less obvious since the generation of differential rotation by the Reynolds stresses of convection is not represented in that approximation. Also, inertial oscillations which play a role in low Prandtl
Figure 16. (a) Kinetic energy densities $E_{\rho}^m$ (upper left panel), $E_{\rho}^f$ (upper right panel), $E_{\eta}^m$ (lower left panel) and $E_{\eta}^f$ (lower right panel) all multiplied by $P^2$ as a function of the Prandtl number $P$ in the case $\tau = 5 \times 10^3$. The dynamos corresponding to fixed ratios $\kappa/\lambda = 1, 2, 5, 0.5$ are indicated by solid, long-dashed, dash-dotted and dotted lines, respectively, and cases of pure convection by a short-dashed line. The values of the Rayleigh number $R = 5 \times 10^5, 6 \times 10^5, 8 \times 10^5, 10^6$ are denoted by empty circles and squares and full circles and squares, respectively. (b) Same as Fig. 16a but for the magnetic energy densities $M_{\rho}^m$ (upper left panel), $M_{\rho}^f$ (upper right panel), $M_{\eta}^m$ (lower left panel) and $M_{\eta}^f$ (lower right panel) all multiplied by $P$. 

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number convection cannot be represented in the magnetostrophic approximation. In Figs. 16a and b the energy densities have been plotted for fixed values of $\kappa/\lambda$. It can be seen that the kinetic energies tend to become independent of $P$ with increasing $P$ in accordance with the magnetostrophic assumption. The energy $E_m^\kappa$ representing the differential rotation is the only exception as expected. No indication of an approach toward the validity of the magnetostrophic approximation is found, however, when the magnetic energy densities are considered. As has already been mentioned the dynamo process is rather sensitive to the presence of the differential rotation and much higher values of $P$ may be needed before the magnetostrophic regime is approached.

Results obtained on the basis of the magnetostrophic approximation are by definition independent of $P$. In particular the ratio between the magnetic energy and kinetic energy will be proportional to $P$ (Glatzmaier and Roberts, 1995) as is evident from the different scales used for the velocity and for $B/\sqrt{\rho\mu}$ in (10a). The ratio between Ohmic and viscous dissipation would be independent of $P$ and would depend only on $\kappa/\lambda$. This latter parameter seems to be even more important than the magnetic Prandtl number for convection driven dynamos. At least for Prandtl number of unity or less $\lambda \lesssim \kappa$ appears to be a condition for the dynamo action as is indicated by a comparison of dynamos obtained for $P = 1$ (see Fig. 1 of Grote et al., 2002) and for $P = 0.1$ (see Fig. 17).

Figure 17. Convection driven dynamos of different types as a function of the Rayleigh number $R$ and the magnetic Prandtl number $P_m$ in the case $\tau = 10^5$ and $P = 0.1$. Dipolar and hemispherical dynamos are indicated by squares and triangles, respectively, while the circles correspond to decaying dynamos. The critical Rayleigh number for onset of convection at these parameters is $R_c = 1.035 \times 10^6$. 
Before leaving the magnetostrophic approximation we wish to mention that it is also obtained when the limit $P_m \to \infty$ is approached. This property is easily seen when $d^2/\lambda$ is used as timescale and the magnetic flux density $B$ is scaled with $\sqrt{\rho \mu \lambda / d}$. Instead with $P^{-1}$ the acceleration term in the equation of motion (10a) will then be multiplied by $P_m^{-1}$. Dynamo solution should become independent for $P_m \to \infty$ and the ratio of magnetic to kinetic energy would grow in proportion to $P_m$.

7. Convection driven dynamos in low Prandtl number fluids

A systematic exploration of the onset of dynamos in low Prandtl number spherical shells has been done in the case $P = 0.1$. The results are shown in Fig. 17 and indicate the usual behavior that the Rayleigh number must be increased for decreasing $P_m$ in order to reach a critical magnetic Reynolds number of the order 100 for dynamo action. The property that for larger values $P_m$ dipolar dynamos are realized while for lower values a transition to hemispherical dynamos occurs also agrees with the results obtained earlier for $P = 1$ (Grote et al., 2001). When dynamo at even lower values of $P_m$ than shown in Fig. 17 would be obtained a further transition to quadrupolar dynamos can be expected. Problems of numerical resolution have prevented us so far from reaching Rayleigh numbers beyond $10^7$. It is remarkable, however, that a dynamo at a magnetic Prandtl number as low as 0.1 could be attained which is considerably lower than the lowest value attainable in the case $P = 1$. The time averaged magnetic and kinetic energy densities for the dynamos of Fig. 17 have been plotted in Fig. 18. Also shown are the corresponding values of Ohmic and viscous dissipation. While in both cases magnetic and kinetic values are quite comparable, a tendency toward an increasing ratio of Ohmic to viscous dissipation with decreasing $P_m$ can be noticed. It appears that for the parameter range that has been investigated the Ohmic dissipation is the most important ingredient in determining the equilibrium amplitude of the magnetic field.

As is well known from Rayleigh-Bénard convection the level of turbulence strongly increases with decreasing Prandtl number for a given supercritical Rayleigh number owing to the increasing importance of the momentum advection term in the equation of motion. This effect is also observed in rotating systems, but it is moderated by the presence
Figure 18. Time averaged magnetic (heavy symbols) and kinetic (light symbols) energy densities (left side) and corresponding dissipations (right side) have been plotted for $P = 0.1$, $\tau = 10^5$. The magnetic Prandtl numbers are indicated in the figure. Energy densities and dissipation densities of axisymmetric field components are indicated by circles (poloidal) and squares (toroidal) while plus-signs and lying crosses denote the corresponding quantities for fluctuating poloidal and toroidal components, respectively. The mean poloidal kinetic energy densities have been multiplied by the factor 100, its dissipation by factor 10.

of the magnetic field. The latter appears to damp the time dependence of convection and the effective Prandtl number appears to be increased by the contribution of Ohmic dissipation. In fact computations of dynamos can be performed more efficiently than corresponding cases of convection without magnetic variables because of lower demands for numerical resolution.

The breaking of the differential rotation by the Lorentz force found in earlier work (Grote and Busse, 2001) continues to be an important effect in low Prandtl number dynamos. The anticorrelation between the strength of the magnetic field and the energy of differential rotation which is given by $E_{\theta}^{\text{rot}}$ is clearly evident in the time records shown in Fig. 19. Also apparent is the intermittent nature of the dynamos in this case which varies between a more dipolar and a more hemispherical structure. While the latter type dynamos always exhibit an oscillatory character, changes in the polarity of the dipolar state only occur in an aperiodic fashion in the case of Fig. 19. An example of such a reversal is shown in Fig. 20. It is remarkable to see that the reversal occurs in a
Figure 19. Time series of a convection driven dynamo with $\tau = 3 \times 10^4$, $R = 8.5 \times 10^5$, $P = 0.1$, $P_m = 1$. The first, second and third plot from the top show energy densities of the dipolar and quadrupolar components of the magnetic field and of the velocity field, respectively. The mean and fluctuating toroidal energy densities are indicated by red and blue lines, while the mean and fluctuating poloidal energy densities are black and green, respectively.

similar way as an oscillation in that magnetic flux with the new polarity is created close to the equatorial plane while the old flux is pushed to higher latitudes and dissipated in the polar regions. We shall return to the phenomenon of dipole oscillations in Section 9.

8. Torsional oscillations

Torsional oscillations have long been postulated for the Earth’s core (Braginsky, 1970) and there is some growing evidence that they manifest themselves in the special phenomenon of geomagnetic secular variation known as “jerks”. See the recent paper by Bloxham et al. (2002) and references quoted therein. The abrupt changes in the time derivative of the declination of the geomagnetic field have been noticed at
Figure 20. A magnetic field polarity reversal shown by a time sequence of equidistant plots (top to bottom) covering the time span from $t = 5.777$ to $t = 6.017$ of the time series of Fig. 19. The first column shows lines of constant $u_r$ on the surface $r = r_i + 0.5$. The second column exhibits lines of constant $B_r$ on the surface $r = r_o + 0.7$. The circles in the right column indicate lines of constant $B_\phi$ in their left halves and meridional field lines, $r \sin \theta \partial h/\partial \theta = \text{const.}$, in their right halves.
several observatories in the past, but they were usually interpreted as a more local phenomenon since some observatory records do not exhibit them. The analysis of Bloxham et al. (2002) has demonstrated, however, that the morphology of the local geomagnetic field determines whether the “jerks” are observed or not. The fact that we have found spontaneous torsional oscillations in our dynamo computations opens new possibilities for a connection between geomagnetic observations and dynamo theory. An example of a torsional oscillation is shown in Fig. 21. The dashed and solid lines correspond to left and right sides of the dissipationless equation of torsional oscillations, which have been extracted from the results of the dynamo computations. Since the latter include the dissipation effects in the presence of turbulent convection which are neglected in (11), only a rough agreement between left- and right-hand sides of (11) can be expected. The $s$–coordinate in (11) refers to the distance from the axis and $z_{\pm}$ denote the values of $\pm \sqrt{r_0^2 - s^2}$, where coaxial cylinders with radius $s$ intersect the outer spherical boundary.
9. Oscillating dipolar dynamos

In one of the first studies (Grote et al., 2000) of the present series the impression was gained that dipolar dynamos are non-oscillatory while hemispherical and quadrupolar always occur in an oscillatory manner. Since then it has become apparent that dipolar dynamos also often oscillate. An example is shown in Fig. 22. As in the case of hemispherical and quadrupolar dynamos magnetic flux of the new polarity first appears close to the equatorial plane and then propagates to higher latitudes where it is finally dissipated. A characteristic feature of dipolar oscillations appears to be that they are not entirely symmetric with respect to the two signs of the magnetic field. This asymmetry can already be noticed in the case of Fig. 22, but it is more strongly exhibited in the case of Fig. 23. Finally, the asymmetry may become so strong that the oscillations can hardly be inferred from watching the poloidal field at some distance from the outer boundary of the shell as is evident from Fig. 24. We are calling this version of the oscillatory dipolar dynamo the “invisible” one. Only close to the equator does the poloidal field exhibit a reversal at distances not too far from the sphere.

The “invisible” oscillating dynamo occurs at Prandtl numbers larger than unity. These dynamos are characterized by strong polar tubes of

\[ \frac{\partial \phi}{\partial \theta} = \text{const.} \]

Figure 22. Oscillating dipolar dynamo with \( \Delta t = 0.003 \) (clockwise) in the case of \( P = 1, \tau = 5 \times 10^3, R = 1.4 \times 10^6 \) and \( P_m = 1 \). The left half of each plot shows lines of constant \( B_\phi \) while the right half indicates meridional field lines, \( r \sin \theta \phi / \phi = \text{const.} \).
Figure 23. Oscillating dipolar dynamo for $P = 1$, $\tau = 10^4$, $R = 5.4 \times 10^5$ and $P_m = 4$. The time sequence with $\Delta t = 0.04$ starts with upper left plot and continues clockwise to the lower left plot such that about a period is covered. The left half of each plot shows lines of constant $B_\phi$ while the right half indicates meridional field lines, $r \sin \theta \partial h / \partial \theta = \text{const.}$

Zonal magnetic flux which do not participate in the oscillation. The latter is confined to the region outside the tangent cylinder. Although the poloidal field is also located mostly outside the tangent cylinder the presence of the polar flux tubes prevents its participation in the oscillation except for the small reversal region close to the equator.

Another type of dipole oscillations which is frequently encountered in low Prandtl number dynamos is shown in Fig. 25. The magnetic field structure is characterized by a strong hemispherical component. But this component switches nearly periodically between the northern and the southern hemisphere. While the mean azimuthal field becomes almost quadrupolar during the switchover (see the 5. plot of Fig. 25) the poloidal field attains its most dipolar structure at this point.

In general oscillating dipolar dynamos are found in the region of the parameter space between the regimes of non-oscillatory dipolar dynamos and of hemispherical dynamos. This distinction becomes blurred, of course, at higher values of the Rayleigh number when the onset of convection in the polar region tends to destroy the symmetry properties of dynamos.
Figure 24. “Invisible” oscillating dynamo with $\Delta t = 0.06$ in the case of $P = P_m = 5$, $\tau = 5 \times 10^3$ and $R = 6 \times 10^5$. The circles in the first column indicate lines of constant $B_\phi$ in their left halves and meridional field lines, $r \sin \theta \partial h/\partial \theta = \text{const.}$, in their right halves. The second column exhibits lines of constant $B_r$ on the surface $r = r_0 + 0.4$. 
10. Discussion and outlook

Throughout the preceding sections an electrically insulating core has been assumed. A finite conductivity of the same order as that of the fluid shell may be expected to influence the time dependence of the dynamo (Hollerbach and Jones, 1995). When such a finite core conductivity was introduced in the course of the present computations, however, little influence was seen. The magnetic field penetrating into the inner core is rather weak and has little effect on the chaotic dynamo in the outer core. This conclusion was reached previously by Wicht (2002).

A strong effect is exerted by no-slip conditions in place of stress-free conditions at the outer boundary of the spherical shell. In particular, the differential rotation is significantly reduced in the case of $P = 1$ (Kutzner and Christensen, 2002) and the $\omega$-effect of the dynamo is nearly absent. Dynamos with a no-slip condition on the boundary thus resemble those obtained at much larger values of $P$. It must be expected, however, that the difference between results for stress-free and those for no-slip boundaries will vanish in the limit of large $\tau$. It therefore appears to be preferable to use stress-free conditions at the outer boundary for planetary applications.

The ultimate goal of dynamo simulations are results which can be compared quantitatively with the properties of the geomagnetic field and its variations in time. Reversals and torsional oscillations appear
to be particularly attractive possibilities in this respect. The low Prandtl number regime is well suited for this purpose because viscous dissipation plays a lesser role than in the case of \( P \geq 1 \). Higher values of the Coriolis number can thus be reached in the numerical computations. While current estimates of the effective Prandtl number in the outer core of the Earth range around \( P = 0.1 \) it could well be that the diffusivity of light elements should be used instead of the thermal diffusivity. Compositional buoyancy rather than thermal buoyancy is generally believed to be the major energy source of convection (Lister and Buffett, 1995) and the role played by multiple sources of buoyancy will thus require more detailed studies. The possibility of new dynamical phenomena associated with double buoyancy has been pointed out (Busse, 2002b) and is presently investigated numerically.

References

CONVECTION IN ROTATING SPHERICAL FLUID SHELLS


