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ANGULAR VELOCITY GRADIENTS IN THE SOLAR CONVECTION ZONE

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ABSTRACT

We test the hypothesis that the weak influence of rotation upon solar supergranulation, resulting in fluid particles conserving their angular momentum while moving radially, is responsible for the outward decrease in angular velocity inferred from the difference between photospheric plasma and sunspot rotation rates. This test is performed using numerical integrations of a Boussinesq spherical convection model for a thin shell at small Taylor number (implying weak influence of rotation). We find that the convection does maintain an outward decrease in angular velocity, which approaches the limit implied by angular momentum conservation as the Rayleigh number or driving for convection is increased.

By examining the energetics of the motion, we verify that the dominant process maintaining the calculated angular velocity profile against viscous diffusion is the inward transport of angular momentum by the convection. Axisymmetric meridional circulation plays virtually no role in this process. We further find there is no tendency for convection weakly influenced by rotation to form an equatorial acceleration, contrary to earlier calculations by Busse for similar velocity boundary conditions.

We argue from these and earlier calculations that the origin of the Sun's latitudinal gradient of angular velocity is deep in the convection zone. At these depths there may be a strong tendency for angular velocity to be constant on cylinders, implying a positive radial gradient of angular velocity. The latitude gradient is transmitted to the photosphere by supergranulation which locally produces the negative radial gradient in the top layers. We suggest from the rotation of various magnetic features that the transition from negative to positive radial angular velocity gradient occurs near the bottom of the supergranule layer.

Although the convection model we have used is Boussinesq, we argue that angular momentum conservation in radially moving fluid particles should produce a similar angular velocity profile in compressible convecting fluid layers.

Subject headings: convection — Sun: atmospheric motions — Sun: granulation — Sun: rotation

I. INTRODUCTION

Doppler observations of the photosphere (e.g., Howard and Harvey 1970) show that the mean rotation rate of the gas is about 5% lower than the rate observed from rotation of most magnetic tracers (such as sunspots) at the same latitude. This difference might arise (Foukal 1972) if the magnetic tracers are anchored in a deeper layer of higher angular velocity, and can slip through higher layers with relatively little resistance. The observed sign and magnitude of the difference are reproduced (Foukal and Jokipii 1975) if the gas conserves angular momentum in its radial motion over a depth $d \sim 1.5 \times 10^4$ km, corresponding to the depth expected for supergranular convection.

It is interesting that a very similar rotation profile with ω increasing inward by about 5% in the top layers of the convection zone has been obtained more recently in the preliminary analysis of the splitting of the mode structure of the 5 minute oscillation in the (k, ω)-plane (Deubner 1977).

It is likely that the convection of largely nonmagnetic gas in this layer is effectively decoupled dynamically from the magnetic field (Foukal and Jokipii 1975). This suggests (Foukal 1977) that supergranulation offers a relatively direct opportunity (that is, uncomplicated by the influence of Lorentz forces) to compare solar convection with the models of Boussinesq nonaxisymmetric convection in a rotating spherical shell, developed by Gilman (1975, 1976, 1977, 1978).

In particular, integrations performed with this model indicate (Gilman 1977) that for Prandtl numbers $P \lesssim 1$, in a slowly rotating shell, the gas tends to conserve angular momentum in radial motion. These results emerged from calculations carried out on a geometrically thick shell and with a stress-free

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lower boundary. The integrations reported in this paper were carried out to determine whether a geometrically thin layer with a nonslip lower boundary might still exhibit the tendency to conserve angular momentum that we have inferred in the supergranular layer from the observations described above.

We also discuss the implication of these calculations in the broader context of radial and latitudinal angular velocity gradients throughout the convection zone, as well as model calculations by others.

We argue finally that, although the model treats convection in the Boussinesq approximation, a similar rotation profile may be established in a convecting layer of nonuniform density as on the Sun.

II. CONVECTION MODEL

The model equations actually integrated are the standard Boussinesq equations for a rotating stratified liquid. Density variations are ignored except where coupled with gravity. Fluid accelerations at a point are produced in response to the following forces: pressure gradient, buoyancy, Coriolis and viscous, as well as nonlinear inertia forces due to the fluid transporting its own momentum. Temperature is changed by diffusion and nonlinear transports. Temperature and density fluctuations are linearly related by the equation of state for a liquid. The flow has no divergence. Viscous and thermal diffusion are represented linearly, with scalar constant diffusion coefficients.

All fluid variables are represented by Fourier series in longitude and the resulting amplitude functions are integrated forward in time on a finite difference grid in the meridian plane. The flow is not axisymmetric about the rotation axis, since a large group of nonzero longitudinal wavenumbers are retained, as well as the axisymmetric (wavenumber zero) part. The equations are solved in dimensionless form, in which four nondimensional parameters appear, namely, the Prandtl number P, Taylor number T, Rayleigh number R, and a parameter β, which is the ratio of inner radius of the shell to the shell thickness. As defined in publications cited above, $P = \nu/\kappa$, in which ν is the kinematic viscosity and κ the temperature diffusivity; $T = 4\Omega^2 d^4/\nu^2$, in which Ω is the rotation rate of the reference frame, and d the shell depth; $R = g\alpha\Delta\theta d^3/\kappa\nu$, in which g is gravity, α the coefficient of volume expansion, and $\Delta\theta$ the temperature difference across the shell.

We regard this convection model not as representing the supergranular layer directly, but rather as an analog containing much of the relevant physics needed for studying how convection redistributes angular momentum. We are interested here in the role played by supergranule scale convection, and so choose ranges of the parameters P, T, R, and β with this in mind. In particular, we assume granules and other small-scale motions act in an eddy diffusive way on the supergranule scale motions. Therefore ν and κ are identified as eddy diffusivities and taken to be of similar magnitude. Thus we focus on cases with $P = 1$. From mixing-length arguments, we can estimate $\nu \sim 10^{12}$–10^{13} cm^2 s^{-1}. Then with $d = 2 \times 10^9$ cm, $\Omega = 2.8 \times 10^{-6}$ s^{-1}, the Taylor number for supergranules falls in the range 5 to 500. R is very difficult to estimate, since on the Sun $\Delta\theta$ corresponds to the excess temperature difference above the adiabatic difference across the layer. Obviously, we must choose R large enough to get convection. In the actual numerical experiment, we will scan through a range of R values between 1500 and 48,000.

Previous studies with this convection model have focused on the problem of how convection influenced by rotation could produce equatorial acceleration. We found (Gilman 1972, 1976, 1977, 1978) that in order to get significant equatorial acceleration, the convection must be strongly influenced by rotation. As argued in Gilman (1977), this may be expressed as a statement that $PT/R > 1$, or the Coriolis frequency is comparable to or larger than the buoyancy frequency or the turnover rate for the convection. For supergranules, clearly the turnover rate is greater than the Coriolis frequency, so that supergranules are at best weakly influenced by rotation. Thus they cannot be responsible for the solar equatorial acceleration. As we shall show, they can, however, be largely responsible for the radial gradient of angular velocity in this thin layer below the photosphere.

III. NUMERICAL EXPERIMENT

The supergranule layer on the Sun probably has a depth no greater than 2%–3% of the solar radius. To model this thin a layer over a whole spherical shell requires very high horizontal resolution to resolve the convective cells properly. We choose to take a somewhat thicker layer, 10% ($\beta = 9$), to greatly reduce the computation required, while still illustrating the basic effect. For this layer we adopt a 2°.5 grid in latitude, with eight interior grid intervals in radius, and the lowest 20 longitudinal wavenumbers (0 through 19). The bottom of the layer is assumed to be one of constant heat flux; the top, constant temperature. The bottom is taken to rotate rigidly, to simulate a strong coupling to the angular momentum of the denser layer below. The top is assumed to be stress-free. We could have applied the observed latitudinal differential rotation at the bottom, but we choose to illustrate the radial gradient effect in its simplest form. Also, a nonslip boundary condition on the convective velocities represents the most restrictive condition, in that the viscous boundary should impede the convection from conserving angular momentum in radial motion. An alternative, but fluid dynamically somewhat artificial, boundary condition would have been to retain the nonslip condition on the axisymmetric east-west flow, and allow stress-free conditions on all other tangential velocities.

The calculation is then begun from random numbers in the temperature field, at a given P, T, R, and allowed to continue until the average properties of the convective flow seem stationary. Among other statistics, the model computes the average linear (as opposed to angular) rotational velocity u_0 as a function of level above the bottom. If the rotational velocities produced at all levels have the same angular momentum as the rigidly rotating bottom of the shell at the same latitude, then, at each level above the bottom, there will be a rotational velocity relative to the bottom which is negative. Expressed in dimensionless form in terms of the parameters defined above, it is given by $u_0(K)$ at the level K,

$$u_0(K) = -PT^{1/2}(2K-3)\frac{\Delta}{4}\frac{4\beta+(2K-3)\Delta}{2\beta+(2K-3)\Delta}\cos\phi\,, \tag{1}$$

in which Δ is the grid interval in the radial direction in fractions of the shell depth, and ϕ is the latitude. In this case $\Delta = 1/8$. Note that the Rayleigh number R does not enter into (1). The K levels are defined as follows: $K = 1$ is $\Delta/2$ below the inner boundary, $K = 2$ is $\Delta/2$ above the inner boundary, and higher K values are successively $(K-2)\Delta$ above. The lowest two levels, which straddle the physical boundary, are used to set the boundary conditions.

The object, then, of the calculation is to see how close the values of u_0 obtained from the model calculation compare to the constant angular momentum values given in equation (1).

IV. RESULTS

Figure 1 shows the calculated linear rotational velocity at K level 9 (near the top of the convective shell) for several Rayleigh numbers, at a Taylor number of 300 and Prandtl number of 1, compared to the smooth curve for constant angular momentum obtained from equation (1). We see that there is an obvious tendency toward conserving angular momentum along local radii. At $R = 3000$, which is only about 3 times critical for convection to occur, about half the required amplitude is achieved, while by $R = 24{,}000$, almost 80% of the constant angular momentum profile is attained. At $R = 48{,}000$ (not shown) the agreement is even better. The effect is about the same at all latitudes except near the poles, where the effect is less clear. This should not be surprising, because at higher latitudes, the local moment arm changes more due to horizontal than radial motions.

The jagged nature of the computed profiles is due to the finite size of the convection elements introducing local fine structure in the profiles. The latitudinal scale of these features would be considerably reduced for a layer of supergranule depth rather than the 10% we have chosen.

How closely does the rotational profile approach that of constant angular momentum along radii for deeper levels in the shell? Figure 2 plots results for

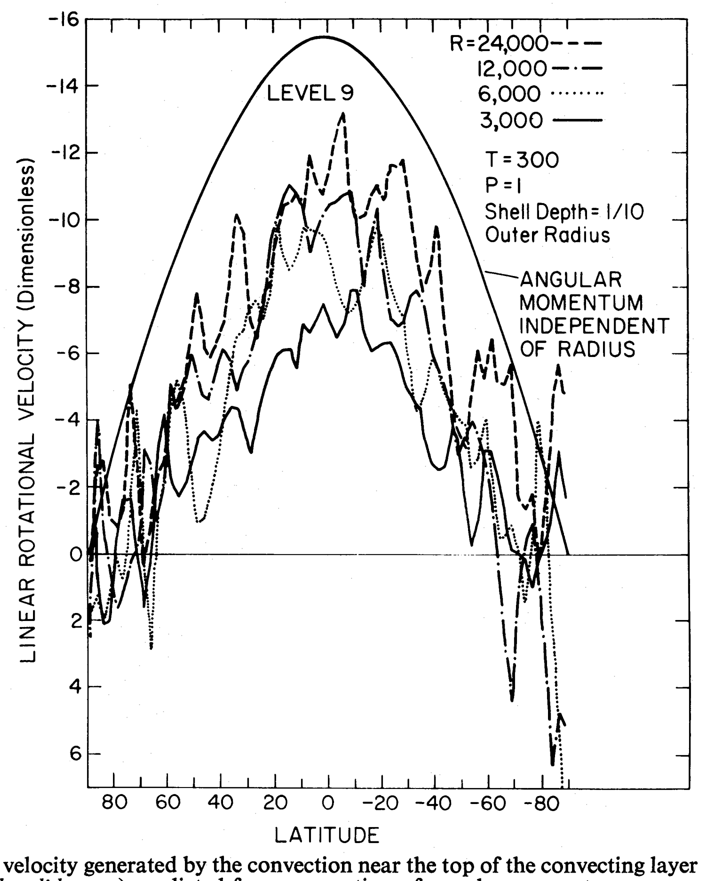

FIG. 1.—Linear rotational velocity generated by the convection near the top of the convecting layer for several Rayleigh numbers, compared with value (*smooth solid curve*) predicted for conservation of angular momentum.

four levels at $R = 24{,}000$, to be compared with the constant angular momentum values at the same levels in Figure 3. We see that levels 9 and 7 compare rather well, level 5 somewhat less so, and level 3 not well. We interpret this result as indicating that at level 3, the influence of the viscous boundary layer at the bottom is quite large, resulting in little tendency for fluid particles moving radially to conserve angular momentum through these deepest levels. Instead, level 3 rotates at nearly the same rate as the bottom, and so has a higher angular momentum. Above level 3, the tendency becomes much stronger, so little further angular momentum is gained. For example, at low latitudes in both Figure 2 and 3 the linear velocity becomes 7 or 8 units more negative in going from level 5 to level 9. Had we chosen stress-free bottom boundary conditions for the convective velocities, while retaining a nonslip condition for the differential rotation there, the effect of the viscous boundary layer at the bottom would have been greatly reduced.

We have done other calculations at even lower Rayleigh numbers, and at other small Taylor numbers, for example, $T = 100$, and found very similar results.

It appears from our calculation that as R is increased, the calculated rotational profile asymptotically approaches the profile of constant angular momentum along local radii. This is a rather general result, indicating we need not know precisely what the effective Rayleigh number is for supergranules in order to determine whether fluid particles moving radially in them are conserving angular momentum.

Busse (1973) predicted from linear theory at low Taylor numbers for a thin shell with rigidly rotating bottom that an equatorial acceleration would be formed. We find no evidence of this, even at an $R = 1500$ for $T = 300$, which is only about 1.5 times critical. An equatorial acceleration would show up as positive values (below the zero line) of rotational velocity in Figures 2 and 3, which is the opposite of what we find. Clearly the tendency of fluid particles moving radially to conserve their angular momentum predominates over any latitudinal angular momentum transport toward the equator which would be needed to spin it up. This effect of radially moving particles is apparently not properly captured in Busse's (1973) calculations.

V. MAINTENANCE OF THE CALCULATED RADIAL DIFFERENTIAL ROTATION PROFILE

In the absence of a process to maintain it, the radial angular velocity gradient implied by Figure 2 would relax to zero due to viscous stresses. This relaxation would constitute an outward diffusion of angular momentum. Since an equilibrium has been reached in which angular velocity decreases outward, this out-

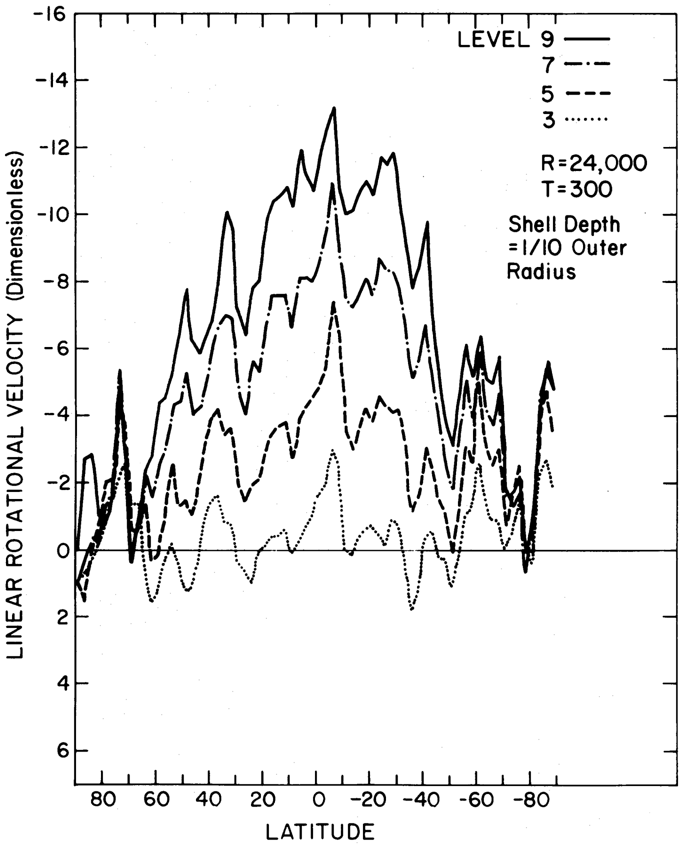

FIG. 2.—Linear rotational velocity profile for several levels in the convecting layer at a Rayleigh number of 24,000

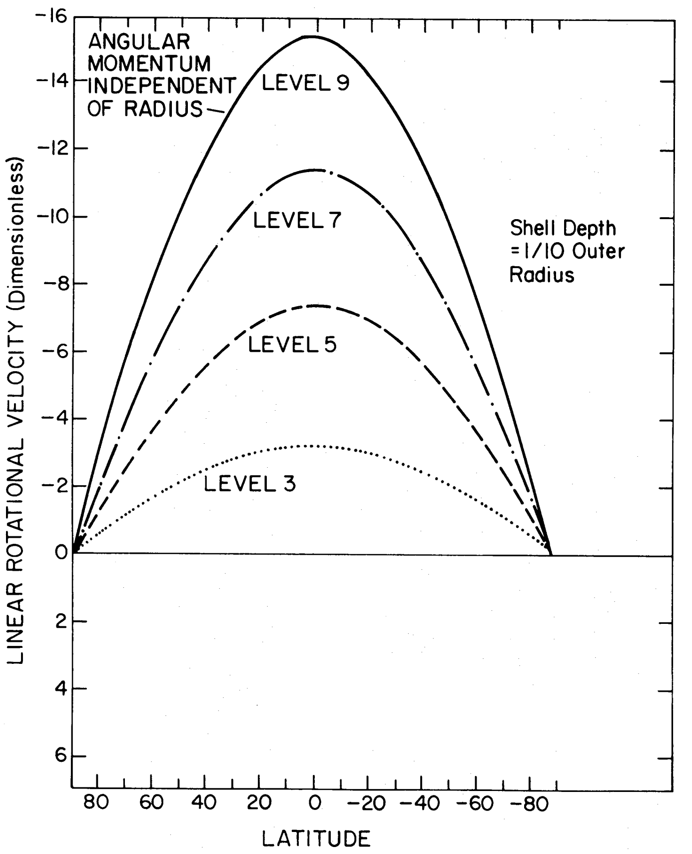

FIG. 3.—Linear rotational velocity profiles for the same levels as shown in Fig. 2 which have the same angular momentum as the bottom of the convecting layer.

ward diffusion of angular momentum must be balanced by inward angular momentum transports due to the convective motions. A check of the radial angular momentum transport by the convection in the model indeed shows that, virtually everywhere, the transport is inward.

That this inward radial transport is the dominant mechanism for maintaining the angular velocity gradients can be verified by computing the work done to maintain the kinetic energy of differential rotation u_0 which we denote by DRKE. The details of this kind of calculation have been outlined earlier in Gilman (1977, 1978). As described there the integral equation describing the work done can be written in symbolic form as

$$\frac{\partial}{\partial t}(\text{DRKE}) = \text{HMT} + \text{VMT} + \text{CURV} + \text{HCF} + \text{VCF} - 1 \,. \quad (2)$$

All the rates of work done on the right are normalized with respect to the viscous dissipation rate, so it appears as a -1. HMT represents the work done by the convergence of latitudinal or horizontal momentum transport by Reynolds stresses associated with the convection, VMT the same for vertical or radial transport. CURV represents work done by stresses associated with the curvature of the coordinate system. HCF represents work done by Coriolis forces due to the horizontal or latitudinal component of axisymmetric meridional circulation, and VCF the same for the vertical or radial component of the meridional circulation.

For energy balance to be achieved, the sum of the first five terms on the right in equation (2) must add up to about $+1$. Figure 4 plots the magnitude of these terms as function of the Rayleigh number R for the calculations we have done. We see that for all R, the dominant work term is indeed VMT, indicating that the primary mechanism for maintaining the angular velocity gradient is the inward radial transport of angular momentum by Reynolds stresses induced in the convection. This is the inevitable consequence of inward-moving fluid particles having higher angular momentum than their outward-moving counterparts at the same level.

By contrast, any axisymmetric meridional circulation present plays a negligible role in maintaining the differential rotation. This implies it is risky to infer much about differential rotation maintenance from purely axisymmetric models, as numerous authors have done in the past.

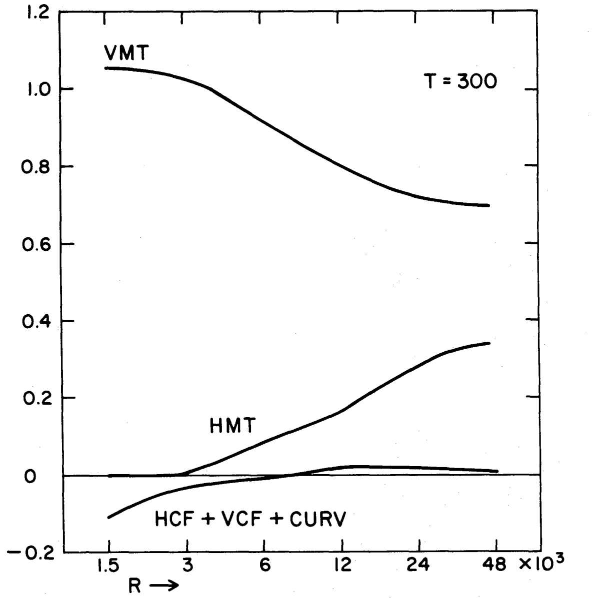

FIG. 4.—Graphical representation of eq. (2) for rates of maintenance of differential rotation kinetic energy scaled relative to the kinetic energy dissipation rate plotted against the log of the Rayleigh number. See text for explanation of curve labels.

At large R, we see that HMT grows to about one-third of the total work done. Examination of the profile of latitudinal angular momentum transport reveals that it is highly structured, with no global pattern of equatorial transport. This is in contrast to the results obtained by Busse (1973) from linear theory for weak rotational influence on convection in a thin rotating spherical shell with nonslip bottom, and illustrates the risks in extrapolating very far from linear theory into the nonlinear region. At small R, HMT is negligible compared with VMT in maintaining the angular velocity profile, also contrary to the inferences made by Busse from his linear results. The reason why Busse's (1973) model does not work well is that the mode selection mechanism found by him to favor modes which produce an equatorial acceleration is an extremely weak one when the Taylor number is small. It is easily overcome even quite close to the critical Rayleigh number required for convection to occur, once a spectrum of convective modes are allowed to grow and seek their own relative amplitudes, as happens in our model calculation.

VI. COMMENTS ON ANGULAR VELOCITY PROFILES WITH LATITUDE AND DEPTH ON THE SUN

We have shown that convection in a thin spherical shell weakly influenced by rotation can produce a substantial outward decrease of rotational velocity, approaching the limit predicted if radially moving fluid particles conserve their angular momentum. This provides a plausible explanation for the difference in angular velocity between sunspots and the photospheric plasma. It is clear, however, that these same motions cannot be responsible for the maintenance of an equatorial acceleration such as the Sun has. Our calculations are for a nonslip bottom, and it might be argued this prevented such an equatorial acceleration from forming. However, similar calculations with a stress-free bottom at weak rotational influence indicate quite the opposite; in particular, with a stress-free bottom boundary, the polar regions accelerate much more.

On the other hand, numerous calculations by Gilman (1976, 1977, 1978) indicate that convection in a rotating spherical shell will generate substantial equatorial acceleration provided the rotational influence on the convection is strong enough. Strong rotational influence on convection cannot be obtained for supergranules, because their turnover time is simply too short compared with the rotation time. But we would expect giant cells, extending to the bottom of the solar convection zone, to be strongly influenced by rotation. These global motions could then be the origin of the latitudinal gradient in angular velocity seen on the Sun.

In summary, then, our qualitative picture of the angular velocity gradient in the solar convection zone is as follows: Near the top, the angular velocity decreases outward, due to the action of supergranules. The profile of angular velocity decreasing with latitude is formed at much deeper levels, and is efficiently transmitted to the surface by the supergranules. In the deeper layer, underneath the supergranule shell, the angular velocity may be nearly constant on cylinders concentric with the axis of rotation, as implied by the convection calculations of Gilman (1972, 1976, 1977, 1978). These arguments would predict that there is an intermediate depth in the convection zone, at which the angular velocity reaches a maximum for the latitude. What this depth is is not clear, but we suspect that it is near the bottom of the supergranule layer.

Observational evidence for the existence of a maximum in the angular velocity profile with depth may come from measurements of rotational frequency shifts of global oscillations. We point out here that rotational velocities of sunspots and X-ray emission features of different sizes, together with surface Doppler velocities, may have already provided us with such evidence. In particular, Golub and Vaiana (1978) have shown that the shortest-lived X-ray emission features rotate at essentially the surface Doppler rate, while the largest, longest-lived features rotate up to 5% faster, or at the sunspot rate. On the other hand, Ward (1966) found that small, short-lived sunspots rotate up to 2% faster than large, long-lived spots, particularly the recurrent spots. Thus small sunspots have the fastest rotation rate of all features in low latitudes. Let us assume, with Golub and Vaiana (1978), that the larger X-ray features have magnetic field roots which go deeper than for small features, make a similar assumption for large spots as compared with small, and finally assume that sunspots, with their higher field strengths spread over larger areas, reach deeper than the magnetic roots of X-ray features. Then the various tracers would reach to depths as shown schematically in Figure 5, with long-lived, large sun-

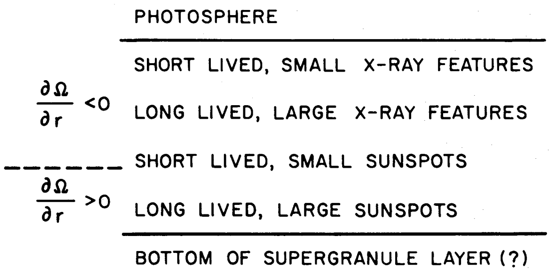

FIG. 5.—Schematic of supergranule layer illustrating depths to which various magnetic features may reach, together with the sign of the radial angular velocity gradient inferred.

spots extending deepest, and short-lived, small X-ray regions shallowest. Therefore the radial angular velocity gradient would change sign at the depth of small sunspots, probably near the bottom of the supergranular layer, about 1.5×10^4 km below the photosphere. We acknowledge that the fast rotation rate of small spots compared with large ones may also in part be due to expansion of young emerging active regions and the tendency for small short-lived spots to occur in the preceding part of such a dipole, as suggested by Kiepenheuer (1953).

VII. EFFECTS OF COMPRESSIBILITY

We have demonstrated a strong tendency for radial motions in Boussinesq convection in a slowly rotating thin spherical shell to conserve their angular momentum. This tendency increases with increases in the Rayleigh number. To determine in detail how strong the same effect is in convection of a compressible fluid, we need to do comparable simulations with a compressible model. However, we can argue heuristically why it should also occur in the compressible case, as follows. Conservation of angular momentum by a fluid element, whether compressible or incompressible, axisymmetric or nonaxisymmetric, implies

$$\rho r \cos\phi(2\Omega r \cos\phi + u_0)\Delta V = \text{const.} \tag{3}$$

in which ρ is the element density, ΔV its volume, Ω the rotation rate of the coordinate system, u_0 the linear rotational velocity of the element relative to the coordinate frame, and r the distance to the center of the sphere. The mass of the same fluid element is also conserved, so

$$\rho\Delta V = \text{const.} \tag{4}$$

But then dividing equation (4) into (3) gives

$$r\cos\phi(2\Omega r\cos\phi + u_0) = \text{const.} \tag{5}$$

so that the angular momentum per unit mass for the fluid element is conserved, for both compressible and incompressible fluids. Therefore if viscous and other torques are sufficiently weak in a compressible, convecting, slowly rotating fluid, our present calculations suggest the angular velocity should decrease outward in it at a rate approaching that predicted by angular momentum conservation.

Parallel arguments could be made by considering a vorticity equation for the differential rotation, but this is beyond the scope of the present work.

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