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Effects of shell thickness on cross-helicity generation in convection-driven dynamos

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1 Introduction

1.1 *The cross-helicity effect*

Because of the coupling between the velocity and magnetic fields, magnetohydrodynamic (MHD) turbulence shows very interesting nonlinear behaviour. Magnetic fields are induced by turbulent fluid motion, and the generated fields influence the turbulence. These are strong effects in MHD flows. At large magnetic Reynolds number ($R_m = uL/\lambda$, u: characteristic velocity, L: characteristic length, λ: magnetic diffusivity), the induced magnetic field is sometimes much larger in magnitude than the original or imposed field. If the tendency of the magnetic field to be advected by fluid motion is very strong, the magnetic field can be considered as frozen into the plasma motion. In such a high R_m flow, the turbulent cross helicity, the velocity-magnetic-field correlation of turbulence, is a quantity of primary importance. The most important property of the cross helicity is related to the turbulent dynamo. The mean magnetic field obeys the mean induction equation,

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times \mathbf{E}_{\mathrm{M}} + \lambda \nabla^2 \mathbf{B}. \tag{1}$$

Here $\mathbf{E}_{\mathrm{M}}$ is the turbulent electromotive force defined by

$$\mathbf{E}_{\mathrm{M}} = \langle \mathbf{u}' \times \mathbf{b}' \rangle,$$

with $\langle ... \rangle$ denoting an appropriate averaging of small-scale velocity and magnetic field fluctuations $\mathbf{u}'$ and $\mathbf{b}'$. This quantity is of primary importance in MHD turbulence theory since it contains the effect of turbulence in the mean magnetic induction equation (1). In the presence of large-scale vorticity $\boldsymbol{\Omega} = \nabla \times \mathbf{U}$, cross helicity in the turbulence may lead to a turbulent electromotive force aligned with the large-scale vorticity as schematically illustrated in the Figure 1. We consider a fluctuating fluid element with large-scale vorticity and assume there is a positive correlation between the velocity and magnetic fields

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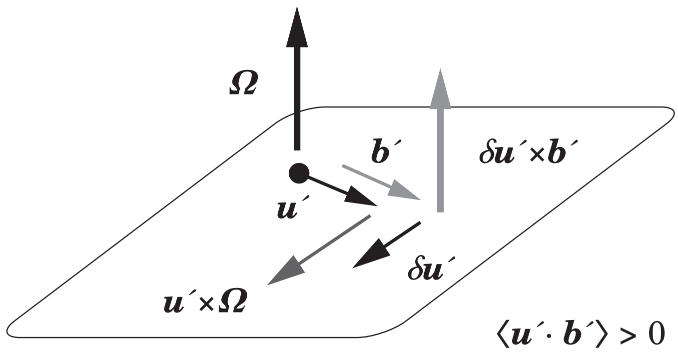

Figure 1. Physical interpretation of the cross-helicity effect. Redrawn from Yokoi (1999).

in the turbulence, $\langle \mathbf{u}' \cdot \mathbf{b}' \rangle > 0$. The fluid element is subject to a Coriolis-like force due to local angular-momentum conservation, and the induced velocity is, $\delta\mathbf{u}' = \tau \mathbf{u}' \times \mathbf{\Omega}$, where τ demotes the appropriate time scale. By assumption, the magnetic fluctuation $\mathbf{b}'$ is statistically aligned with the velocity fluctuation $\mathbf{u}'$. As a consequence, the contribution to the electromotive force is written as

$$\langle \delta\mathbf{u}' \times \mathbf{b}' \rangle = \tau \langle \mathbf{u}' \cdot \mathbf{b}' \rangle \mathbf{\Omega}.$$

Thus, this assumption has the result that the combination of large-scale vortical motion and positive (or negative) turbulent cross helicity gives rise to an electromotive force parallel (or anti-parallel) to the mean vorticity.

With this effect in mind, it is quite natural that the turbulent electromotive force is expressed as

$$\mathbf{E}_{\mathrm{M}} = \alpha \mathbf{B} - \beta \mathbf{J} + \gamma \mathbf{\Omega}. \tag{2}$$

Here we should note that electromotive force, $\mathbf{E}_{\mathrm{M}}$, and the electric current density, $\mathbf{J} = \nabla \times \mathbf{B}$, are polar vectors whereas the magnetic field B and the vorticity, $\mathbf{\Omega}$ are axial vectors. As a result, the coefficients α and γ are pseudo-scalar whereas β is a pure scalar.

It is known that the coefficients α, β and γ can be expressed in terms of the turbulent residual helicity, $H = \langle \mathbf{b}' \cdot \mathbf{j}' - \mathbf{u}' \cdot \omega' \rangle$, the turbulent MHD energy, $K = \langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle / 2$, and the turbulent cross-helicity $W = \langle \mathbf{u}' \cdot \mathbf{b}' \rangle$, respectively (Pouquet et al. 1976; Krause & Rädler 1980; Yoshizawa 1990). They are modelled as

$$\alpha = C_\alpha \tau \langle \mathbf{b}' \cdot \mathbf{j}' - \mathbf{u}' \cdot \omega' \rangle = C_\alpha \tau H, \tag{3}$$

$$\beta = C_\beta \tau \langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle = C_\beta \tau K, \tag{4}$$

$$\gamma = C_\gamma \tau \langle \mathbf{u}' \cdot \mathbf{b}' \rangle = C_\gamma \tau W, \tag{5}$$

with C_α, C_β and C_γ being model constants. Here τ is the characteristic time of turbulence, which is often expressed as

$$\tau = K/\epsilon,$$

with the dissipation rate of the turbulent MHD energy, ϵ, defined by

$$\epsilon = \nu \left\langle \left(\frac{\partial u'^a}{\partial x^b} \right)^2 \right\rangle + \lambda \left\langle \left(\frac{\partial b'^a}{\partial x^b} \right)^2 \right\rangle .$$

Substituting (2) into (1), we have

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times (\alpha \mathbf{B} + \gamma \boldsymbol{\Omega}) - \nabla \times [(\eta + \beta) \nabla \times \mathbf{B}] \tag{6}$$

This shows that the β-related term in (2) represents the enhancement of the magnetic diffusivity due to turbulence, since $\lambda \rightarrow \lambda + \beta$. Thus, the β-related term is called the turbulent magnetic diffusivity or turbulent resistivity. On the other hand, the $\alpha-$ and γ-related terms represent possible magnetic-field generation mechanisms due to pseudo-scalars in turbulence. They are called the α- or helicity effect and the cross-helicity effect, respectively.

1.2 *Motivation and outline*

In the history of turbulent mean-field dynamo research, the cross-helicity effect has been neglected. This is in strong contrast to the α- or helicity effect, which has been studied extensively **??**. However, as we have just seen, the presence of turbulent cross helicity in the large-scale vortical field leads to the cross-helicity effect. Large-scale rotational motion is ubiquitous in astro- and geophysical phenomena. Thus, it is highly desirable to examine the validity of the cross-helicity dynamo in the astro- and geophysical contexts. The most important problem is to see how and how much cross helicity can exist in turbulence in the presence large-scale inhomogeneities such as velocity shear, rotation, density stratication, etc. This problem is addressed in the present work with the aid of numerical simulations of chaotic convection-driven dynamos in rotation spherical shells. Through these simulations, we examine how the cross helicity is spatially distributed and how it is generated. We also investigate the relative strength of the cross-helicity γ–effect in comparison with the helicity α–effect.

The main goal this work is to investigate the relative importance of the two dynamo effects as a function of the thickness of the convective spherical shell. Intuitive arguments suggest that the helicity α–effect is important in the case of the Geodynamo and the cross-helicity γ–effect is important in the case of the Solar dynamo. Indeed, the geodynamo is believed to operate in a relatively thick spherical shell where large-scale columnar structures (convective banana cells) are likely to develop. The coherent columnar structures are characterised by relatively large vorticity and they will generate a strong helicity α–effect. In contrast, the Solar dynamo is believed to operate in a thinner spherical shell where columnar structures are difficult to support and so vorticity may be relatively weak thus increasing the relative importance of the cross-helicity γ–effect. To investigate this question we present in this work a set of dynamo simulations in which the main parameter variation is in shell thickness η, while we have made an effort to keep the other governing parameters of the problem fixed. We also use this opportunity to remark on several other properties of this set, most notably the existence of bistability and hysteresis as a function of η which has not been reported previously.

The organization of this paper is as follows. Modify this: In section 2, the evolution equation of the cross helicity is presented with special emphasis on the generation mechanisms of the turbulent cross helicity. In section 3, some results of three numerical simulations are presented. In section 4, the cross-helicity generation mechanisms and the magnitude of the turbulent cross helicity scaled by the turbulent MHD energy are discussed in comparison with the theoretical prediction on the magnitude of the torsional oscillation. Concluding remarks are given in section 5.

ACTION: Rado Simitev

2 Cross-helicity evolution equation

Please, formulate the cross-helicity evolution equation. Suggest, terms in that equation that can be evaluated from the numerical simulations in order to quantify cross-helicity generation and the intensity of the γ–effect.

ACTION: Nobu Yokoi

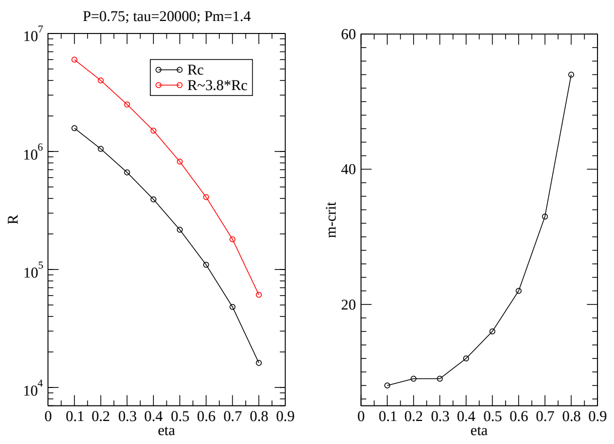

Figure 2. Plot of the critical Rayleigh number and wave number for the linear onset of convection as a function of shell thickness η.

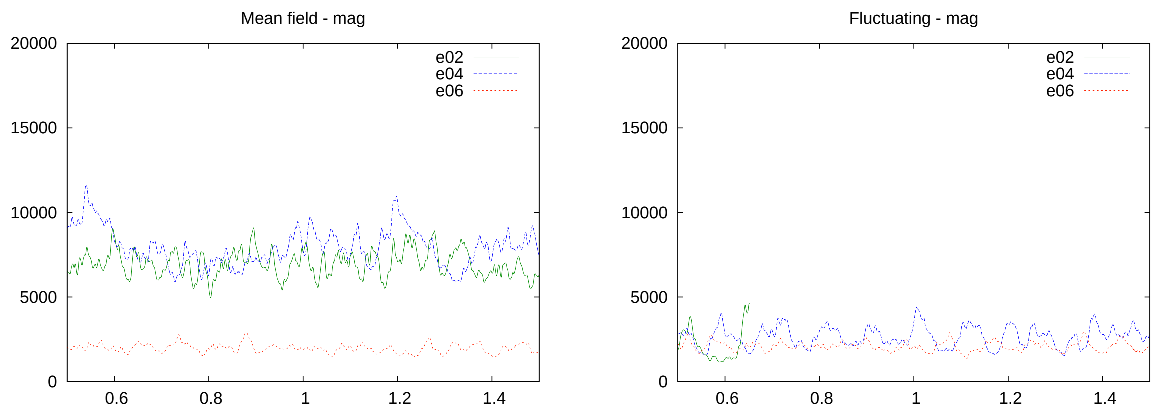

Figure 3. Typical kinetic and magnetic energy density components for one representative case in the dataset of simulations. Something similar to Figure 4 of Busse & Simitev (2011).
We should look for a case that is relatively interesting and also shows other phenomena - oscillations, reversals. Alt, we can prepare a figure like Figure 2 of Simitev & Busse (2009) showing two different dynamos at identical parameters, later that may be useful in relation to bistability discussion.

ACTION: Luis Silva

3 Outline of the numerical simulations

In this section we provide a brief outline of the dynamo simulations carried out in relation to this paper. Discuss Figures 2, 3, 4, 5.

4 Cross-helicity

Discuss Figures 6 and 7.

5 Further dependences on shell thickness

Discuss Figures 8 and 9.

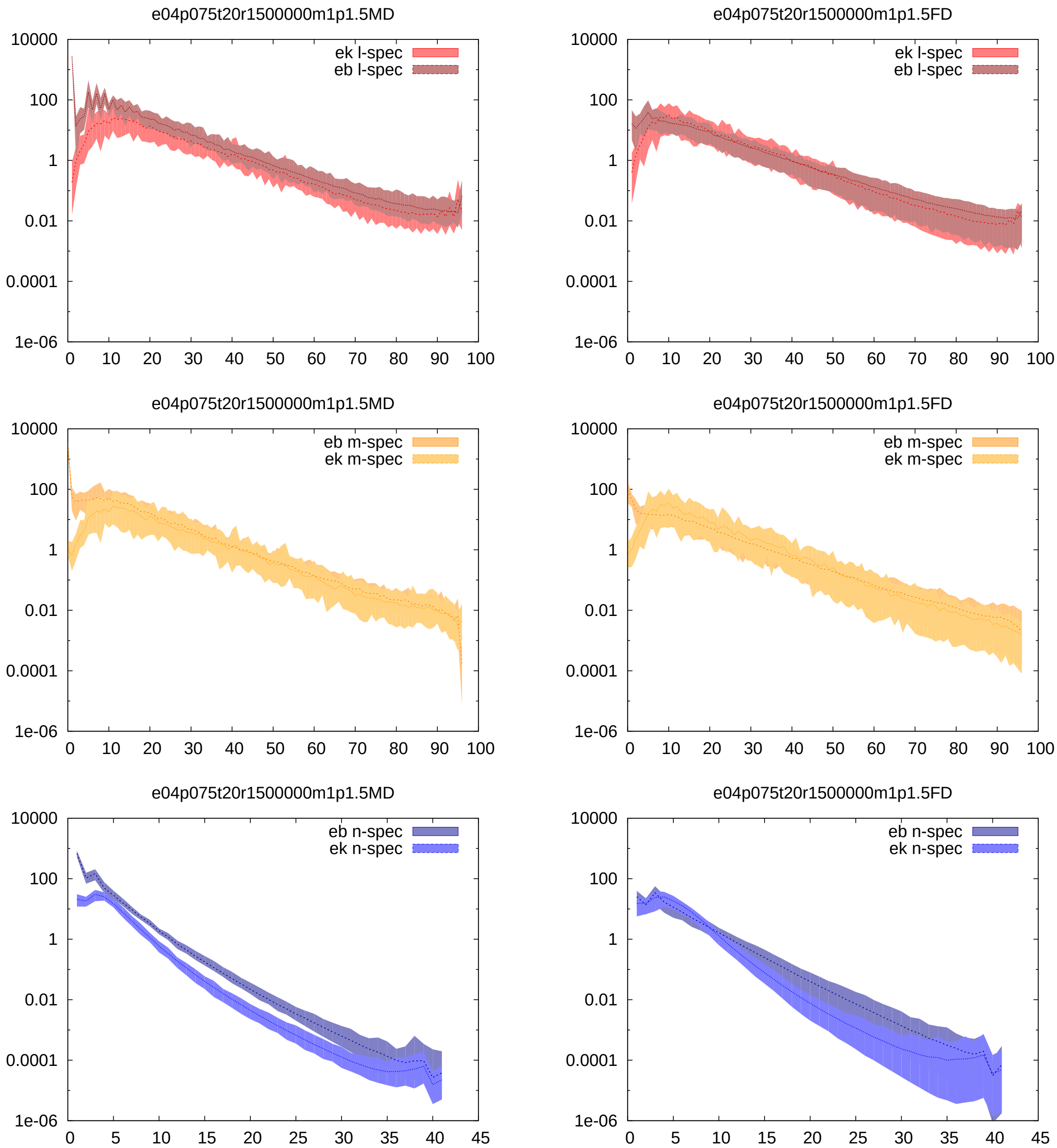

Figure 4. Typical power spectra of velocity and magnetic field for one representative case in the dataset of simulations. Something similar to Figure 1 of Simitev & Busse (2009).

This will serve to illustrate the level of turbulence, the dominant length scales and the adequate resolution.

ACTION: Luis Silva

6 Conclusions

Appendix A. Mathematical formulation of the dynamo problem

Here a brief outline of the mathematical formulation is given which is employed for the simulations of convection driven dynamos referred to in this paper. It is assumed that a static state exists with the temperature distribution $T_S = T_0 - \beta d^2 r^2/2$ where d is the thickness of the spherical shell, rd is the length of the position vector with respect to the center of the sphere and the gravity field is given by $\mathbf{g} = -d\gamma\mathbf{r}$. In addition to the length d, the time d^2/ν, the temperature $\nu^2/\gamma\alpha d^4$ and the magnetic flux

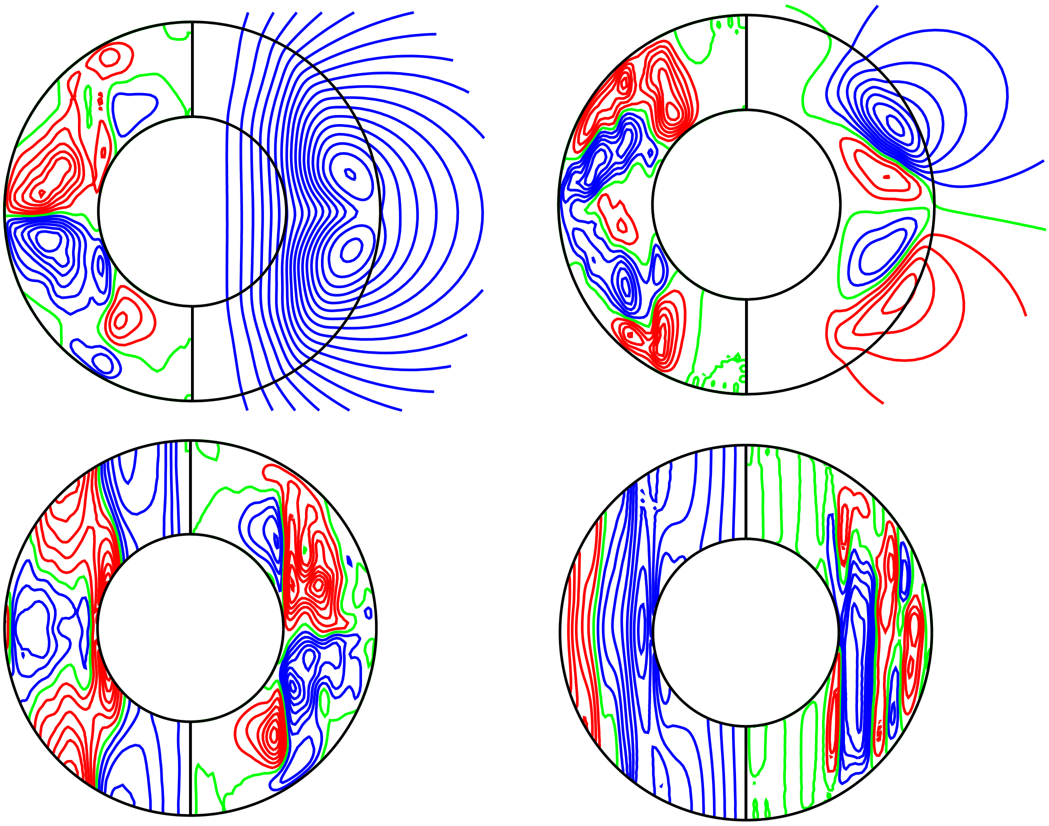

Figure 5. Typical spacial structures of velocity and magnetic field for one or two representative case in the dataset of simulations. Something similar to Figure 3 of Simitev & Busse (2009).

We can plot an example of good oscillations or a reversal sequence to illustrate time evolution.

ACTION: Luis Silva

density $\nu(\mu_0\varrho)^{1/2}/d$ are used as scales for the dimensionless description of the problem where ν denotes the kinematic viscosity of the fluid, κ its thermal diffusivity, ϱ its density and μ_0 is its magnetic permeability. The equations of motion for the velocity vector $\mathbf{u}$, the heat equation for the deviation Θ from the static temperature distribution, and the equation of induction for the magnetic flux density $\mathbf{B}$ are thus given by

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \tau\mathbf{k}\times\mathbf{u} = -\nabla\pi + \Theta\mathbf{r} + \nabla^2\mathbf{u} + \mathbf{B}\cdot\nabla\mathbf{B}, \tag{A.1a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{A.1b}$$

$$P(\partial_t\Theta + \mathbf{u}\cdot\nabla\Theta) = R\mathbf{r}\cdot\mathbf{u} + \nabla^2\Theta, \tag{A.1c}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{A.1d}$$

$$\nabla^2\mathbf{B} = P_m(\partial_t\mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{u}), \tag{A.1e}$$

where ∂_t denotes the partial derivative with respect to time t and where all terms in the equation of motion that can be written as gradients have been combined into $\nabla\pi$. The Boussinesq approximation has been assumed in that the density ϱ is regarded as constant except in the gravity term where its temperature dependence given by $\alpha \equiv -(\mathrm{d}\varrho/\,\mathrm{d}T)/\varrho =$*const* is taken into account. The dimensionless parameters are defined as

$$R = \frac{\alpha g \Delta T d^3}{\nu\kappa}, \qquad \tau = \frac{2\Omega d^2}{\nu}, \qquad P = \frac{\nu}{\kappa}, \qquad P_m = \frac{\nu}{\lambda}. \tag{A.2}$$

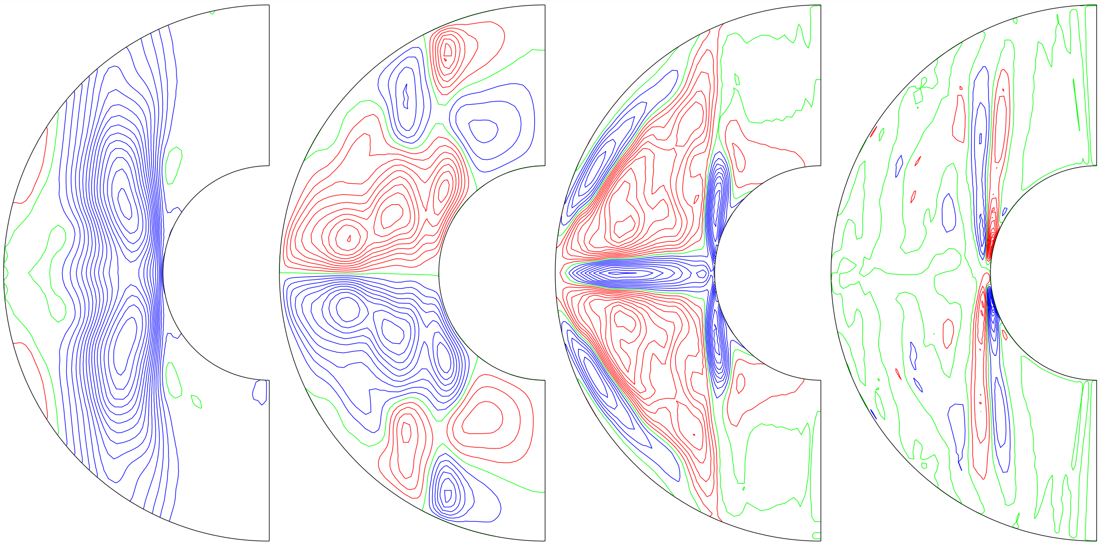

Figure 6. 1st column: Spatial distribution of the turbulent EMF $E_{\rm M}\phi = \langle \mathbf{u}' \times \mathbf{b}' \rangle$ (full quantity by definition).
2nd column: Spatial distribution of αB_ϕ
3rd column: Spatial distribution of $-\beta J_\phi$
4th column: Spatial distribution of $\gamma \Omega_\phi$

This figure will illustrate the balance of therms that contribute to EMF according to the model assumption (2). We should plot a thick and a thin shell case in two rows one over the other.

ACTION: Luis Silva

Figure 7. Relative strength of α– and γ–effects as a function of the shell thickness η.

This should be a XY-plot. On the abscissa we put values of $\eta = 0.1, 0.2, .., 0.8, 0.9$. On the ordinate we put values of the ratio $Q = |\alpha \mathbf{B}|/|\gamma \mathbf{\Omega}|$. We may need to develop some global (integral) measure of this quantities.

ACTION: Luis Silva

Because the velocity field $\mathbf{u}$ as well as the magnetic flux density $\mathbf{B}$ are solenoidal vector fields, the general representation in terms of poloidal and toroidal components can be employed,

$$\mathbf{u} = \nabla \times (\nabla v \times \mathbf{r}) + \nabla w \times \mathbf{r} \ , \tag{A.3a}$$

$$\mathbf{B} = \nabla \times (\nabla h \times \mathbf{r}) + \nabla g \times \mathbf{r} \ . \tag{A.3b}$$

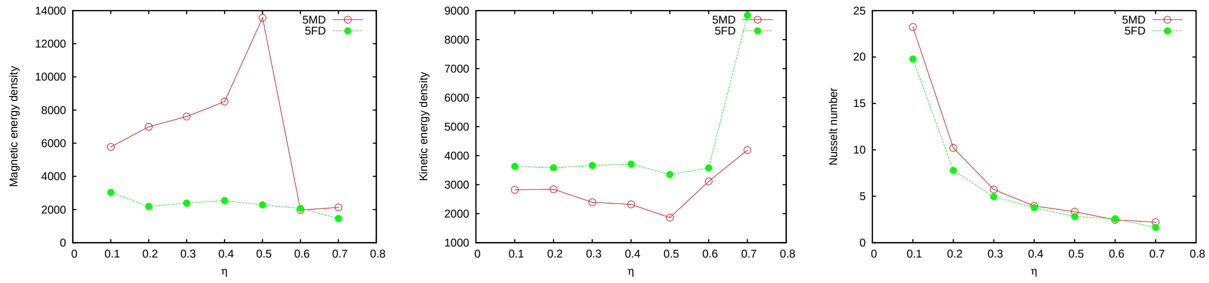

Figure 8. Averaged magnetic and kinetic energy density components, and the Nusselt numbers as a function of the shell thickness η. Similar to Figures 5, 6 etc. of Busse & Simitev (2006).

Here we may need to deal with cases of bistability.

ACTION: Luis Silva

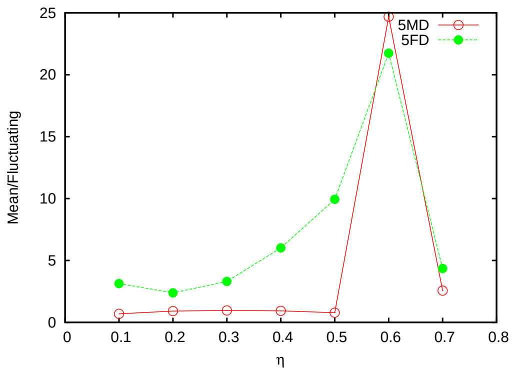

Figure 9. Bistability as a function of the shell thickness η. Similar to Figure 5 of Simitev & Busse (2009). Plot ratio $\widetilde{M}_p/\overline{M}_p$ on the ordinate.

ACTION: Luis Silva

Stress-free boundaries with fixed temperatures are used,

$$v = \partial^2_{rr} v = \partial_r(w/r) = \Theta = 0 \qquad \text{at} \quad r = r_i \equiv 2/3, \quad \text{and at} \quad r = r_o \equiv 5/3, \tag{A.4}$$

where the radius ratio is fixed at the value $r_i/r_o = 0.4$. For the magnetic field electrically insulating boundaries are assumed such that the poloidal function h must be matched to the function $h^{(e)}$ which describes the potential fields outside the fluid shell

$$g = h - h^{(e)} = \partial_r(h - h^{(e)}) = 0 \qquad \text{at } r = r_i \text{ and } r = r_o. \tag{A.5}$$

The energy densities are defined by

$$\overline{E}_p = \frac{1}{2}\langle \mid \nabla \times (\nabla \bar{v} \times \mathbf{r}) \mid^2 \rangle, \quad \overline{E}_t = \frac{1}{2}\langle \mid \nabla \bar{w} \times \mathbf{r} \mid^2 \rangle, \tag{A.6a}$$

$$\hat{E}_p = \frac{1}{2}\langle \mid \nabla \times (\nabla \hat{v} \times \mathbf{r}) \mid^2 \rangle, \quad \hat{E}_t = \frac{1}{2}\langle \mid \nabla \hat{w} \times \mathbf{r} \mid^2 \rangle, \tag{A.6b}$$

where the angular brackets indicate the average over the fluid shell and $\bar{v}$ refers to the azimuthally averaged component of v, while $\hat{v}$ is defined by $\hat{v} = v - \bar{v}$. The Nusselt numbers at the inner and outer

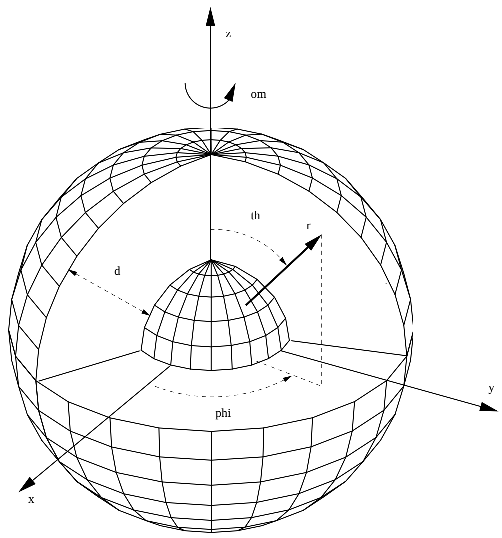

Figure A1. Geometrical configuration of the problem. A part of the outer spherical surface is removed to expose the interior of the shell to which the conducting fluid is confined.

spherical boundaries Nu_i and Nu_o are defined by

$$Nu_i = 1 - \frac{P}{r_i R} \left. \frac{\mathrm{d}\overline{\overline{\Theta}}}{\mathrm{d}r} \right|_{r=r_i}, \qquad Nu_o = 1 - \frac{P}{r_o R} \left. \frac{\mathrm{d}\overline{\overline{\Theta}}}{\mathrm{d}r} \right|_{r=r_o}, \tag{A.7}$$

where the double bar indicates the average over the spherical surface. (The factor $1/R$ has accidentally been dropped in previous papers of the authors.) The ratio of external heating to internal heating is given by

$$\frac{r_i^3 Nu_i}{r_o^3 Nu_o - r_i^3 Nu_i}. \tag{A.8}$$

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