Quasi-geostrophic approximation of anelastic convection

By F. H. BUSSE[1]† AND R. D. SIMITEV[2]

[1]Institute of Physics, University of Bayreuth, Bayreuth D-95440, Germany,
[2]Schoo of Mathematics and Statistics, University of Glasgow, Glasgow G12 8QW, UK

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The onset of convection in a rotating cylindrical annulus filled with a compressible fluid is studied in the anelastic approximation. Thermal Rossby waves propagating in the azimuthal direction are found as solutions. The analogy to the case of Boussinesq convection in the presence of conical end surfaces of the annular region is emphasized. As in the latter case the results can be applied as an approximation for the description of the onset of anelastic convection in rotating spherical fluid shells. Reasonable agreement with three-dimensional numerical results for the latter problem is found.

1. Summary

- Rotating ANNULUS WITH CONICAL CAPS see figure 1.
- Significant BACKGROUND DENSITY VARIATION with depth described in within the anelastic approximation.
- ANALYTICAL RESULTS on the LINEAR ONSET of convection, e.g. expressions for the critical Rayleigh number R_c, critical mode n, and drift frequency ω.
- COMPARISON of the annulus results with linear onset in SPHERICAL SHELLS.

2. Introduction

The tendency of fluid motions in rapidly rotating systems to develop nearly two-dimensional structures has often been exploited to simplify the theoretical analysis. The description of convection flows in systems where gravity vector and rotation axis are not parallel provides typical examples (Busse, 1970, 2002). In applications of convection problems to rotating planets and stars the tendency towards two-dimensionality is partly obscured by the strong variation of fluid density as function of radius in the

† busse@uni-bayreuth.de

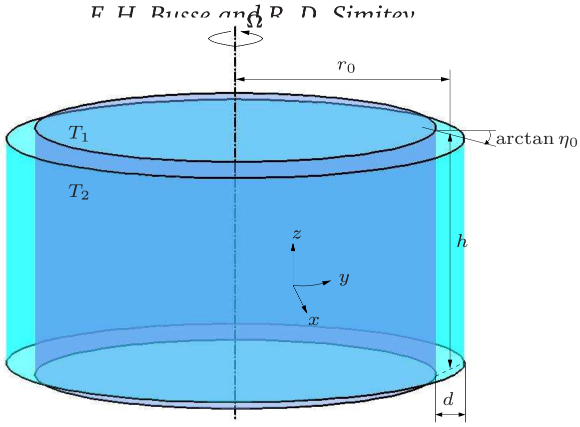

FIGURE 1. Sketch of the rotating cylindrical annulus with conical end caps. Note that the sketch is not to scale with the limits of a small gap and a small angle of inclination of the conical caps.

nearly spherical systems. It is thus of interest to investigate the extent to which the two-dimensional description can still provide an approximation for three-dimensional convection in rapidly rotating systems with strong variations of density.

In the case of the Boussinesq approximation in which the density is regarded as constant except in connection with the gravity term the results derived from the approximate two-dimensional analysis of the onset of convection in rotating spherical fluid shells compares well with the results of the three-dimensional numerical analysis (Simitev and Busse, 2003). Here the two-dimensional model was based on the problem of convection in a rotating cylindrical annulus with conical end boundaries (Busse, 1970, 1986).

In recent years the anelastic approximation (Gough, 1966) has been widely used to obtain more realistic descriptions of convection in the atmospheres of planets and stars with strong variations of density. In the paper by Busse (1986, which will be referred to by B86 in the following) the analogy between the effect of changing height induced by the conical boundaries and the effect of a radial variation of density had already been pointed out. In the present paper we intent to demonstrate quantitatively that the two-dimensional analysis provides a reasonable approximation for the onset of convection in the presence of strong anelastic density variations in rotating spherical fluid shells.

The main purpose of this paper is not the demonstration of a high accuracy of the two-dimensional approximation. Instead we wish to emphasize the insights into anelastic convection in rotating spheres gained from the analytical model. In the following section we first introduce the narrow-gap cylindrical annulus and derive the two-dimensional solution describing convection. In section 3 the model is modified for applications to the onset of anelastic convection in rotating spherical shells. Comparisons with numerical solutions are evaluated in section 4. Some nonlinear problems are discussed in the final section of the paper.

3. Mathematical description of two-dimensional anelastic convection

We consider a cylindrical annulus rotating about its axis with the angular velocity Ω. The gap width d in the radial direction of the annular region is small in comparison with its inner radius r_i such that a cartesian system of dimensionless coordinates x, y, z in the radial, azimuthal, and axial direction, respectively, can be used for a local description of convection. The corresponding unit vectors are $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$. The annular gap is filled with an ideal gas the state of which differs little from an isentropic reference state in the presence of gravity pointing in the negative x-direction. The small deviation from the isentropic state is described by the small positive excess entropy Δs by which the entropy at the inner cylinder exceeds the entropy at the outer cylindrical boundary. In experimental realizations gravity could be replaced by the centrifugal force. The dynamical problem would then be identical when a negative value of Δs is assumed.

Using d as length scale, d^2/κ as time scale and Δs as scale of the entropy we obtain the dimensionless form of the anelastic equations as introduced in the benchmark paper (Jones et al., 2011),

$$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \tau\boldsymbol{k}\times\boldsymbol{u} = -\nabla\pi - \frac{R}{Pr}\boldsymbol{i}s + \boldsymbol{F}, \tag{3.1a}$$

$$\nabla\cdot\boldsymbol{u} = \bar{\rho}\boldsymbol{u}\cdot\nabla\frac{1}{\bar{\rho}}, \tag{3.1b}$$

$$Pr(\frac{\partial}{\partial t}s + \boldsymbol{u}\cdot\nabla s) = \nabla^2 s + \nabla s\cdot\frac{1}{\kappa\bar{\rho}\bar{T}}\nabla\kappa\bar{\rho}\bar{T} + \hat{Q}, \tag{3.1c}$$

The Rayleigh number R, the Prandtl number Pr and the Coriolis number τ are defined by

$$R = \frac{gd^3\Delta s}{\kappa\nu c_p}, \qquad Pr = \frac{\nu}{\kappa}, \qquad \tau = 2\Omega\frac{d^2}{\nu}, \tag{3.2}$$

Here κ is the entropy diffusivity and ν is the kinematic viscosity. For simplicity we have assumed that the material properties are constant except for $\bar{\rho}$ which represents the x-dependent density of the isentropic reference state made dimensionless through division by its average value. The constant gravity vector is given by $\boldsymbol{g} = -g\boldsymbol{i}$ and s can be separated into two parts,

$$s = -x + \tilde{s}, \tag{3.3}$$

such that the boundary condition $\tilde{s} = 0$ holds at $x = \pm 1/2$. We thus arrive at the same equations as in the case of Boussinesq convection in the annulus with conical end boundaries with the only difference that the equation of continuity is modified and the second and third term on the right side of equation (2.1c) are missing in the latter case.

We now consider two-dimensional solutions of eqs. (2.1) which are independent of z and thus satisfy the Proudman-Taylor condition. Assuming $\boldsymbol{u} = (1/\bar{\rho})\nabla\psi(x,y,t)\times\boldsymbol{k}$ we obtain for the $z-$component of the vorticity of eq. (2.1a)

$$\frac{\partial}{\partial t}\zeta + \frac{1}{\bar{\rho}}\Big(\frac{\partial}{\partial x}\zeta\frac{\partial}{\partial y}\psi - \frac{\partial}{\partial y}\zeta\frac{\partial}{\partial x}\psi - \frac{1}{\bar{\rho}}(\tau+\zeta)(\frac{d}{dx}\bar{\rho})\frac{\partial}{\partial y}\psi\Big) = -\frac{R}{Pr}\frac{\partial}{\partial y}\tilde{s} + \Delta_2\zeta, \tag{3.4}$$

where $\zeta = \boldsymbol{k}\cdot\nabla\times((\nabla\psi\times\boldsymbol{k})/\bar{\rho})$ is the z-component of the vorticity and $\partial^2/\partial x^2 + \partial^2/\partial y^2$ has been denoted by Δ_2. Following Evonuk and Glatzmaier (2004) the friction term has been reduced to its main contributor. Assuming that $\bar{\rho}$ varies slowly such that the absolute value of

$$\eta_\rho \equiv -\frac{1}{\bar{\rho}}\frac{d}{dx}\bar{\rho} \tag{3.5}$$

is a small constant we find that the absolute value of

$$\eta^*_\rho \equiv \eta_\rho\tau \tag{3.6}$$

is a parameter of the order unity or larger for $\tau \gg 1$.

The linearized versions of equations (2.1c) and (2.4) assume the form

$$\frac{\partial}{\partial t}\zeta + \frac{\eta^*_\rho}{\bar{\rho}}\frac{\partial}{\partial y}\psi = -\frac{R}{Pr}\frac{\partial}{\partial y}\tilde{s} + \Delta_2\zeta, \tag{3.7a}$$

$$Pr\frac{\partial}{\partial t}\tilde{s} - \frac{Pr}{\bar{\rho}}\frac{\partial}{\partial y}\psi = \Delta_2 \tilde{s}. \tag{3.7b}$$

After elimination of $\tilde{s}$, neglect of terms of the order η, and multiplication of the equation of motion by $\bar{\rho}$ we obtain

$$(Pr\frac{\partial}{\partial t} - \Delta_2)[(\frac{\partial}{\partial t} - \Delta_2)\Delta_2 - \eta_\rho^* \frac{\partial}{\partial y}]\psi) = Ra\frac{\partial^2}{\partial y^2}\psi. \tag{3.8}$$

This equation is easily solved when stress-free conditions at the boundaries $x = \pm 1/2$ are assumed,

$$\psi = \sin\pi(x + 1/2)\exp\{i\alpha y + i\omega t\}, \qquad \tilde{s} = \frac{-i\alpha\psi}{i\omega Pr + \alpha^2 + \pi^2}. \tag{3.9}$$

This solution yields the dispersion relation

$$R_o\alpha^2 = (i\omega Pr + \alpha^2 + \pi^2)[(i\omega + \alpha^2 + \pi^2)(\alpha^2 + \pi^2) + i\alpha\eta_\rho^*]. \tag{3.10}$$

Real and imaginary parts of this equation determine the neutral curve $Ra_o(\alpha)$ and the frequency of the thermal Rossby wave,

$$\omega_o = \frac{-\alpha\eta_\rho^*}{(1 + Pr)(\alpha^2 + \pi^2)}, \qquad R_o = (\alpha^2 + \pi^2)^3\alpha^{-2} + \left(\frac{\eta_\rho^* Pr}{1 + Pr}\right)^2 /(\alpha^2 + \pi^2). \tag{3.11}$$

The angular frequency ω_o resembles that of ordinary Rossby waves from which it differs only through the appearance of Pr in the denominator. The Rayleigh number is determined by two terms. The first is the familiar expression from Rayleigh-Bénard convection which is independent of the Coriolis number. The second term is introduced by the density variation caused by the compressibility.

The critical value Ra_c and the corresponding wavenumber α_c are obtained through minimizing $Ra_o(\alpha)$ which yields in the limit of high values of $|\eta_\rho^*|$

$$\alpha_c = \eta_P^{1/3}(1 - \frac{7}{12}\pi^2\eta_P^{-2/3} + ...), \qquad R_c = \eta_P^{4/3}(3 + \pi^2\eta_P^{-2/3} + ...), \tag{3.12}$$

where η_P is defined by

$$\eta_P \equiv \frac{|\eta_\rho^*|Pr}{\sqrt{2}(1 + Pr)}. \tag{3.13}$$

As in the Boussinesq case of the cylindrical annulus with conical axial boundaries, the onset of convection becomes independent of the gap width d in the limit of high $|\eta^*|$ and the Rayleigh numbers for modes with $\sin l\pi(x + 1/2)$ with $l = 2, 3, 4$ etc hardly differ from that for $l = 1$. The neglected second term on the right hand side of equation (11c) would contribute only a negligible amount in the limit of high α_c. The other neglected term $\hat{Q}$ does not enter the linear problem, of course.

4. **Application to three-dimensional geometries**

In applying the two-dimensional solution to a three-dimensional configuration we follow the corresponding analysis in the case of Boussinesq convection. In particular we shall consider the case of a rotating spherical fluid shell of thickness d such that the inner and outer radii, r_i and r_o, are given by $\beta/(1-\beta)$ and $1/(1-\beta)$, respectively, where β is defined by $\beta = r_i/r_o$. Simitev & Busse (2003) have demonstrated that a good approximation for the onset of convection in rotating spherical shells can be obtained by solutions of the form (2.8) through (2.10) as described in B86. In this case the parameter η^* is

TABLE 1. Comparison in the case $\tau = 10^5$, $\beta = 0.5$, $N = 2$ and $n = 2$.

	$P = 1$, $x = 1.43$		$P = 0.1$, $x = 1.6$	
	Approximate	Numerical	Approximate	Numerical
η^*	$3.89 \cdot 10^4$		$4.17 \cdot 10^4$	
η^*_ρ	$2.40 \cdot 10^5$		$2.79 \cdot 10^5$	
η^*_P	$9.864 \cdot 10^4$		$2.062 \cdot 10^4$	
α	46.2	38.5	27.4	18.1
R	$13.67 \cdot 10^6$		$1.696 \cdot 10^6$	
Ra	$3.569 \cdot 10^7$	$3.3262 \cdot 10^7$	$4.191 \cdot 10^6$	$4.685 \cdot 10^6$
ω	3019	1844	10643	6901

defined by

$$\eta^* \equiv \frac{\tau \tan\theta}{\cos\theta r_o}. \tag{4.14}$$

where θ is the colatitude on the spherical surface with respect to the axis of rotation and $r_o \sin\theta$ represents the distance from the axis at which convection sets.

In the presence of density variation the contribution η^*_ρ as defined in the preceding section must be added. Since the density in the spherical configuration varies not only with distance from the axis, but parallel to the axis as well an average over the latter dimension must be taken. The same procedure must be applied to the gravity. Following the example of the benchmark case (Jones et al., 2011) we shall use

$$\bar{\rho} \equiv \xi^n \quad \text{with} \quad \xi = c_0 + c_1/r \tag{4.15}$$

$$\text{and with} \quad c_0 = \frac{2\xi_o - \beta - 1}{1-\beta}, \quad c_1 = \frac{(1+\beta)(1-\xi_o)}{(1-\beta)^2} \tag{4.16}$$

$$\text{where} \quad \xi_o = \frac{\beta + 1}{\beta \exp(N_\rho/n) + 1}, \quad \xi_i = \frac{\beta + 1 - \xi_o}{\beta} \tag{4.17}$$

where ξ_i and ξ_o are the values of ξ at the inner and the outer boundary and where N_ρ is the number of density scale heights, $N_\rho = n \ln(\xi_i/\xi_o)$.

The definition (2.5) thus becomes modified

$$\eta_\rho = \frac{n}{\sqrt{r_o^2 - x^2}} \int_0^{\sqrt{r_o^2 - x^2}} \frac{c_1 x dz}{(c_0\sqrt{x^2+z^2} + c_1)(x^2+z^2)}, \tag{4.18}$$

where x denotes the distance from the axis of rotation. An analytical expression for this integral can be obtained, but it is lengthy and will not be given here. In a similar fashion the variation of gravity together with the r.derivative of basic entropy must be accommodated through an average,

$$\frac{R}{Ra} = \frac{nc_1 x}{\sqrt{r_o^2 - x^2}} \int_0^{\sqrt{r_o^2 - x^2}} \frac{dz}{\sqrt{(x^2+z^2)^5}(c_0 + c_1/\sqrt{x^2+z^2})(\xi_o^{-n} - \xi_i^{-n})}, \tag{4.19}$$

where Ra is the Rayleigh number as used in the benchmark (Jones et al., 2011).

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