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DIFFERENTIAL ROTATION IN STELLAR CONVECTION ZONES

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ABSTRACT

The dynamical effects of rotation on thermal convection in a fluid layer of spherical shape induce an equatorial acceleration. Special properties of the convection zone assumed in earlier theories on the differential rotation of the Sun are not required. To demonstrate the mechanism, the mathematical problem of convection in a rotating Boussinesq fluid subject to temperature and gravity fields of spherical symmetry is considered. At the critical value of the Rayleigh number the onset of convection takes place predominantly in the equatorial region. The dynamical constraint of rotation, together with the geometry of the problem, leads to the equatorial acceleration at finite convection amplitudes. Estimates give reasonable agreement with values observed on the Sun.

I. INTRODUCTION

Since C. Carrington discovered in 1863 that the Sun rotates faster at the equator than at the poles, a variety of explanations have been put forward for the phenomenon of the solar differential rotation. Among the more recent theories two hypotheses have received particular attention and have been investigated mathematically to a considerable extent.

Biermann (1951, 1958) has proposed the concept of an anisotropic viscosity tensor to describe the dynamical consequences of the turbulent motion in the solar convection zone. In his work and in later work by Kippenhahn (1963) it was shown that, owing to the anisotropic viscosity tensor, a complex flow, including differential rotation, replaces rigid-body rotation as the asymptotically approached state.

A recent reexamination of sunspot data by Ward (1965, 1966) suggests that the transport of angular momentum by eddies of rather large scale is responsible for the maintenance of the differential rotation. Spectroscopic measurements by Plaskett (1966) indicate similar phenomena. These observations have stimulated Ward (1965), Plaskett (1966), and Starr and Gilman (1965) to refer to the analogous situation in the Earth's atmosphere, in which azimuthal wind systems are caused by large-scale eddies with the dynamical properties of Rossby waves. The energy source for these eddies is the baroclinic conversion of potential energy which is generated by the temperature difference between pole and equator. Gilman (1966) proposes that a similar temperature difference exists on the Sun, at least in the deeper zones of the convection layer if not in the photosphere. At present no conclusive evidence for such a temperature gradient does exist.

In contrast to these and other theories, we propose in this paper that the dynamical effects of rotation on convection in a spherical geometry are sufficient to produce a mean azimuthal flow of the observed form. Neither the anisotropic effects of thermal turbulence nor large-scale temperature differences are necessary to explain the maintenance of the solar differential rotation, although they may be important in the actual situation on the Sun.

To exhibit the characteristic features of the problem, we adopt the policy of neglecting all properties of stellar convection zones which are not necessary to explain the mechanism of the equatorial acceleration. It will become apparent that the present work can be extended in a straightforward way for calculations of more realistic models. The mathematical formulation of the problem is given in § II. We shall use the Boussinesq approximation and restrict the analysis to small convection amplitudes. In § III it is shown how the linear part of the equations can be solved by an expansion in powers of the rotation rate. The nonlinear terms which induce the differential rotation will be considered in § IV. An explicit solution of the problem is given in § V in the case of the thin shell. The assumption of stress-free boundaries with fixed temperature will allow us to obtain a simple analytical solution. The conclusions which can be drawn from the highly idealized model are necessarily of a qualitative nature. The numerical estimate given in § VI should be regarded solely as a suggestion that a more realistic picture of the proposed mechanism can provide values for the differential rotation in reasonable agreement with observations of the Sun. The effects of compressibility, in particular, will strongly modify the results. The justification for studying the model of an incompressible fluid is based on the experience that the dynamical properties of incompressible fluids are reflected by compressible fluids under corresponding conditions as long as all velocities are subsonic.

II. FORMULATION OF THE MATHEMATICAL PROBLEM

We consider a spherical shell of an incompressible fluid subjected to a gravitational force of spherical symmetry. A spherically symmetric distribution of heat sources in the core underlying the shell, and possibly in the shell itself, leads to a temperature gradient of the same symmetry in the static fluid layer. When the temperature difference ΔT across the shell reaches a sufficiently high value, the buoyancy forces overcome the stabilizing effects of viscous and thermal dissipation. The static state becomes unstable, and convective motions set in.

For the mathematical description of the problem we shall use the equations of motion in the Boussinesq approximation in which homogeneous properties of the fluid are assumed, with the exception of the temperature dependence of the density, which is taken into account only in the gravity term. A more detailed description of the problem can be found in Chandrasekhar's book (1961, chap. 6). It is convenient to introduce dimensionless variables by using the thickness h of the shell, h^2/ν, and $\Delta T \times \mathrm{Pr}$ as scales for length, time, and temperature, respectively, where Pr is the Prandtl number, which is defined as the ratio between the kinematic viscosity ν and the thermal diffusivity κ. We assume further that the entire system is rotating homogeneously about an axis described by the unit vector $\boldsymbol{k}$. Since the centrifugal force can be derived from a potential, it enters the problem analogously to the force of gravity. In consistency with the assumption of a spherically symmetric shell, we shall neglect the centrifugal force in comparison with the force of gravity. Accordingly, the equation for the velocity vector $\boldsymbol{v}$ and the heat equation for the deviation ϑ of the temperature from the static temperature field $T(r)$ are given by

$$-\nabla \times (\nabla \times \boldsymbol{u}) + \mathrm{R}\vartheta \boldsymbol{r}\gamma(r) - \nabla\pi = \lambda \boldsymbol{k} \times \boldsymbol{u} + \frac{\partial}{\partial t}\boldsymbol{u} - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) \,, \quad (2.1)$$

$$\nabla^2\vartheta + \boldsymbol{u}\cdot \boldsymbol{r}T'(r)r^{-1} = \mathrm{Pr}\left(\boldsymbol{u}\cdot\nabla\vartheta + \frac{\partial}{\partial t}\vartheta\right). \quad (2.2)$$

$\nabla\pi$ includes the pressure and other terms which can be written in the form of a gradient, and $\boldsymbol{r}$ denotes the position vector with respect to the center of the shell. The functional dependence $\gamma(r)$ of the gravity force $g_0\boldsymbol{r}\gamma(r)$ has been normalized in such a way that g_0 gives the value of gravity at the inner radius $r = r_0$ of the shell. The Rayleigh number is defined by

$$\mathrm{R} = \frac{\alpha g_0 \Delta T h^3}{\kappa\nu} \,,$$

where α denotes the coefficient of expansion. The rate of rotation enters the problem in the form of the parameter

$$\lambda = \frac{2\Omega h^2}{\nu} .$$

In addition to equation (2.1), the velocity field has to satisfy the equation of continuity

$$\nabla \cdot \boldsymbol{u} = 0 . \tag{2.3}$$

Any vector field $\boldsymbol{u}$ satisfying this equation can be separated into two parts, a poloidal and a toroidal part, each of which is described by a scalar function

$$\boldsymbol{u} = \nabla \times (\nabla \times \boldsymbol{r} v) + \nabla \times \boldsymbol{r} w . \tag{2.4}$$

By applying the operations $\boldsymbol{r} \cdot \nabla \times (\nabla \times$ and $\boldsymbol{r} \cdot \nabla \times$ onto equation (2.1), after the representation (2.4) has been introduced, we obtain

$$\left[L^2\left(\nabla^2 - \frac{\partial}{\partial t}\right) + \lambda \boldsymbol{k} \times \boldsymbol{r} \cdot \nabla\right]\nabla^2 v + \lambda Q w - \mathrm{R}\gamma L^2 \vartheta = \boldsymbol{r} \cdot \nabla \times \{\nabla \times [\boldsymbol{u} \times (\nabla \times \boldsymbol{u})]\} , \tag{2.5}$$

$$\left[L^2\left(\nabla^2 - \frac{\partial}{\partial t}\right) + \lambda \boldsymbol{k} \times \boldsymbol{r} \cdot \nabla\right] w - \lambda Q v = -\boldsymbol{r} \cdot \nabla \times [\boldsymbol{u} \times (\nabla \times \boldsymbol{u})] . \tag{2.6}$$

Equation (2.2) can be rewritten in the form

$$\left\{\nabla^2 - \mathrm{Pr}\frac{\partial}{\partial t}\right\} \vartheta + L^2 v T'(r) r^{-1} = \mathrm{Pr}(\boldsymbol{u} \cdot \nabla \vartheta) . \tag{2.7}$$

We have used the following property of the representation (2.4):

$$\boldsymbol{r} \cdot \boldsymbol{u} = L^2 v \equiv -r^2 \nabla^2 v + \boldsymbol{r} \cdot \nabla(\boldsymbol{r} \cdot \nabla v + v) .$$

In spherical coordinates (r, θ, ϕ) with respect to the axis $\boldsymbol{k}$ the operator L^2 is given by

$$L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} . \tag{2.8}$$

The operator Q is defined by

$$Q = \boldsymbol{k} \cdot \nabla - \tfrac{1}{2}(L^2 \boldsymbol{k} \cdot \nabla + \boldsymbol{k} \cdot \nabla L^2) \tag{2.9}$$

and is identical with the operator $Q^{(3)}$ introduced by Roberts (1968) in a related problem. In the following we shall not refer to specified boundary conditions. We assume that they are of the form

$$v = \left(\frac{\partial^2}{\partial r^2} + a_1 \frac{\partial}{\partial r}\right) v = \left(\frac{\partial}{\partial r} + a_2\right) w = \left(\frac{\partial}{\partial r} + a_3\right) \vartheta = 0 \quad \text{at} \quad r = r_0 , \quad r_0 + 1 , \tag{2.10}$$

although the discussion will hold for more general, not necessarily linear, boundary conditions. The arbitrary constants a_ν may have different values at the two surfaces of the shell.

In order to solve the system of equations (2.5)–(2.7), we introduce an expansion in powers of the convection amplitude ϵ

$$R = R^{(0)} + \epsilon R^{(1)} + \epsilon^2 R^{(2)} + \dots ,$$
$$v = \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \dots , \quad (2.11)$$

and analogous series for w and ϑ. This expansion allows us to replace the nonlinear equations (2.5)–(2.7) by a hierarchy of linear equations which can be solved in turn. The first order in this hierarchy consists of all terms proportional to ϵ and represents a homogeneous eigenvalue problem with $R^{(0)}$ as the eigenvalue. In higher orders inhomogeneous equations have to be solved in which the parameters $R^{(n)}$ ($n > 0$) are to be determined by the solvability conditions. We shall not extend the analysis beyond the second order. The second-order equations, however, are important, since the nonlinear terms enter here for the first time. It will become evident in § IV that they introduce effects which are qualitatively different from those described by the linear terms.

In the following section we shall be concerned with the linear problem which is given by equations (2.5)–(2.7) after the right-hand sides have been replaced by zero. Since even the linear problem does not allow a solution in the form of simple analytic functions, we shall introduce another expansion in powers of the parameter λ.

III. THE LINEAR PROBLEM

It is convenient to assume in the discussion of the linear part of equations (2.5)–(2.7) the special functional dependence

$$\gamma(r) = T'(r)r^{-1} = r_0^{-1} . \quad (3.1)$$

The equations for $v^{(1)}$, $\vartheta^{(1)}$, and $w^{(1)}$ are then given by

$$L^2(\nabla^2 - i\omega) + im\lambda]\nabla^2 v^{(1)} - R^{(0)} r_0^{-1} L^2 \vartheta = -\lambda Q w^{(1)} ,$$
$$[L^2(\nabla^2 - i\omega) + im\lambda] w^{(1)} = \lambda Q v^{(1)} , \quad (3.2)$$
$$[\nabla^2 - \mathrm{Pr} i\omega]\vartheta^{(1)} + r_0^{-1} L^2 v^{(1)} = 0 .$$

Because of the linearity, a ϕ- and t-dependence of the form

$$e^{im\phi + i\omega t} \quad (3.3)$$

has been assumed. Equations (3.2) have been solved by Chandrasekhar (1953, 1961) for various boundary conditions in the case $\lambda = 0$. In the latter reference it was also shown that the results of the particular case (3.1) do not change qualitatively if more general functions $\gamma(r)$ and $T(r)$ are considered.

The problem is to determine the lowest value R_c of $R^{(0)}$ at which a solution exists satisfying equations (3.2) and the boundary conditions for real values of ω. R_c represents the critical value of the Rayleigh number at which the static fluid layer becomes unstable. The corresponding solution $v^{(1)}$, $\vartheta^{(1)}$, $w^{(1)}$ describes the physically realized solution to the first order of the amplitude, provided that the eigenvalue $R^{(0)} = R_c$ is simple.

Since we are interested in the qualitative features of the solution rather than in its most general description, it is sufficient to consider the case of small values of λ which allow a representation of the solution in the form

$$R^{(0)} = R_0 + \lambda R_1 + \lambda^2 R_2 + \dots ,$$
$$v^{(1)} = v_0 + \lambda v_1 + \lambda^2 v_2 + \dots , \quad (3.4)$$

and analogous expansions for ω, $\vartheta^{(1)}$, $w^{(1)}$. Chandrasekhar (1961) has shown for fairly general cases of boundary conditions that $w^{(1)}$ and ω vanish in equations (3.2) in the case

$\lambda = 0$. Accordingly, the equations of lowest order are

$$\nabla^4 v_0 - R_0 \vartheta_0 r_0^{-1} = 0 \,, \qquad \nabla^2 \vartheta_0 + L^2 v_0 r_0^{-1} = 0 \,. \tag{3.5}$$

These equations allow a solution in the form of spherical harmonics

$$v_0 = V(r) P_l^m(\cos \vartheta) e^{im\phi} \,. \tag{3.6}$$

$V(r)$ and R_0 are determined as a function of l by

$$\left(\frac{d}{dr} r^2 \frac{d}{dr} - \frac{l(l+1)}{r^2}\right)^3 V(r) = -\frac{l(l+1)}{r_0^2} R_0 V(r) \,. \tag{3.7}$$

In the following we shall restrict the use of the symbols R_0, l, $V(r)$ to the particular solution of equation (3.7) with the lowest value of R_0. Since the parameter m does not appear explicitly in equation (3.7), $2l + 1$ independent convection modes correspond to the value R_0. This degeneracy of the eigenvalue problem will be removed as soon as the influence of rotation is considered.

The first-order equations consisting of all terms proportional to λ of equations (3.2) do not yet resolve the degeneracy. The solvability condition, which is obtained in the same way as relation (3.13) below, yields

$$R_1 = 0 \,, \qquad \omega_1 = \frac{m}{l(l+1)(1+\mathrm{Pr})} \tag{3.8}$$

Using the normalization condition $\langle \vartheta^{(1)} L^2 v^{(1)} \rangle = \langle \vartheta_0 L^2 v_0 \rangle$, we find

$$v_1 = \vartheta_1 = 0 \,. \tag{3.9}$$

Relation (3.8) in conjunction with expression (3.3) shows that the convective motion has the time dependence of a wave propagating opposite to the sense of the given rotation. This property corresponds to the dynamical behavior of Rossby waves in a container in which the depth parallel to the axis of rotation increases with the distance from the axis. In the discussion which follows we have to consider the toroidal component of the velocity field which is determined by

$$\begin{aligned} L^2 \nabla w_1^2 &= Q v_0 \\ &= \left[\cos\theta \nabla^2 - \left(L^2 + r\frac{\partial}{\partial r}\right)\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\right] v_0 \\ &= \left[P^m{}_{l+1} \frac{l-m+1}{2l+1} t_+(r,l) + P^m{}_{l-1} \frac{l+m}{2l+1} t_-(r,l)\right] e^{im\phi + i\omega t} \,. \end{aligned} \tag{3.10}$$

The quantities t_+ and t_- denote expressions independent of m. The right-hand side of equation (3.10) indicates the important fact that w and v have a different θ-dependence.

The second order in λ of equations (3.2) is given by

$$L^2 \nabla^4 v_2 - R_0 r_0^{-1} L^2 \vartheta_2 = -Q w_1 + R_2 r_0^{-1} L^2 \vartheta_0 \,, \qquad \nabla^2 \vartheta_2 + r_0^{-1} L^2 v_2 = 0 \,. \tag{3.11}$$

We want to show that R_2 is always positive and that it reaches its lowest value for $m = l$. By multiplying the first of equations (3.11) by v_0^*, the second by $R_0 \vartheta_0^*$, adding the result, and integrating it over the volume of the shell, we obtain after partial integration

$$0 = -\langle v_0^* Q w_1 \rangle + R_2 \langle v_0^* L^2 \vartheta_0 \rangle r_0^{-1} \,. \tag{3.12}$$

The angular brackets denote the average over the shell, and the asterisk indicates the complex conjugate. The right-hand side of relation (3.12) has vanished because v_0^*, ϑ_0^* satisfy the homogeneous part of equations (3.11). For the partial integration, boundary conditions of the form (2.10) have been assumed. To prove that R_2 is positive, we note that

$$\langle v_0^* L^2 \vartheta_0 \rangle = \langle \vartheta_0 L^2 v_0^* \rangle = r_0 \langle |\nabla \vartheta_0|^2 \rangle \tag{3.13}$$

is positive. The term

$$\langle v_0^* Q w_1 \rangle = - \langle w_1 L^2 \nabla^2 w_1^* \rangle = \langle \nabla L^2 w_1 \cdot \nabla w_1^* \rangle \tag{3.14}$$

is positive since w_1 can be written according to equation (3.10) as the sum of two parts

$$w_1 = \left[\frac{l - m + 1}{2l + 1} P^m{}_{l+1} T_+(r,l) + \frac{l + m}{2l + 1} P^m{}_{l-1} T_-(r,l) \right] e^{i(m\phi + \omega t)} , \tag{3.15}$$

for which the operation L^2 is equivalent to multiplication by $(l + 1)(l + 2)$ and $l(l - 1)$, respectively. Hence R_2 exceeds zero. The fact that T_+ and T_- do not depend on m allows us to write the integral (3.14) in the form

$$- \langle w_1 L^2 \nabla^2 w_1^* \rangle = c_+ \left(\frac{l - m + 1}{2l + 1} \right)^2 \langle |P^m{}_{l+1}|^2 \rangle + c_- \left(\frac{l + m}{2l + 1} \right)^2 \langle |P^m{}_{l-1}|^2 \rangle , \tag{3.16}$$

where c_+ and c_- are positive constants independent of m. Similarly, the integral (3.13) can be written

$$\langle v_0^* L^2 \vartheta_0 \rangle r_0^{-1} = c \langle |P_l^m|^2 \rangle \tag{3.17}$$

with a positive constant c independent of m. Finally, the relation

$$\langle |P^m{}_{l+1}|^2 \rangle = \frac{2l + 1}{2l + 3} \frac{l + m + 1}{l - m + 1} \langle |P_l^m|^2 \rangle$$

has to be used, and the proof that $m = l$ yields the lowest value of R_2 follows immediately from

$$c R_2 = c_+ \frac{(l + 1)^2 - m^2}{(2l + 1)(2l + 3)} + c_- \frac{l^2 - m^2}{(2l - 1)(2l + 1)} .$$

Accordingly, it can be concluded that the convection which is physically realized at low supercritical Rayleigh numbers and small but finite rates of rotation corresponds to the mode $P_l^l(\cos \theta)$, a mode which is characterized by a pronounced maximum at the equator.

This result is not surprising if it is remembered that the critical Rayleigh number of a plane fluid layer increases strongly with Ω when the axis of rotation is perpendicular to the layer, yet does not change at all when the axis of rotation is parallel. Because of the curvature of the shell the convection cannot be strictly independent of the $\boldsymbol{k}$-direction and consequently the critical Rayleigh number depends on Ω even at the equator. The fact that the least axisymmetric mode yields the initial Rayleigh number indicates that computations of axisymmetric convection in a rotating spherical shell (Durney 1968) have limited physical significance.

We have considered only terms up to the order λ^2 in the expansion in powers of λ. Yet we expect that the description based on those terms is qualitatively correct even at large values of λ. This belief is supported by the corresponding results in the plane geometry (Chandrasekhar 1961). The terms up to the order λ^2 yield the exact solution in this case if "free" boundaries are assumed and if the horizontal wavenumber is determined by minimizing $R_0 + \lambda^2 R_2$ as a function of the wavenumber.

IV. NONLINEAR EFFECTS

We return to the end of § II where the expansion in powers of the amplitude ϵ was formulated. A number of studies have shown that the nonlinear features of convection in a plane layer can be described successfully for a rather wide range of Rayleigh numbers above the critical value if terms up to the order ϵ^2 are considered. For the particular aspect of this problem in a rotating system we refer to Veronis (1959). It can be expected that the relations which have been obtained in the case of a plane layer hold, with minor modifications, in the case of a spherical shell. The dependence of the amplitude of convection on the supercritical Rayleigh number is the most important relation of this kind. In addition, however, there may possibly be effects which depend on the particular geometry of the shell. One of these possibilities is a differential rotation arising from the nonlinear momentum transport of the convection. In the following we shall restrict our attention to this most striking aspect of the problem.

The equation for the second order of the toroidal part of the velocity field is given according to equation (2.6) by

$$\left[L^2\left(\nabla^2 - \frac{\partial}{\partial t}\right) + \lambda \boldsymbol{k} \times \boldsymbol{r}\cdot\nabla\right]w^{(2)} - \lambda Q v^{(2)} = -\boldsymbol{r}\cdot\nabla \times [\boldsymbol{u}^{(1)} \times (\nabla \times \boldsymbol{u}^{(1)})] . \quad (4.1)$$

The evaluation of the inhomogeneous term on the right-hand side of equation (4.1) yields

$$\begin{aligned} \boldsymbol{r}\cdot\nabla \times [\boldsymbol{u}^{(1)} \times (\nabla \times \boldsymbol{u}^{(1)})] &= -\boldsymbol{r}\cdot(\nabla \boldsymbol{r}\cdot\nabla D v_0 \times \nabla\nabla^2 v_0) \\ &+ \lambda \boldsymbol{r}\cdot\nabla \times [(\boldsymbol{r} \times \nabla D v_0)\nabla^2 w_1 - (\boldsymbol{r} \times \nabla D w_1)\nabla^2 v_0 + \nabla D v_0 \times \nabla D w_1] + \ldots , \end{aligned} \quad (4.2)$$

where terms of the order λ^2 and higher have not been denoted explicitly. The operator D is defined by

$$D \equiv \boldsymbol{r}\cdot\nabla + 1 .$$

In equation (4.2) and in the rest of this section we assume that v_0 and w_1 are described by the real part of the complex description which was derived in the preceding section. The term independent of λ in expression (4.2) vanishes, since both factors in the vectorial product have the same ϕ- and θ-dependence. The terms linear in λ can be rewritten in the following way

$$\begin{aligned} &\lambda[(\boldsymbol{r} \times \nabla\nabla^2 w_1)\cdot(\boldsymbol{r} \times \nabla D v_0) - \nabla^2 w_1 L^2 D v_0 + \tfrac{1}{2}\boldsymbol{r}\cdot\nabla \times (\nabla D v_0 \times \nabla D w_1)] \\ &\qquad - \lambda\,[\text{preceding terms with } w_1 \text{ and } v_0 \text{ interchanged}] \\ &\qquad = \frac{-\lambda}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{r^2} L^2 w_1 \sin\theta \frac{\partial}{\partial\theta} D v_0\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{r^2\sin\theta} L^2 w_1 \frac{\partial}{\partial\phi} D v_0\right)\right] \\ &\qquad + \frac{\lambda}{\sin\theta}\,[\text{preceding terms with } w_1 \text{ and } v_0 \text{ interchanged}] . \end{aligned} \quad (4.3)$$

According to equations (3.6) and (3.15) the expression (4.3) consists of an axisymmetric part and a part with the periodicity $2m$ in the azimuthal direction. We are interested in the axisymmetric part, which is defined as the average of equation (4.3) over the ϕ-dependence and which will be indicated by angular brackets with subscript ϕ. Accordingly, we obtain from equations (4.1) and (4.3), if terms of the order λ^2 are neglected,

$$\frac{\partial}{\partial\theta}\nabla^2\langle w^{(2)}\rangle_\phi = -\lambda\left(r^{-2}\left\langle L^2 w_1 \frac{\partial}{\partial\theta} D v_0\right\rangle_\phi - r^{-2}\left\langle L^2 v_0 \frac{\partial}{\partial\theta} D w_1\right\rangle_\phi\right) . \quad (4.4)$$

It is convenient to rewrite this equation in terms of the mean ϕ-component of the velocity field

$$\langle u_\phi \rangle_\phi = -\epsilon^2 \frac{\partial}{\partial \theta} \langle w^{(2)} \rangle_\phi .$$

Thus, neglecting terms proportional to higher powers of ϵ, we have

$$\left(\frac{\partial}{\partial \theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\sin\theta + \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right)\langle u_\phi \rangle_\phi = \epsilon^2 \lambda \left(L^2 w_1 \frac{\partial}{\partial \theta} D v_0 - L^2 v_0 \frac{\partial}{\partial \theta} D w_1 \right). \quad (4.5)$$

This equation shows that, because v and w differ with respect to their dependence on θ and on r, a differential rotation will be produced by the nonlinear terms. It cannot be easily seen from equation (4.5) which form and which sign the differential rotation does possess. For this purpose we shall study in the following section a special example in more detail.

V. THIN-SHELL APPROXIMATION

To obtain a more detailed picture of the differential rotation, an explicit solution of the linear problem must be inserted in the right-hand side of equation (4.5). We wish to avoid lengthy computation, so we turn to the simplest possible model. We introduce the thin-shell approximation by assuming that the thickness of the convectively unstable shell is small compared with its radius, i.e., we assume $r_0 \gg 1$. Moreover, the case of stress-free boundaries with fixed temperatures will be considered; this is called the "free boundary" case and corresponds to the limit

$$a_1 , \quad a_2 , \quad a_3^{-1} \to 0$$

of the boundary conditions (2.10). As is to be expected, the solution of equation (3.7) is closely related to the analogous solution in a plane convection layer,

$$V(r) = \sin \pi (r - r_0) \quad (5.1)$$

with

$$R_0 \approx \tfrac{27}{4}\pi^4 , \quad l(l+1) \approx l^2 \approx \tfrac{1}{2}\pi^2 r_0^2 . \quad (5.2)$$

Equation (3.10) for w_1 reduces to

$$\begin{aligned} \nabla^2 w_1 &= -\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r_0}\frac{\partial}{\partial \theta}\right) v_0 \\ &= -\frac{1}{2l+1} P^l{}_{l+1}\left[\frac{d}{dr} V(r) - \frac{l}{r_0} V(r)\right] e^{il\phi + i\omega t} , \end{aligned} \quad (5.3)$$

where the result $m = l$ has been used. The integration of this equation yields

$$w_1 = \frac{1}{2l+1} P^l{}_{l+1} \left\{ \frac{2}{3\pi} \cos \pi (r - r_0) - \sum_{\nu=0}^{\infty} \frac{a_\nu}{\pi(4\nu^2 + \frac{1}{2})\sqrt{2}} \cos 2\nu\pi (r - r_0) \right\} e^{il\phi + i\omega t} . \quad (5.4)$$

The coefficients a_ν are determined by the expansion of $\sin \pi(r - r_0)$ in terms of the Fourier modes $\cos 2\nu\pi(r - r_0)$ which satisfy the required boundary conditions

$$a_\nu = -\frac{4}{\pi(4\nu^2 - 1)} \quad \text{for} \quad \nu \geq 1 , \quad a_0 = \frac{2}{\pi} .$$

The solution $\langle u_\phi \rangle_\phi$ of equation (4.5) can be obtained by expanding $\langle u_\phi \rangle_\phi$ and the inhomogeneous right-hand side in terms of a system of orthogonal functions which satisfy the boundary conditions in the θ-direction as well as in the r-direction,

$$\langle u_\phi \rangle_\phi = \Sigma_{\mu,\rho} u_{\mu\rho} \cos \mu\pi(r - r_0) P^1_\rho(\cos\theta) , \tag{5.5}$$

$$\left\{ L^2 w_1 \frac{\partial}{\partial\theta} Dv_0 - L^2 v_0 \frac{\partial}{\partial\theta} Dw_1 \right\} = \Sigma_{\mu,\rho} b_{\mu\rho} \cos \mu\pi(r - r_0) P^1_\rho(\cos\theta) . \tag{5.6}$$

Because of the property

$$\frac{d}{d\theta} \frac{1}{\sin\theta} \frac{d}{d\theta} \sin\theta P^1_\rho(\cos\theta) + \rho(\rho + 1) P^1_\rho(\cos\theta) = 0 \tag{5.7}$$

equation (4.5) yields, in the case of the small-gap approximation with free boundaries, the formal solution

$$u_{\mu\rho} = \epsilon^2 \lambda \frac{b_{\mu\rho}}{\rho(\rho + 1) + (\mu\pi)^2} . \tag{5.8}$$

A more explicit solution can be obtained if we consider only the average of $\langle u \rangle_\phi$ with respect to its r-dependence. We denote this average by a second subscript r and obtain the equation for $\langle u_\phi \rangle_{\phi,r}$ from relation (4.5):

$$\begin{aligned} \frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\langle u_\phi\rangle_{\phi,r}\right) &= \lambda\epsilon^2\left[\left(\frac{d}{d\theta}P_l{}^l\right)\sin\theta\frac{d}{d\theta}P_l{}^l + P_l{}^l\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}P_l{}^l\right)\right]\frac{l}{6} \\ &= \lambda\epsilon^2\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d}{d\theta}(P_l{}^l\sin\theta)^2\right)\frac{l^2}{12(l+1)} , \end{aligned} \tag{5.9}$$

where the relationship

$$\frac{1}{2l + 1} P^l{}_{l+1} = \frac{\sin\theta}{l} \frac{d}{d\theta} P_l{}^l = \frac{1}{l + 1} \frac{d}{d\theta} \sin\theta P_l{}^l \tag{5.10}$$

has been used. Neglecting l^{-1} in comparison with 1, we write the solution of equation (5.9) in the form

$$\langle u_\phi \rangle_{\phi,r} = \lambda\epsilon^2 \frac{l}{12} (P_l{}^l)^2 \sin\theta . \tag{5.11}$$

The important result is the positive sign of the mean azimuthal flow, i.e., the flow has the same direction as the basic rotation. In addition, it is symmetric in both hemispheres and shows a pronounced maximum at the equator. These characteristic features are shown by the differential rotation observed on the Sun.

VI. COMPARISON WITH OBSERVATIONAL EVIDENCE

The knowledge about convection in the solar atmosphere has advanced considerably in recent years owing to the development of new observational techniques. There is now evidence for convective motions of three distinct scales in the outer layer of the Sun. The ordinary granulation can be interpreted as a manifestation of the scale height at the surface as governing scale. An order of magnitude larger is the scale of the supergranulation which was discovered by Leighton (1960) and his coworkers (Leighton, Noyes, and Simon 1962). A third scale of motions, which exceeds the supergranulation scale by another factor of about 10, corresponds to the depth of the convectively unstable layer. The evidence for convection cells of such a large size (they are called "giant cells") is still scarce. Various observations of sunspot motions (Ward 1965, 1966) and of magnetic regions (Bumba, Howard, and Smith 1964), as well as observations by spectroscopic methods (Plaskett 1966), suggest the existence of cells with a horizontal extension of about 4.10^5 km. Theoretical considerations by Simon and Weiss (1968) based on mixing-length arguments tend to support the hypothesis that the observed giant cells correspond to convective motions extending throughout the convection zone.

The giant cells obviously show the closest relation to the idealized convection model considered in the preceding sections. We shall take into account the action of the convective eddies of smaller scale by assuming an eddy viscosity ν_e in place of the molecular viscosity ν. Schwarzschild (1959) gives as a rule of thumb

$$\nu_e = \tfrac{1}{10} VL\,, \tag{6.1}$$

where V represents a typical velocity and L a typical length of the eddies. It was noted by Leighton (1965) that the turbulent diffusion described by the foregoing equation has about the same value for ordinary granulation as for supergranulation. In the following we shall use a value of about 10^{12} cm^2 sec^{-1}. The typical convection velocity of our model

$$V_m \approx \epsilon\pi l v_0 \frac{\nu_e}{h} \tag{6.2}$$

has to be identified with the typical velocity V_g in giant cells which, according to Bumba (1967), has a value of about 3×10^3 cm sec^{-1}. The expression (5.11), together with equations (3.6) and (5.2), yields the following relation for the ratio between the equatorial acceleration and the velocity of convection:

$$\frac{\langle u_\phi \rangle_{\phi, r}}{\epsilon\pi l v_0} \approx \lambda \frac{\sqrt{2}}{\pi^3 r_0 12} \frac{V_m h}{\nu_e} \approx 10\,, \tag{6.3}$$

where $h \approx 10^{10}$ cm has been assumed as the height of the convection zone and $\Omega \approx 3 \times 10^{-6}$ sec^{-1} as the rotation rate of the Sun. The value (6.3) corresponds to an observed value of ≈ 10. The agreement between both values is by no means conclusive because of the large uncertainties which enter such an estimate. Yet the comparison suggests that the proposed driving mechanism for the differential rotation is consistent with the present knowledge about the solar convection zone. For a quantitative comparison with the observed form of the differential rotation, a more detailed model will have to be considered which takes into account the compressibility and the stresses exerted by the magnetic field on the average during the solar cycle. In addition, it would be desirable to replace the concept of eddy viscosity by more reliable results of a theory of turbulence.

In all the qualitative aspects which can be compared, we have found a reasonable correspondence despite the limitations of the model. We therefore conclude that the dynamical mechanism described in this paper is capable of explaining the equatorial acceleration in the solar convection zone.

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