

Dynamics of Rotating Fluids

Equations of motion

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} + \underline{2\boldsymbol{\Omega} \times \mathbf{u}} = \underline{-\nabla \pi} + \nu \nabla^2 \mathbf{u}$$

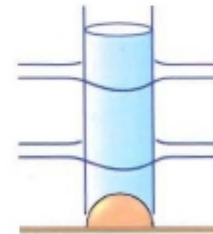
$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \times (2\boldsymbol{\Omega} \times \mathbf{u}) = 2\boldsymbol{\Omega} \nabla \cdot \mathbf{u} - 2\boldsymbol{\Omega} \cdot \nabla \mathbf{u} = 0$$

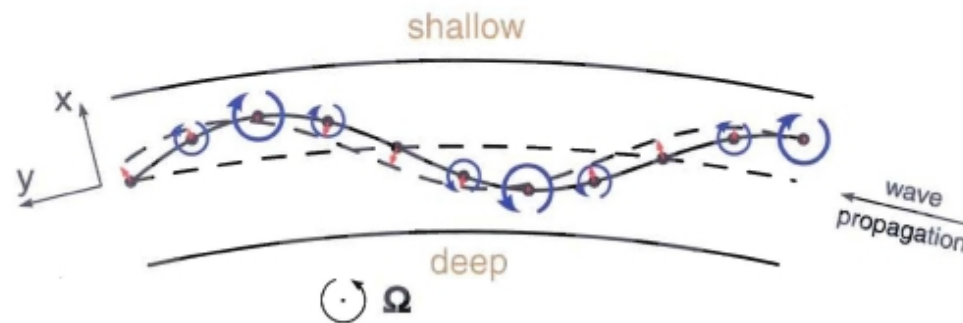
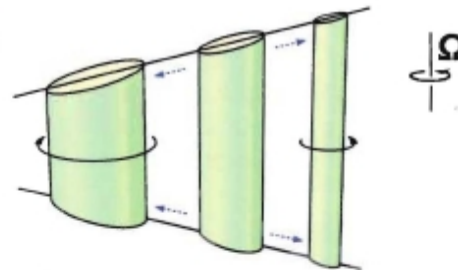
Proudman-Taylor-theorem

$$\boldsymbol{\Omega} \cdot \nabla \mathbf{u} = 0$$

Taylor column

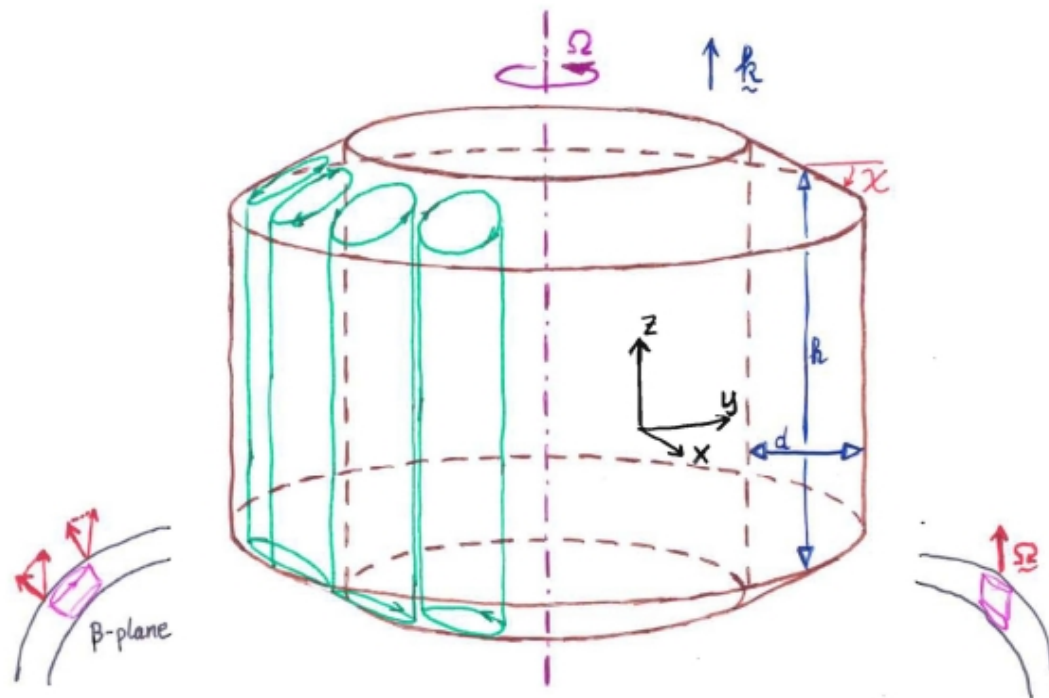


Rossby waves



$$\text{for } u \propto \exp\{i\alpha y + i\omega t\} \cos \gamma x$$

$$\omega = -\frac{2\Omega\alpha\eta}{\alpha^2 + \gamma^2}$$



Rossby Waves in the Rotating Cylindrical Annulus

$$\underline{\hat{h}} \nabla_{\mathbf{x}} \cdot \frac{\partial}{\partial t} \underline{u} + 2\Omega \underline{\hat{h}} \times \underline{u} = -\nabla \pi$$

$$\nabla \cdot \underline{u} = 0$$

$$\underline{u} = \nabla \psi(x, y) \times \underline{\hat{h}} e^{i\omega t} + \dots$$

$$-i\omega \Delta_z \psi - 2\Omega \underline{\hat{h}} \cdot \nabla u_z = 0 \quad \Delta_z = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} dz$$



$$\eta_0 = \tan \chi$$

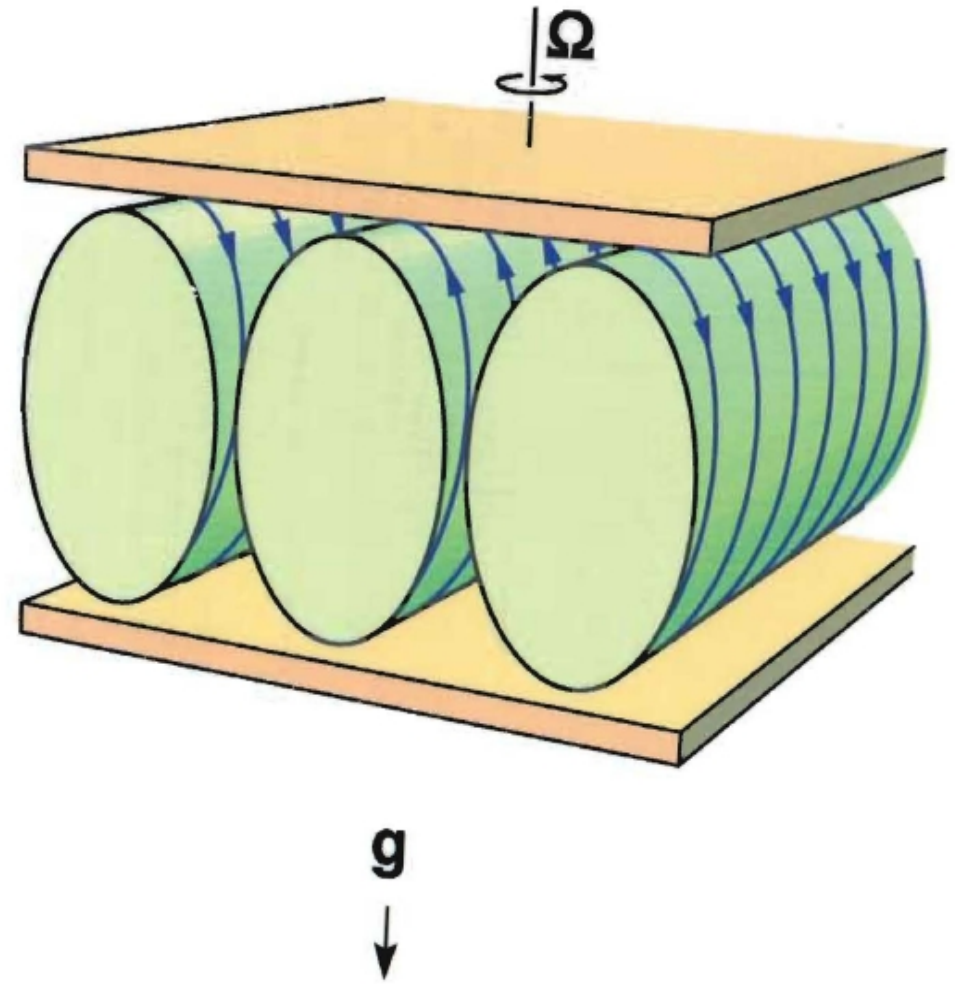
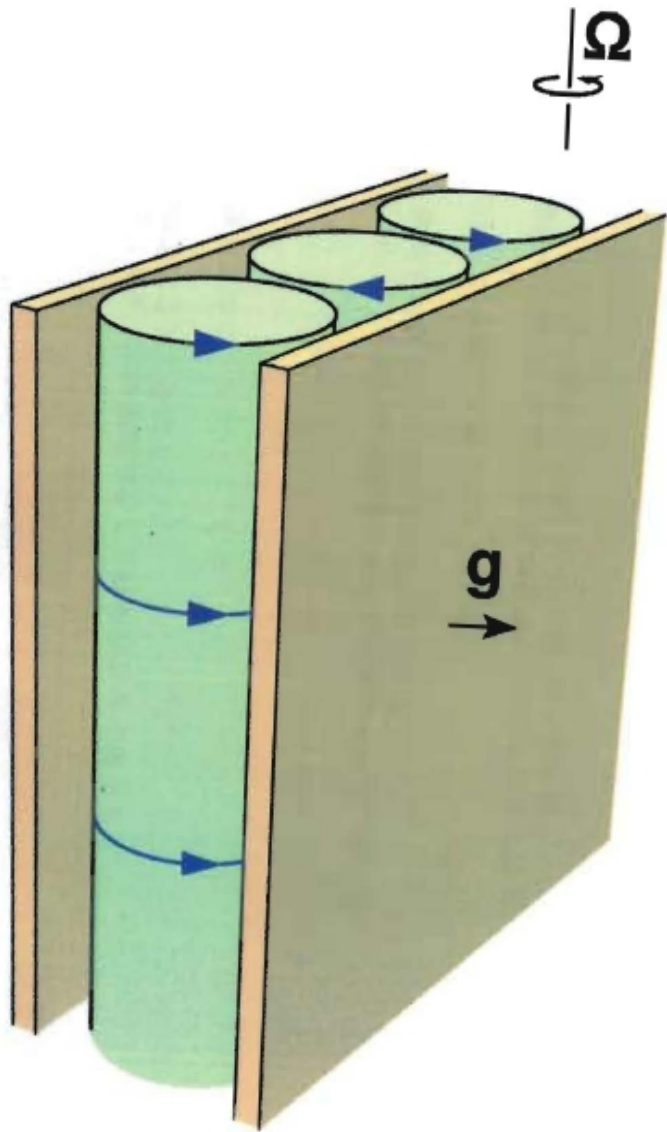
$$u_z \pm \eta_0 \frac{\partial}{\partial y} \psi = 0$$

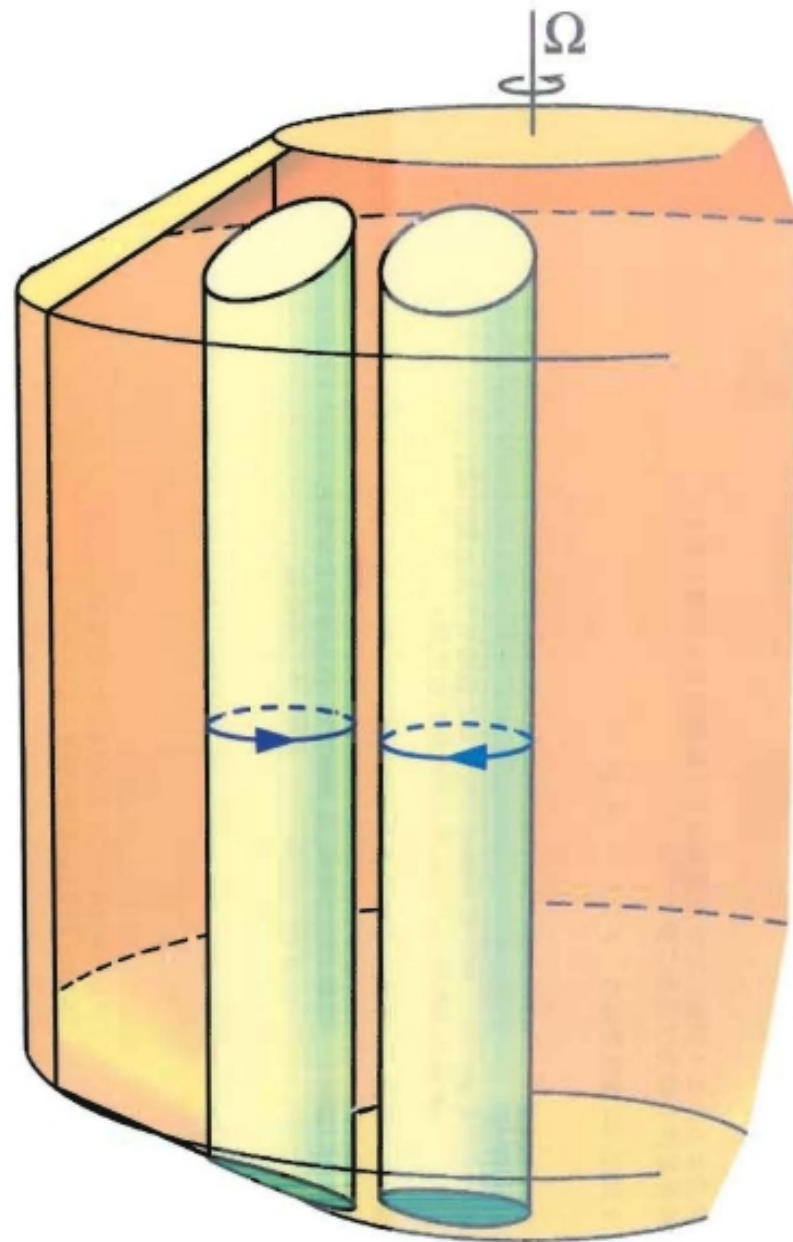
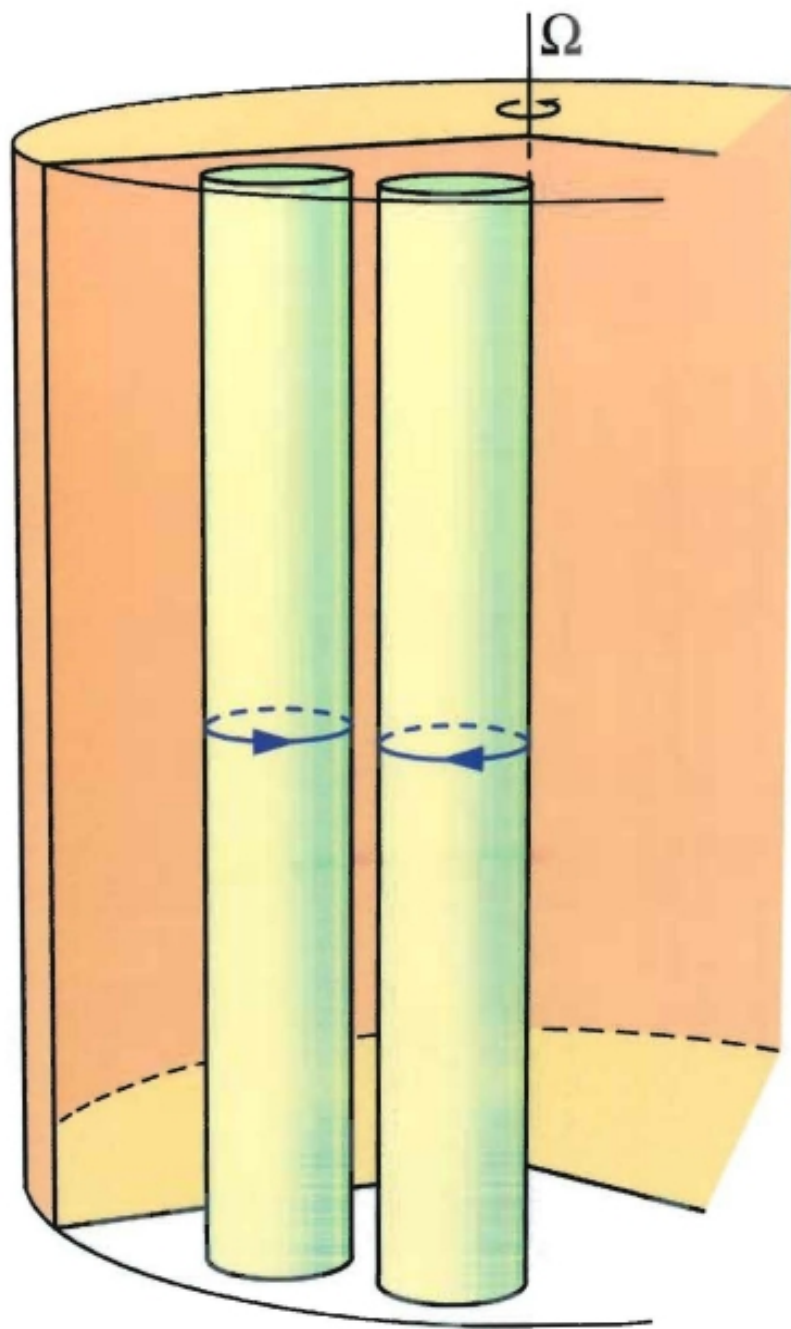
$$\text{at } z = \pm \frac{h}{2}$$

$$-i\omega \Delta_z \psi + \frac{4\Omega \eta_0}{h} \frac{\partial}{\partial y} \psi = 0$$

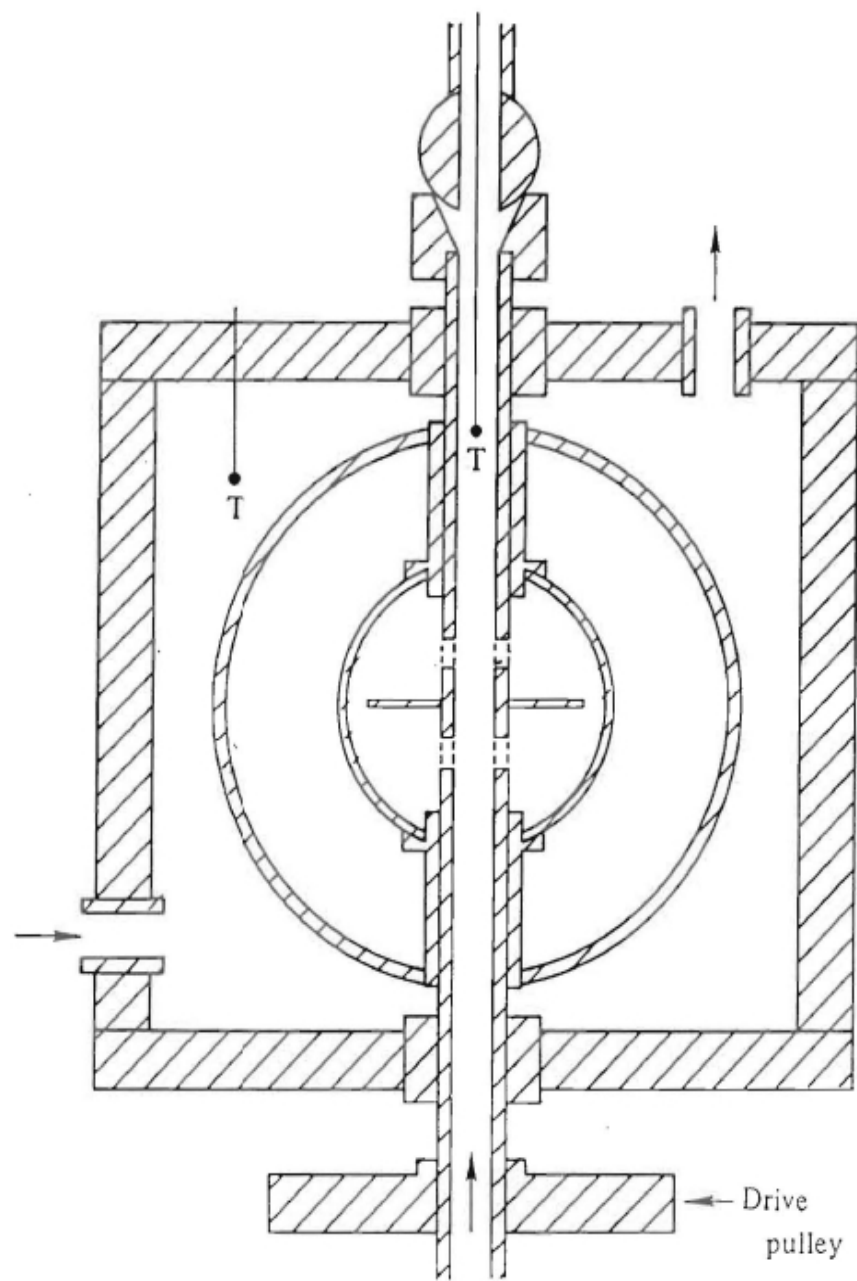
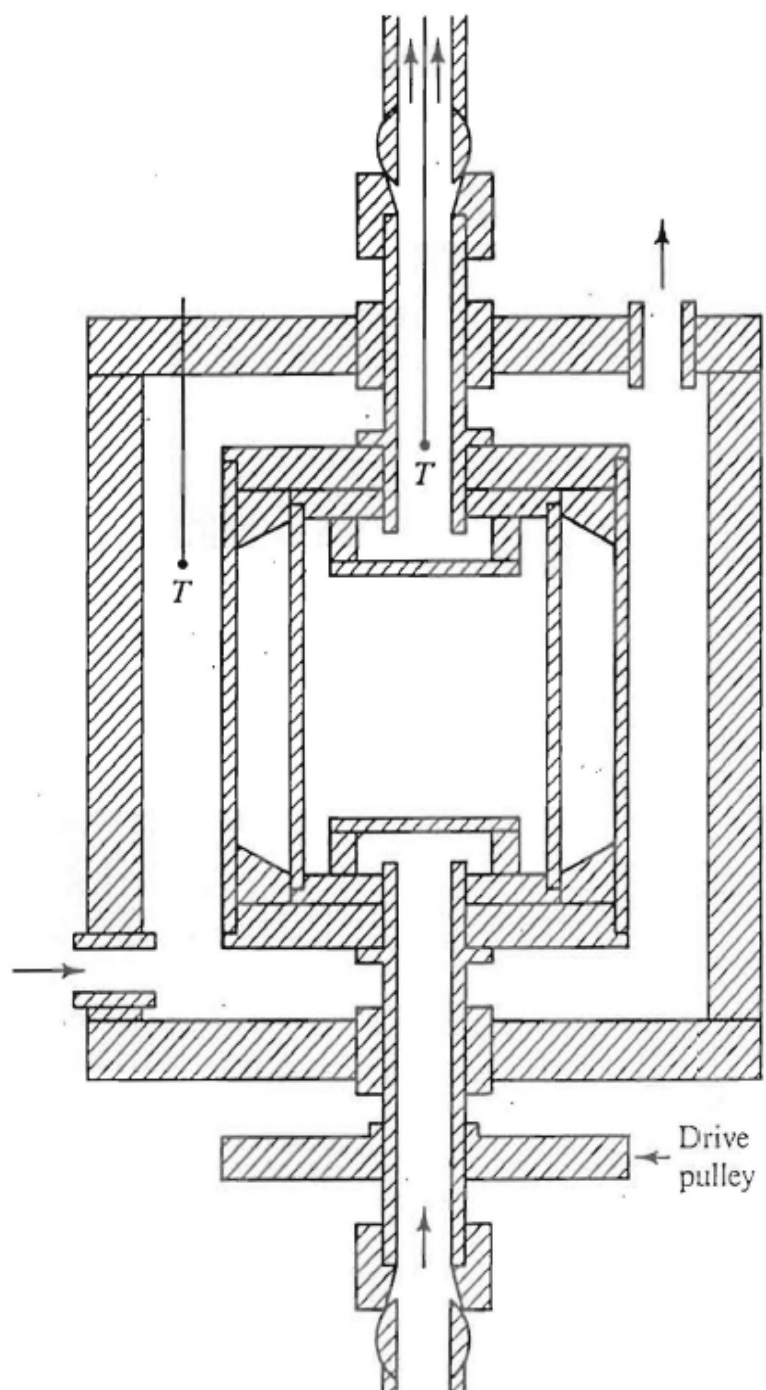
$$\psi(x, y) = \cos \frac{\pi}{a} x \exp\{i\alpha y\}$$

$$\omega = - \frac{4\eta_0 \Omega \alpha}{h \left(\frac{\pi^2}{a^2} + \alpha^2 \right)}$$

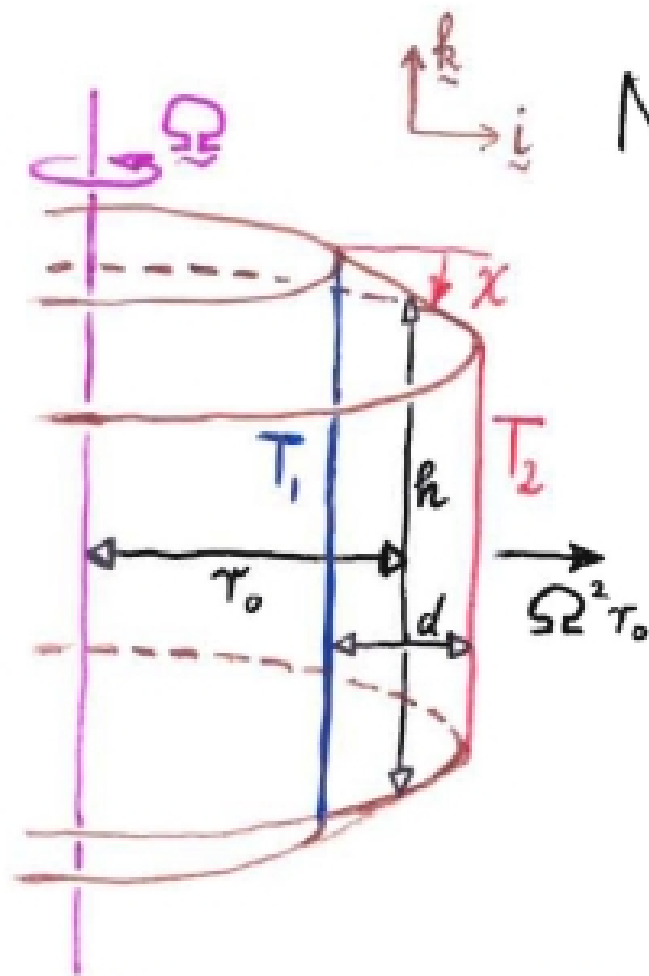








Buoyancy Effects in the Rotating Cylindrical Annulus



Nondimensionalisation

length d , time $\frac{d^2}{\nu}$, temperature $(T_2 - T_1)/\rho$

Basic Equations

$$\nabla \cdot \underline{u} = 0$$

$$\frac{\partial}{\partial t} \underline{u} + \underline{u} \cdot \nabla \underline{u} + 2\Omega \underline{h} \times \underline{u} = -\nabla \pi - i R \Theta + \nabla^2 \underline{u}$$

$$P \left(\frac{\partial}{\partial t} + \underline{u} \cdot \nabla \right) \Theta = -\underline{i} \cdot \underline{u} + \nabla^2 \Theta$$

Dimensionless Parameters: $\Omega = \frac{\Omega_D d^2}{\nu}$, $R = \frac{\gamma (T_2 - T_1) \Omega_D^2 \tau_0 d^3}{\nu \alpha}$, $P = \frac{\nu}{\alpha}$

Inviscid linear analysis: $\underline{u} = \nabla \psi(\alpha y) \times \frac{h}{2} e^{i\omega t}$, $\hat{\Theta} = \hat{\Theta} e^{i\omega t}$

$$(-i\omega \Delta_z + \eta^* \frac{\partial}{\partial y}) \psi - R \frac{\partial}{\partial y} \hat{\Theta} = 0$$

$$\eta^* = 4\eta_0 \Omega d / h$$

$$iP\omega \hat{\Theta} + \frac{\partial}{\partial y} \psi = 0$$

$$\psi = \cos \pi x e^{i\alpha y}$$

$$\hat{\Theta} = -\frac{\alpha}{\omega P} \psi$$

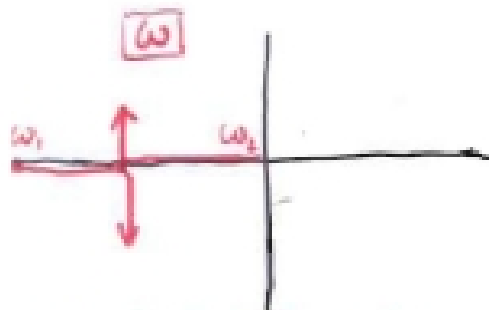
$$Pi\omega (i\omega (\pi^2 + \alpha^2) + i\alpha \eta^*) - R\alpha^2 = 0$$

$$\omega = -\frac{\alpha \eta^*}{2(\pi^2 + \alpha^2)} \pm \sqrt{\frac{(\alpha \eta^*)^2}{4(\pi^2 + \alpha^2)^2} - \frac{R\alpha^2}{P(\pi^2 + \alpha^2)}}$$

$$\omega_1 = -\frac{\alpha \eta^*}{\pi^2 + \alpha^2}$$

$$\omega_2 = -\frac{R\alpha}{\eta^* P}$$

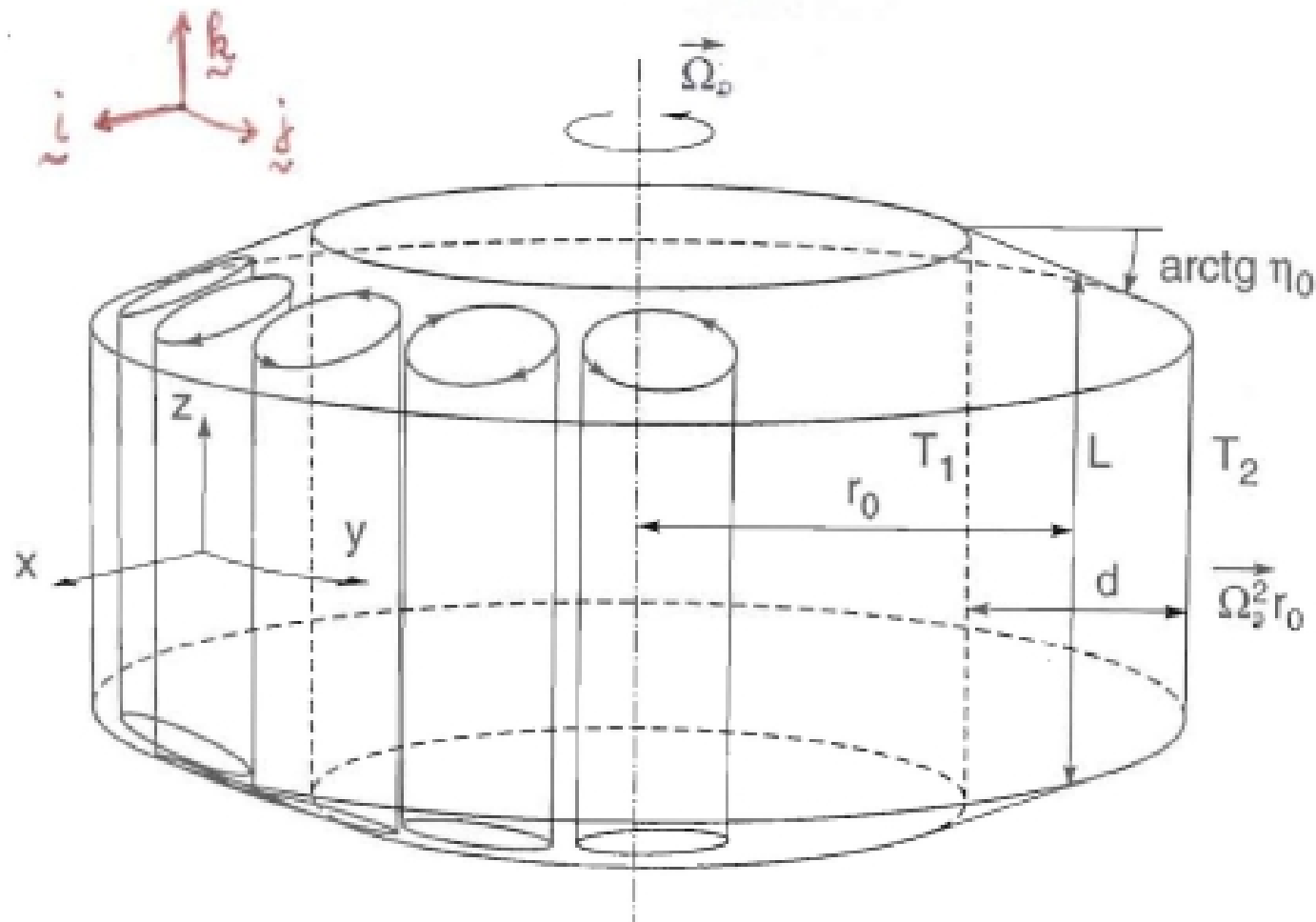
$$\left. \begin{array}{l} \omega_1 \\ \omega_2 \end{array} \right\} \text{ for } R \ll \eta^{*2} P$$



Instability for

$$R > \frac{\eta^{*2} P}{4(\pi^2 + \alpha^2)}$$

Convection in a Rotating Cylindrical Annulus



Scales:

length d
time d^2/ν

temperature
($T_2 - T_1$) ρ

$$\rho = \rho_0 (1 - \gamma (T - T_0))$$

Basic Equations: $\nabla \cdot \underline{u} = 0$

$$\frac{\partial}{\partial t} \underline{u} + \underline{u} \cdot \nabla \underline{u} + 2\Omega \underline{k} \times \underline{u} = -\nabla \pi - \underline{i} R \vartheta + \nabla^2 \underline{u} \quad \otimes$$

$$P \left(\frac{\partial}{\partial t} + \underline{u} \cdot \nabla \right) \vartheta = -\underline{i} \cdot \underline{u} + \nabla^2 \vartheta$$

Dimensionless Parameters: $\Omega \equiv \frac{\Omega_D d^2}{\nu}$, $R \equiv \frac{\alpha(T_i - T_e) \Omega_D^2 \tau_0 d^3}{\nu \kappa}$, $P \equiv \frac{\nu}{\kappa}$

Boundary Conditions: $u_z \pm \eta_0 u_x = \frac{\partial}{\partial z} \vartheta = 0$ at $z = \pm \frac{L}{2d}$, $u_x = \vartheta = 0$ at $x = \pm \frac{1}{2}$

Quasi-geostrophic Balance: $2\Omega \underline{k} \times \underline{u} \approx -\nabla \pi \Rightarrow \underline{u} = \nabla \psi(x, y, t) \times \underline{k} +$

Operation $-\underline{k} \cdot \nabla \times \otimes^2$ yields: $\frac{\partial}{\partial t} \Delta_2 \psi + \underline{u}^T \cdot \nabla \Delta_2 \psi + \frac{2\Omega d}{L} u_z \Big|_{z=\pm \frac{L}{2d}} = R \frac{\partial}{\partial y} \overline{\vartheta}^2 + \Delta_2^2 \psi$
where $\Delta_2 \equiv \partial^2 / \partial x^2 + \partial^2 / \partial y^2$

Two-dimensional
Problem:

with $\eta^* \equiv \frac{4\eta_0 \Omega d}{L}$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} \psi \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \psi \frac{\partial}{\partial y} \right] \Delta_2 \psi - \eta^* \frac{\partial}{\partial y} \psi - \Delta_2^2 \psi = -R \frac{\partial}{\partial y} \vartheta$$

$$P \left[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} \psi \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \psi \frac{\partial}{\partial y} \right] \vartheta + \frac{\partial}{\partial y} \psi - \Delta_2 \vartheta = 0$$

Linear Analysis

Solution for $\varepsilon=0$: $\Psi_0 = \sin\pi(x+\frac{1}{2}) \exp\{i\alpha y + i\omega t\}$, $\mathcal{G}_0 = \frac{-i\omega\Psi_0}{Pi\omega + \pi^2 + \alpha^2}$

Condition for R_0, ω : $[Pi\omega + \alpha^2 + \pi^2] \{ (i\omega + \pi^2 + \alpha^2)(\pi^2 + \alpha^2) + i\alpha\eta^* \} - R_0\alpha^2 = 0$

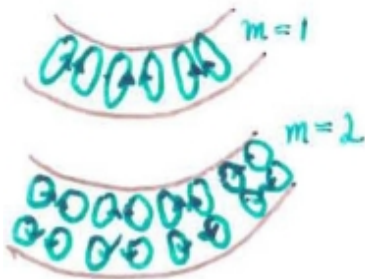
$$\Rightarrow \omega = \frac{-\alpha\eta^*}{(1+P)(\pi^2 + \alpha^2)}, \quad R_0 = \frac{(\pi^2 + \alpha^2)^3}{\alpha^2} + \left(\frac{\eta^* P}{1+P} \right)^2 (\pi^2 + \alpha^2)^{-1}$$

Minimisation of $R_0(\alpha)$ for $\eta^* \rightarrow \infty$: $\alpha_c = \left[\frac{\eta^* P}{\sqrt{2}(1+P)} \right]^{\frac{1}{3}}, \quad R_c = 3\alpha_c^4$

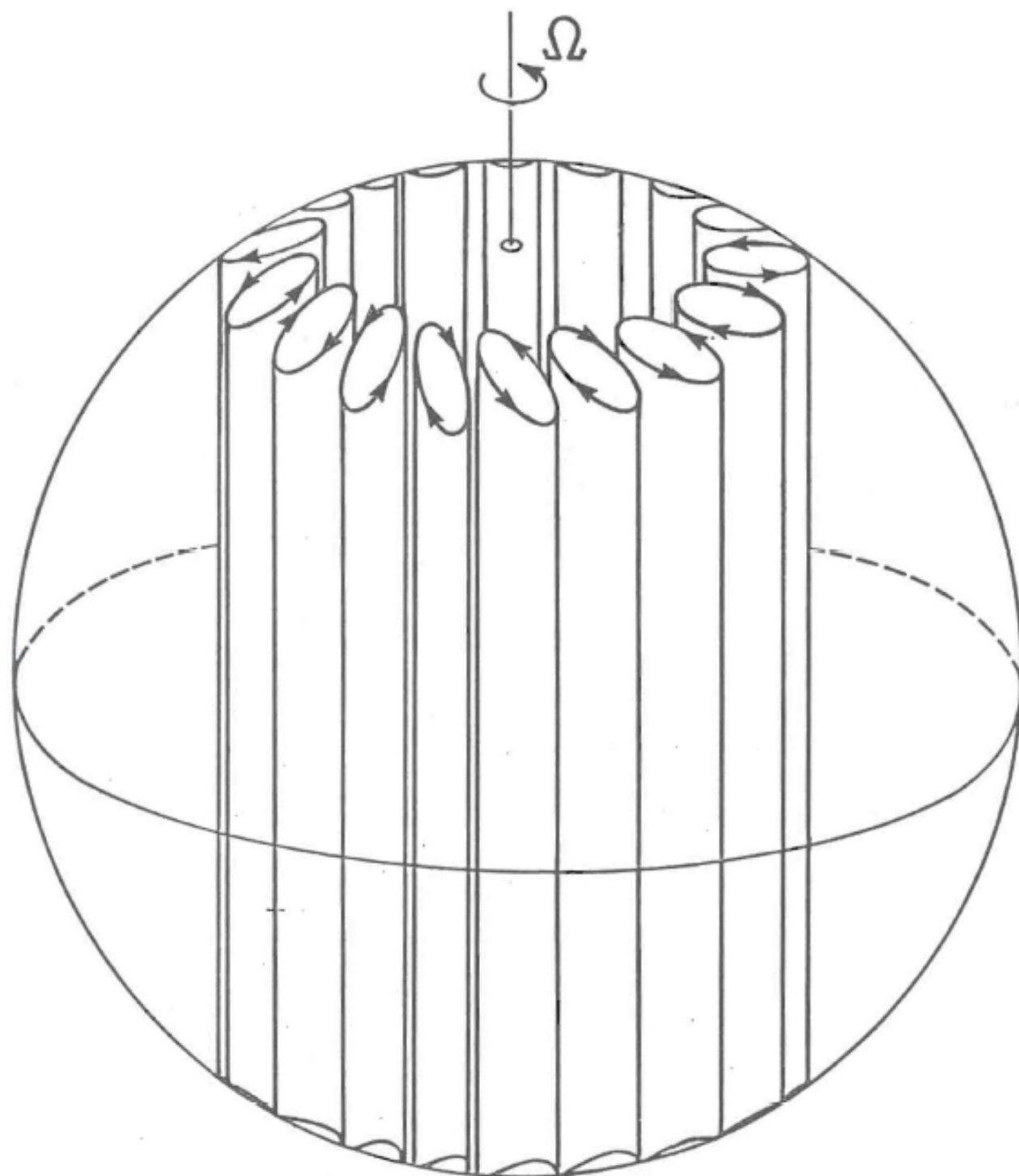
$\Rightarrow$ Onset of convection independent of D for given temperature gradient!

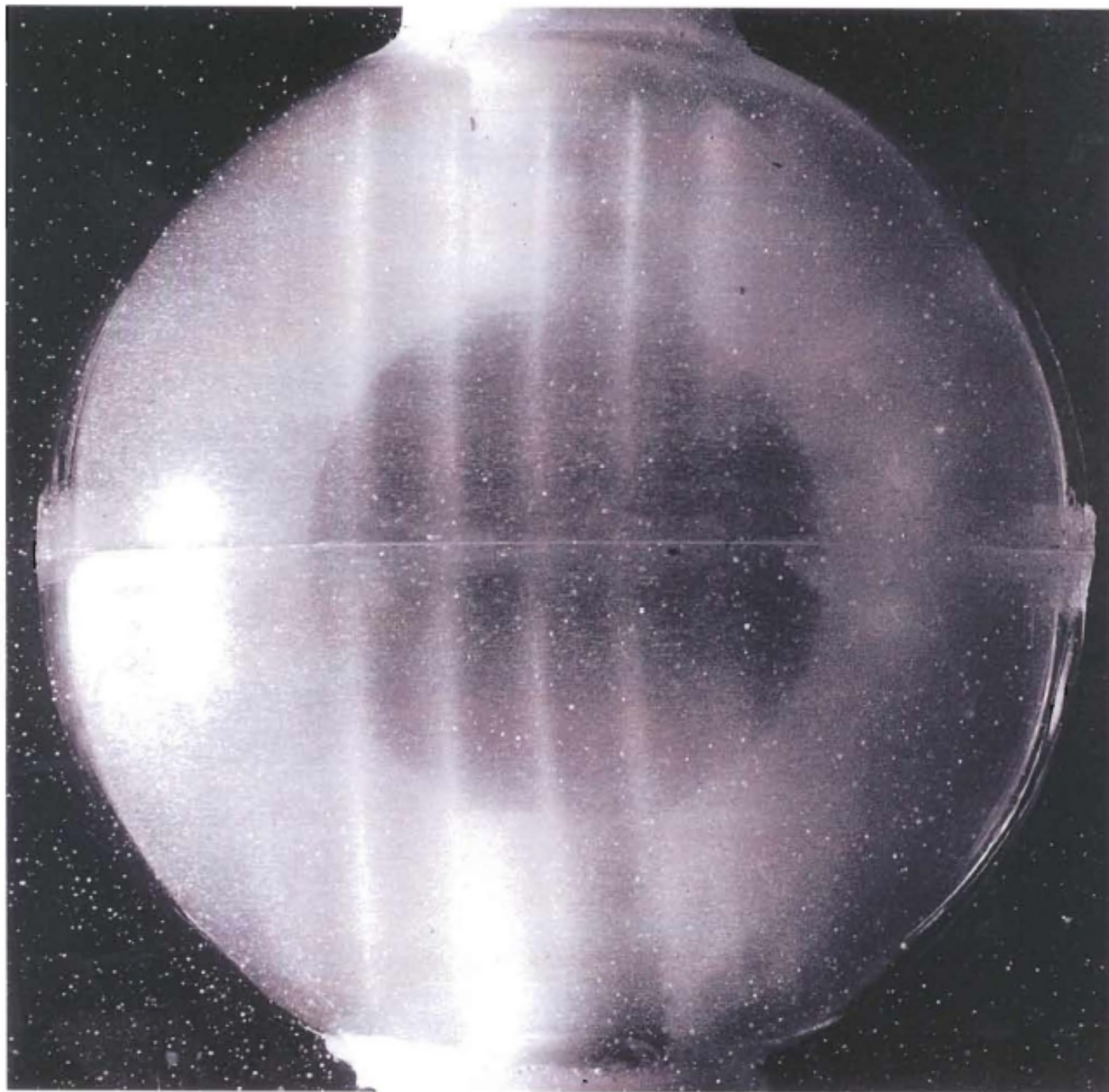
$$\gamma \frac{T_2 - T_1}{D\alpha\nu} \gamma = 3\left(\frac{\alpha_c}{D}\right)^4 = 3\left(\frac{2\pi}{\lambda_c}\right)^4 \quad (\Rightarrow \text{Application to sphere!})$$

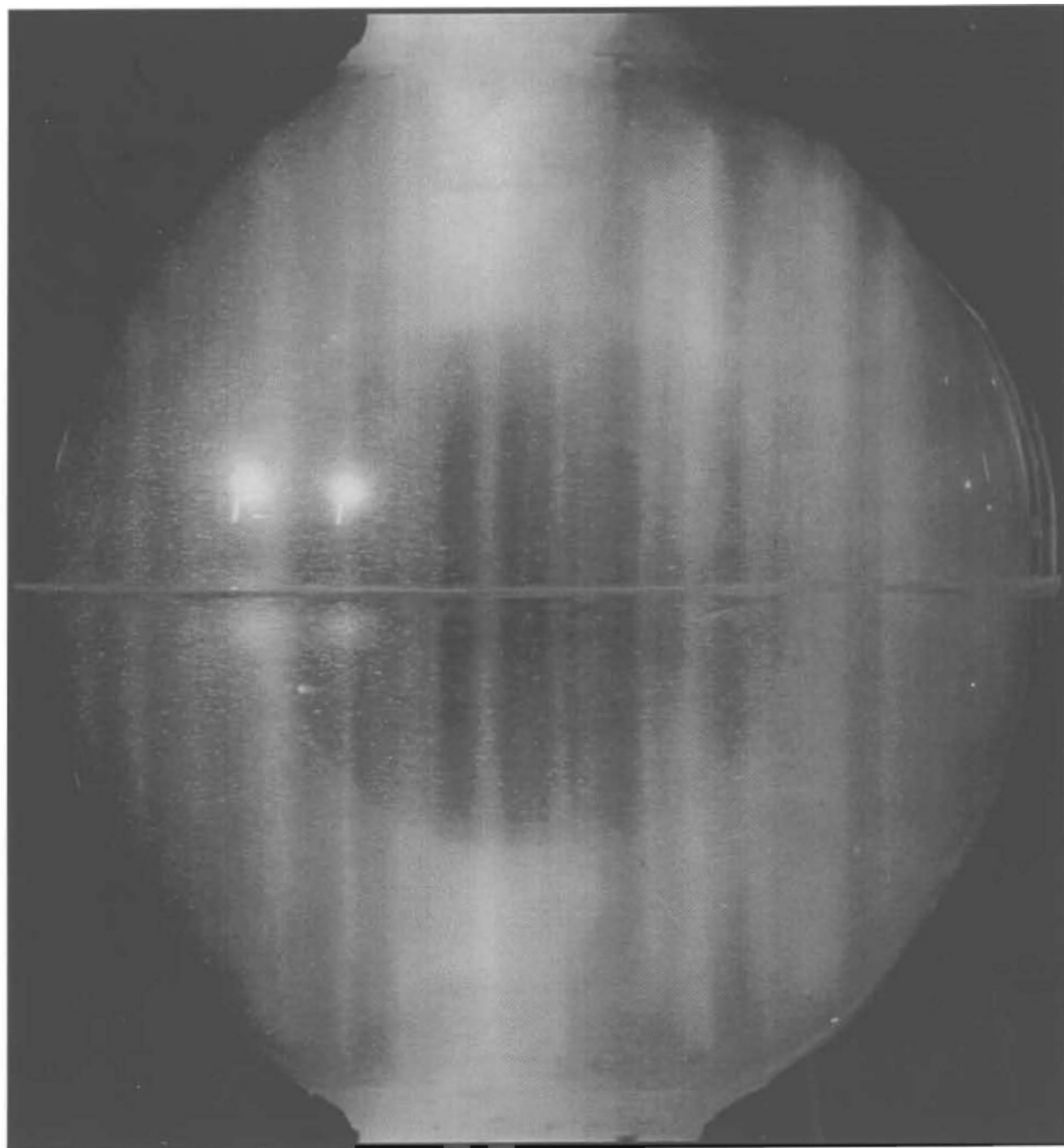
$\Rightarrow$ Modes with $\Psi_0 = \sin m\pi(x+\frac{1}{2}) \exp\{i\alpha y + i\omega t\}$ have nearly the same critical Rayleigh number for $\eta^* \rightarrow \infty$.

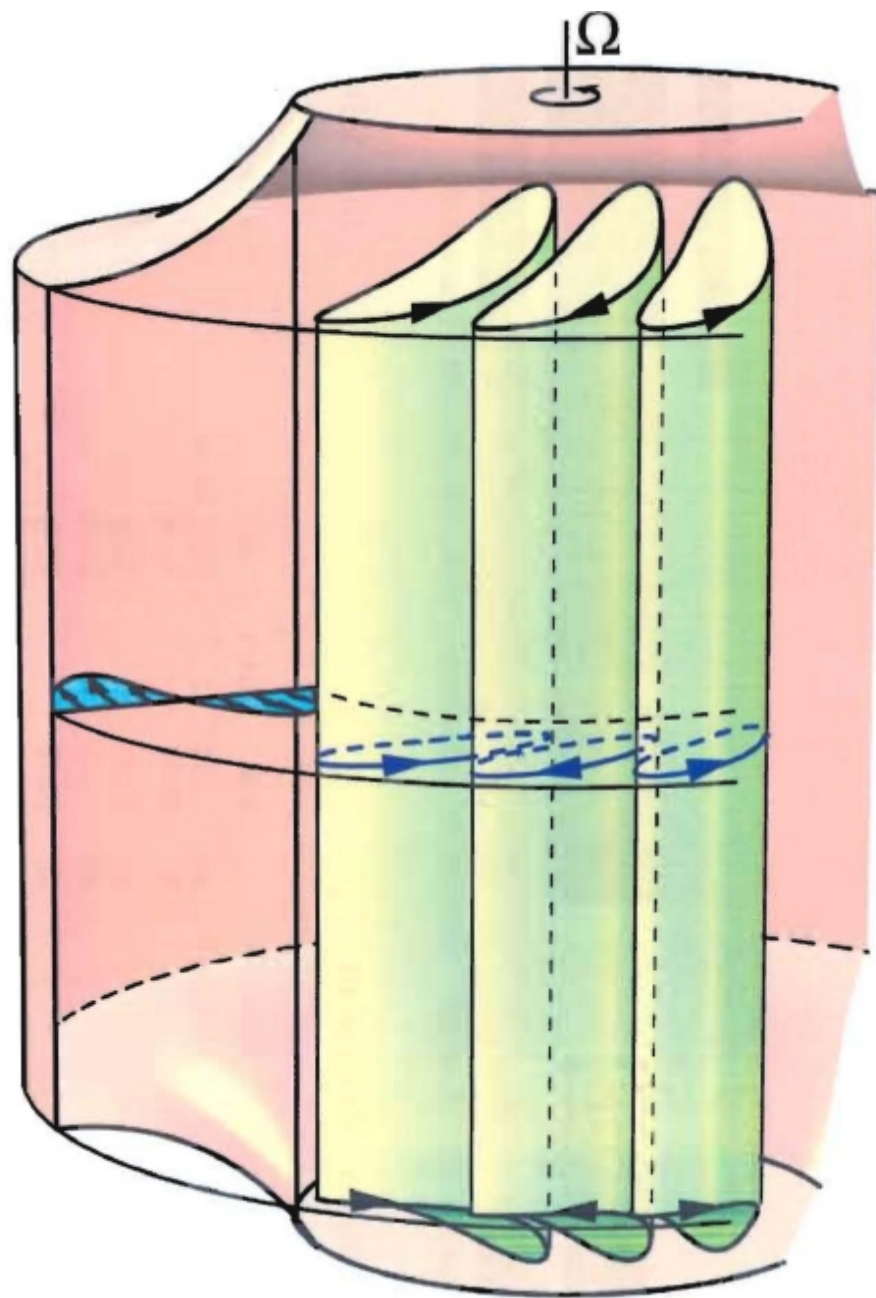
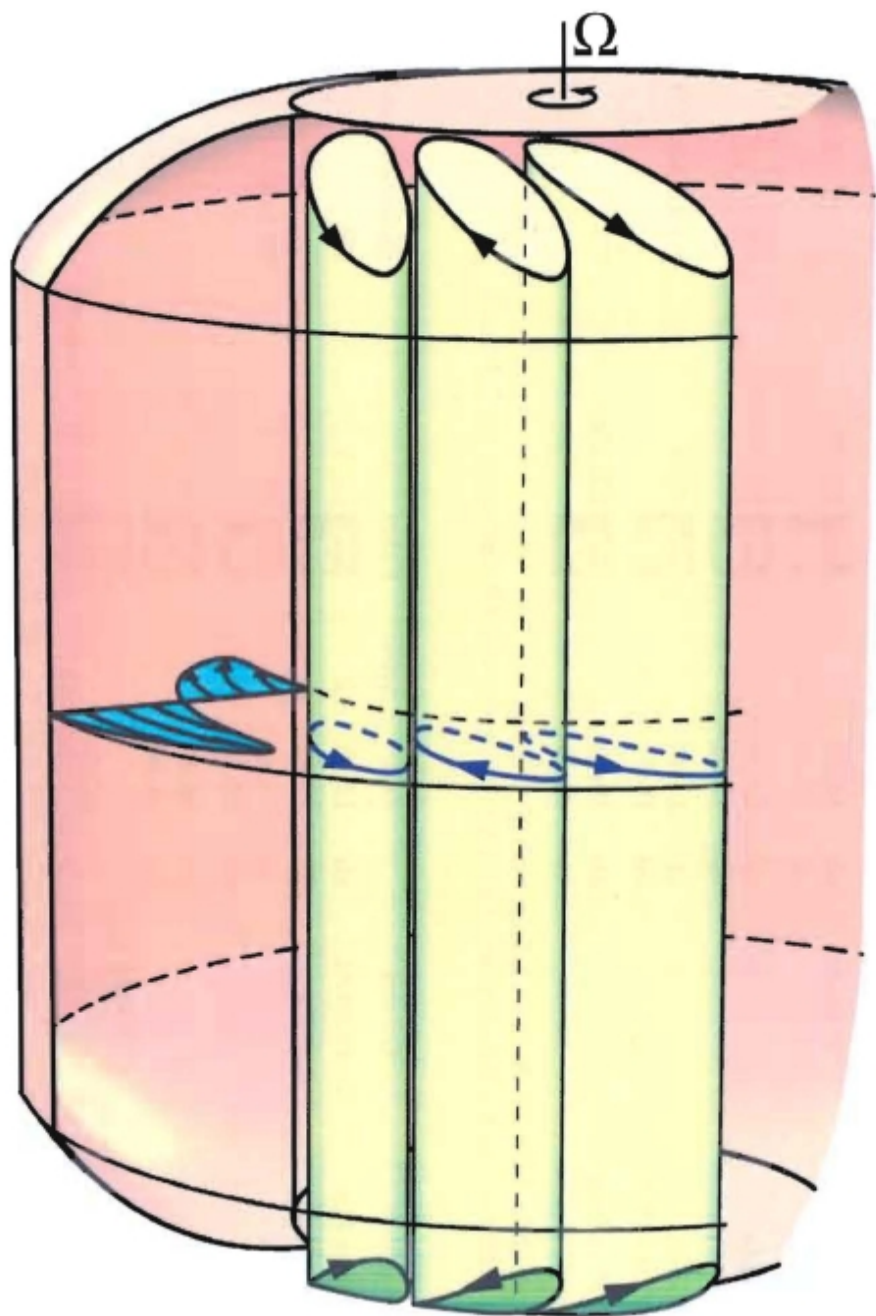


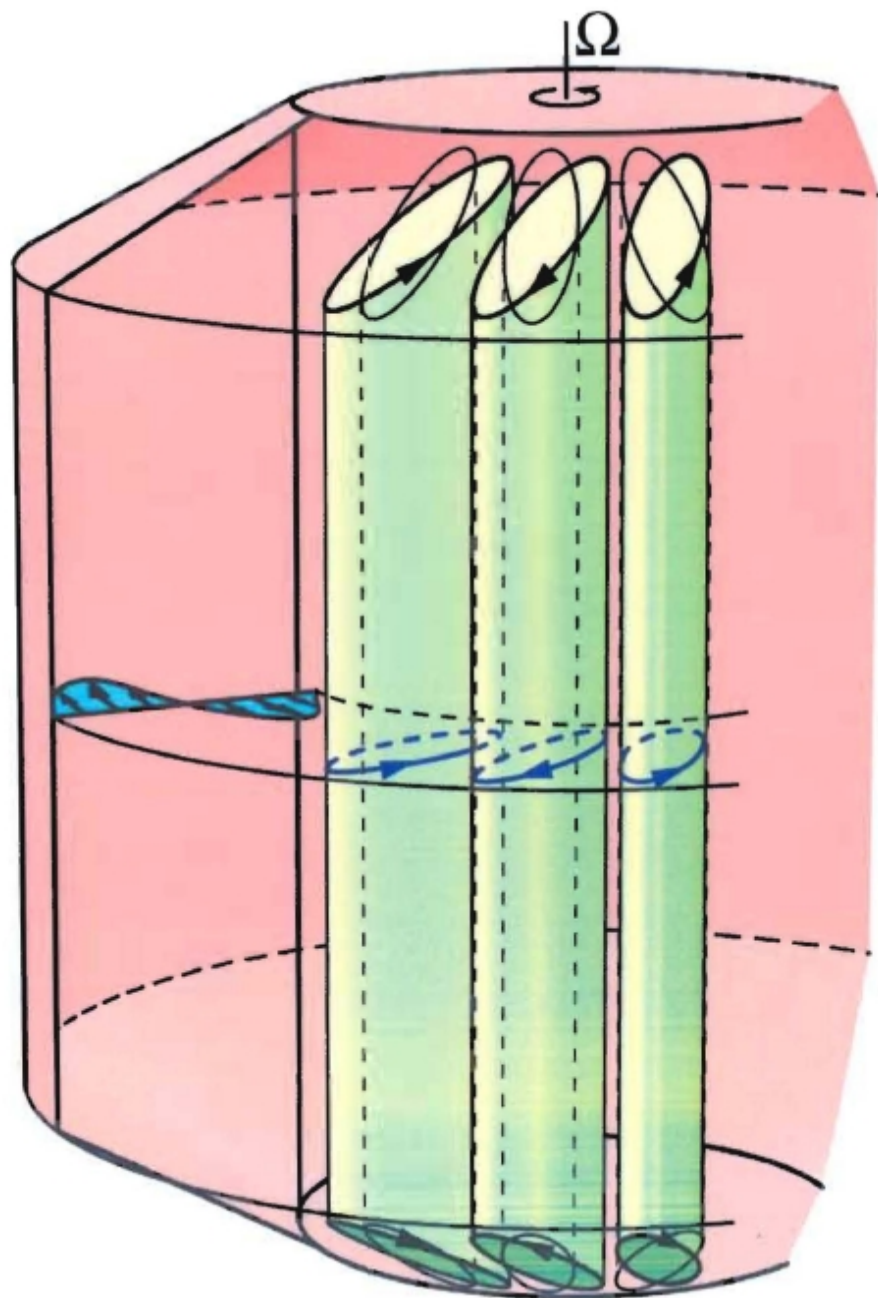
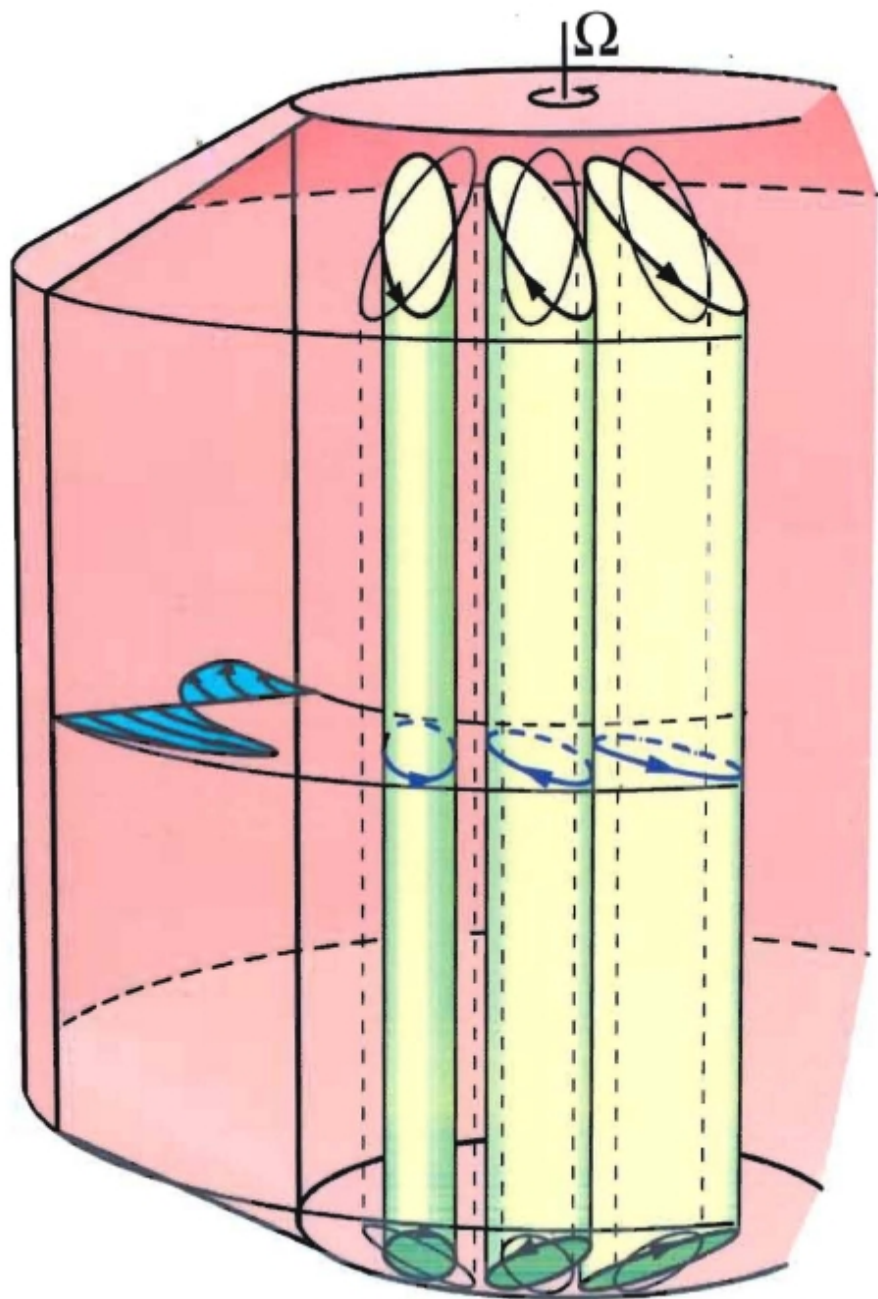
$$R_{mc} = 3\alpha_c^4 \left(1 + \frac{m^2\pi^2}{\alpha_c^2} + \dots \right) \quad \text{for } m=1, 2, 3, \dots$$



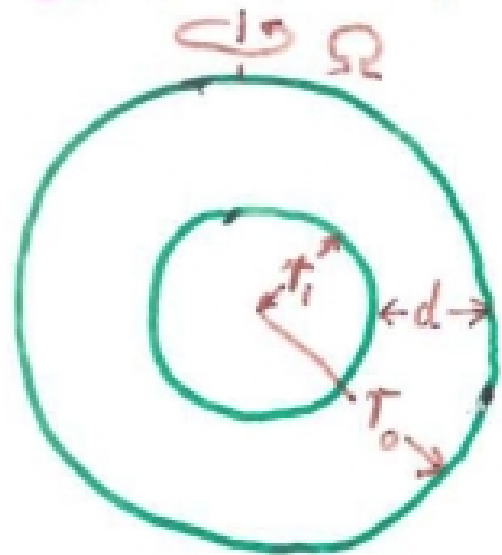








Convection in Rotating Spherical Shells



Basic State: $T_s = T_0 - \frac{1}{2} \hat{r}^2 \beta$

$$\underline{g} = -\gamma \hat{r}$$

$$\rho = \rho_0 (1 - \alpha_c (T - T_0))$$

Scaling: $[d], [\frac{d^2}{\nu}], [\frac{\beta d^2 \nu}{\alpha}]$

Basic Equations:

$$\frac{\partial}{\partial t} \underline{u} + \underline{u} \cdot \nabla \underline{u} + \tau \underline{k} \times \underline{u} = -\nabla \pi + \tau R \underline{\theta} + \nabla^2 \underline{u}$$

$$\nabla \cdot \underline{u} = 0$$

$$\left(\frac{\partial}{\partial t} \underline{\theta} + \underline{u} \cdot \nabla \underline{\theta} \right) P = \tau \cdot \underline{u} + \nabla^2 \underline{\theta}$$

Representation: $\underline{u} = \nabla \times (\nabla \times \underline{r} \Phi) + \nabla \Psi \times \underline{r}$

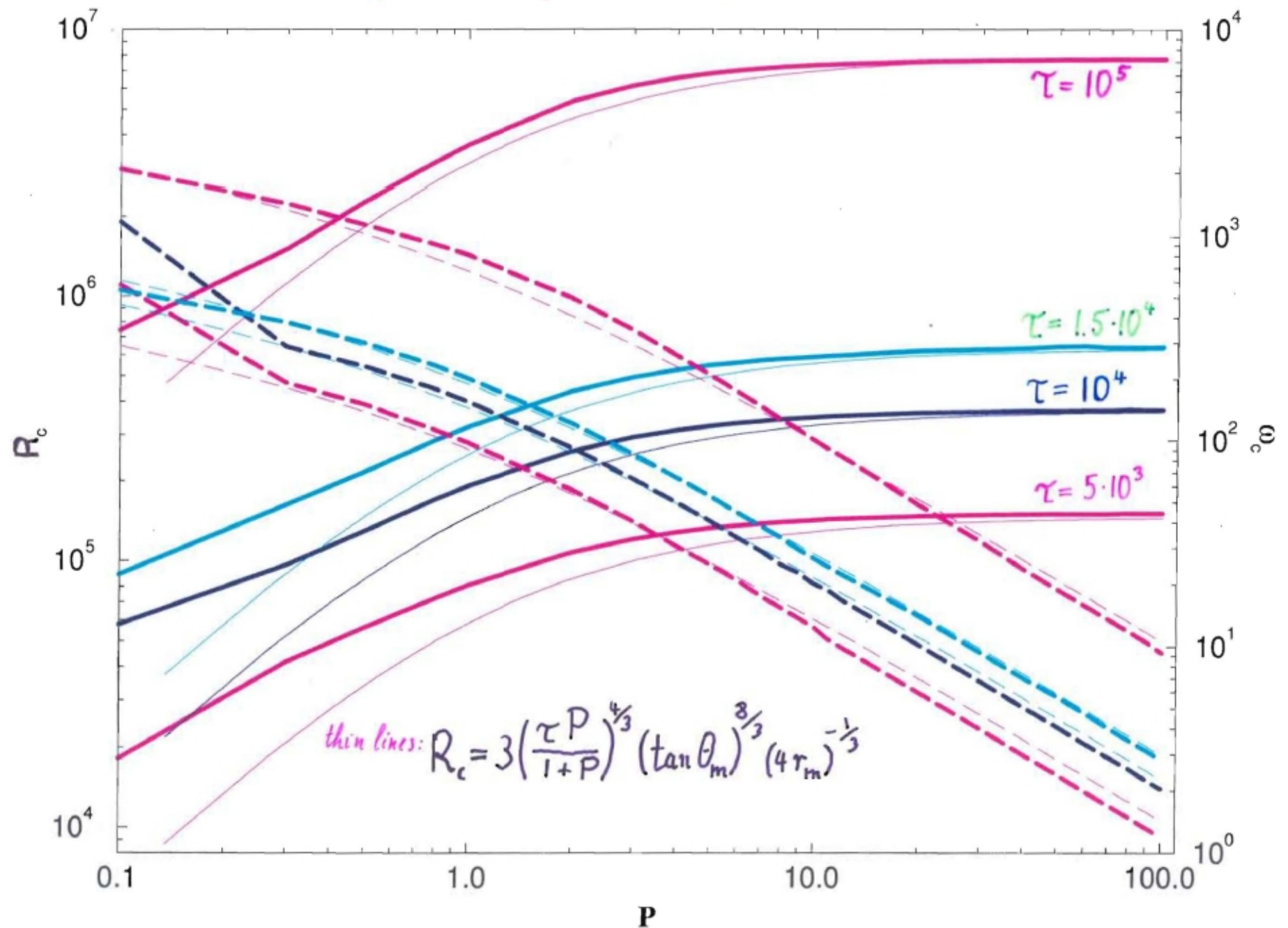
$$\left. \begin{aligned} [(\nabla^2 - \frac{\partial}{\partial t})L_2 + \tau \frac{\partial}{\partial \varphi}] \nabla^2 \Phi + \tau Q \Psi - R L_2 \Theta &= -\underline{r} \cdot \nabla \times (\nabla \times (\underline{u} \cdot \nabla \underline{u})) \\ [(\nabla^2 - \frac{\partial}{\partial t})L_2 + \tau \frac{\partial}{\partial \varphi}] \Psi - \tau Q \Phi &= \underline{r} \cdot \nabla \times (\underline{u} \cdot \nabla \underline{u}) \\ (\nabla^2 - P \frac{\partial}{\partial t}) \Theta + L_2 \Phi &= P (\underline{u} \cdot \nabla \Theta) \end{aligned} \right\} \otimes$$

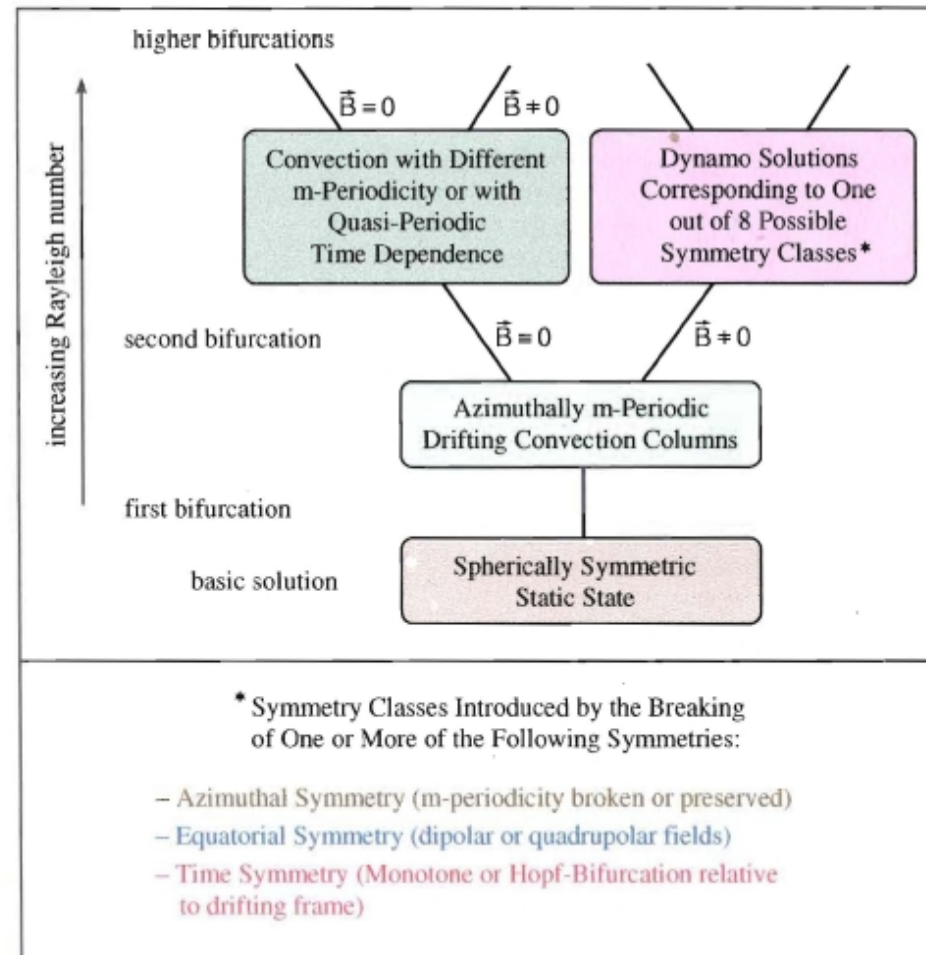
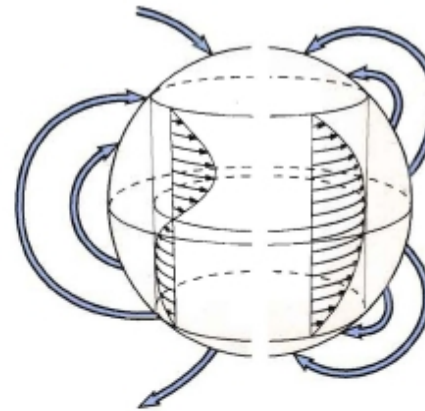
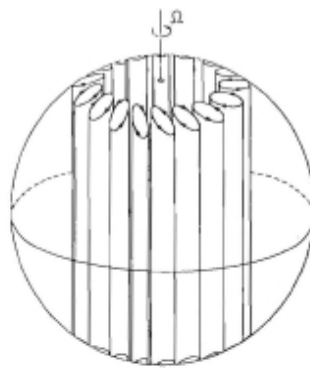
$$L_2 \equiv r \frac{\partial^2}{\partial r^2} r - \hat{r} \nabla^2, \quad Q \equiv \underline{k} \cdot \nabla - \frac{1}{2} (L_2 \underline{k} \cdot \nabla + \underline{k} \cdot \nabla L_2)$$

Stress-free, isothermal boundaries: $\Phi = \frac{\partial^2}{\partial r^2} \Phi = \frac{\partial}{\partial r} \frac{\Psi}{r} = \Theta = 0$ at $r = r_i, 0$

Dimensionless parameters: $R = \frac{\alpha_i \gamma \beta d^6}{\nu \kappa}$, $\tau = \frac{2 \Omega d^2}{\nu}$, $P = \frac{\nu}{\kappa}$

Rayleigh number R_c and frequency ω_c for onset of convection in rotating spherical fluid shells with $\eta=0.4$

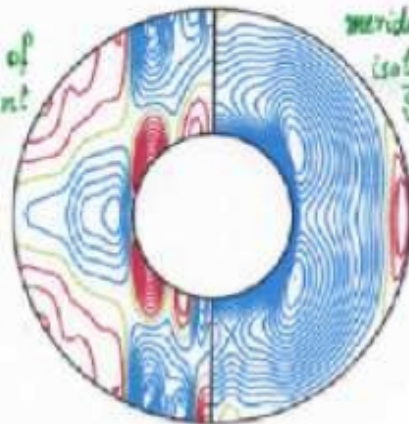




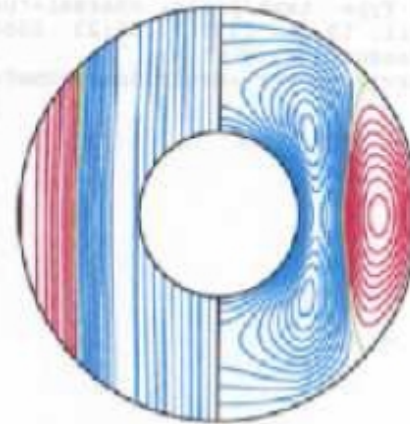
Thermal Convection in Spherical Fluid Shells

$$P=15, \tau=5 \cdot 10^3, R=8 \cdot 10^5 \quad P=1, \tau=10^4, R=4 \cdot 10^5$$

lines of constant u_φ



meridional isotherms $\bar{T} = \text{const}$

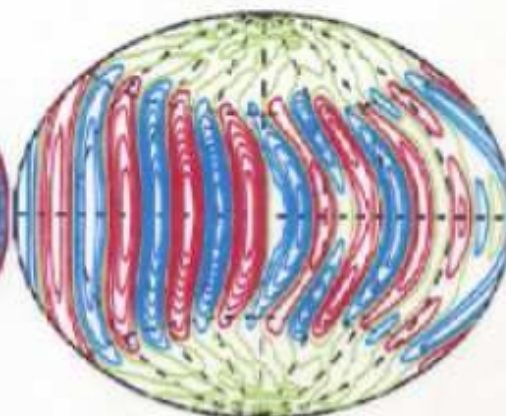
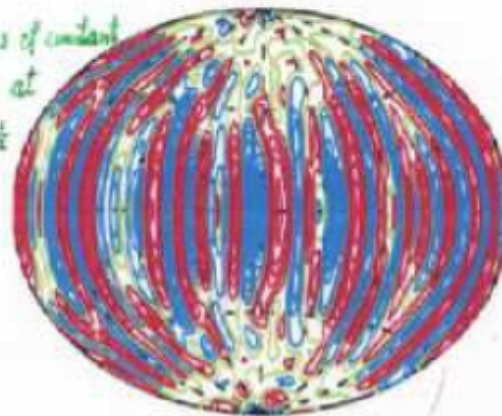


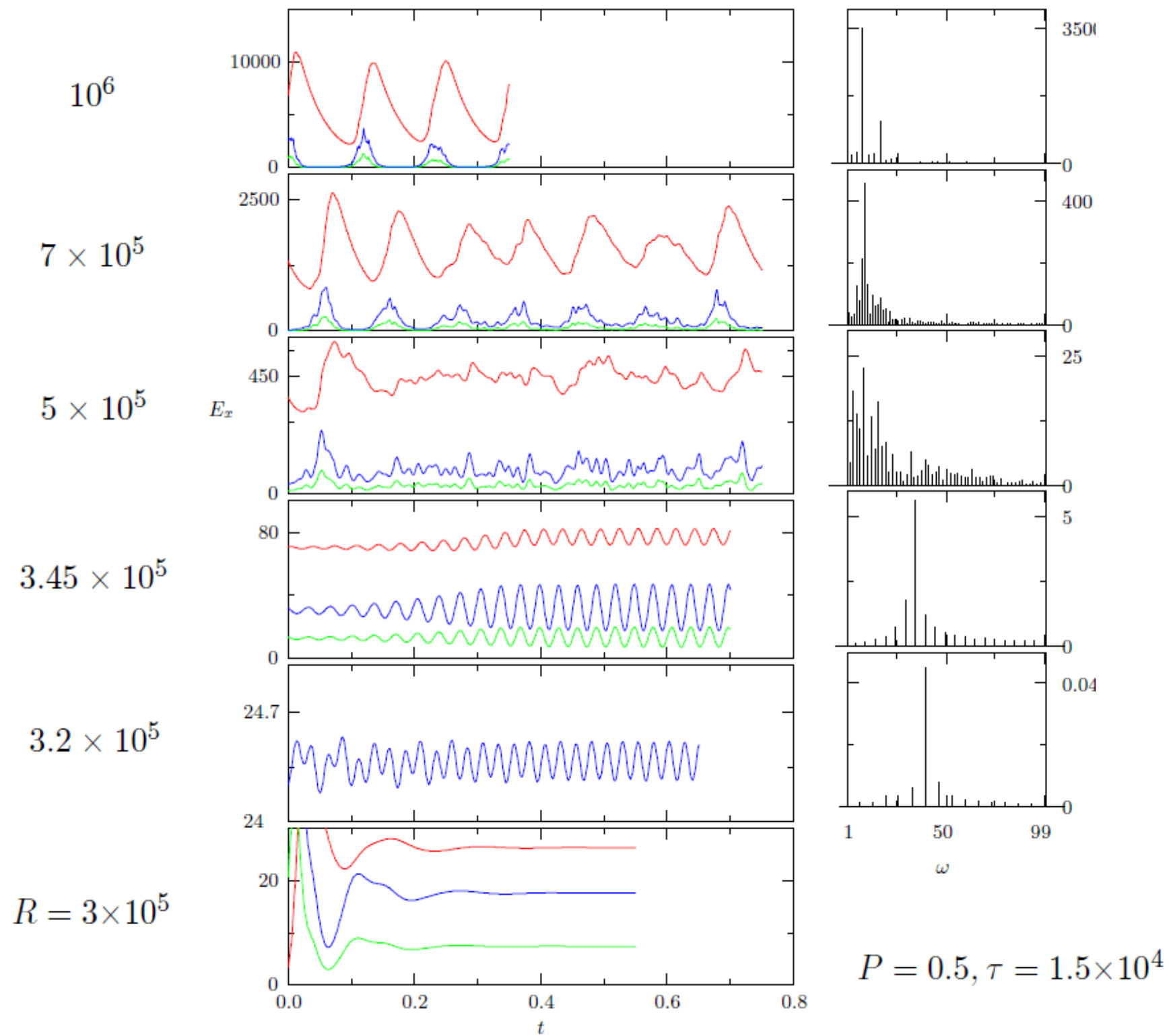
$\tau \frac{\partial v}{\partial \varphi} = \text{const}$

in equatorial plane

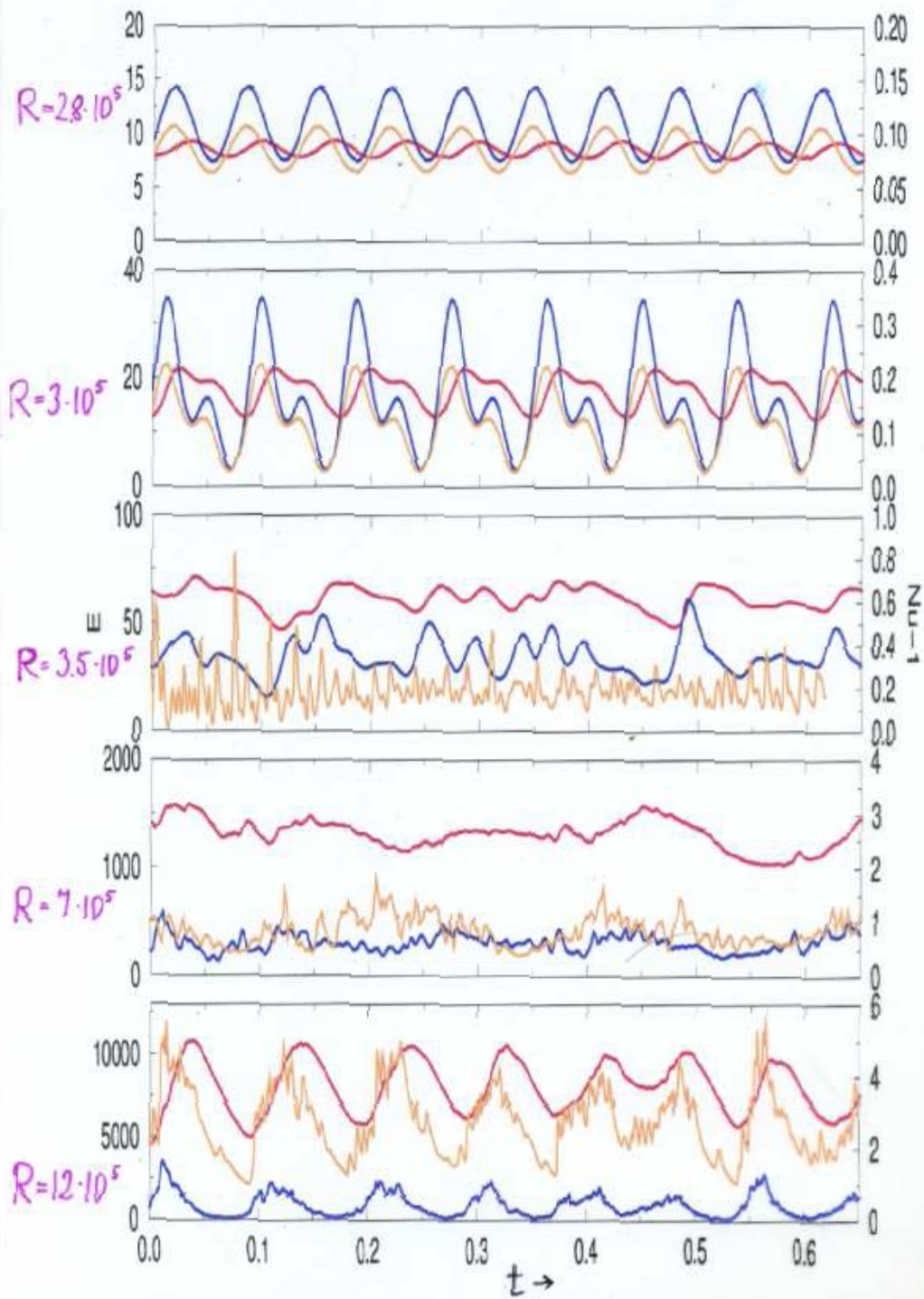


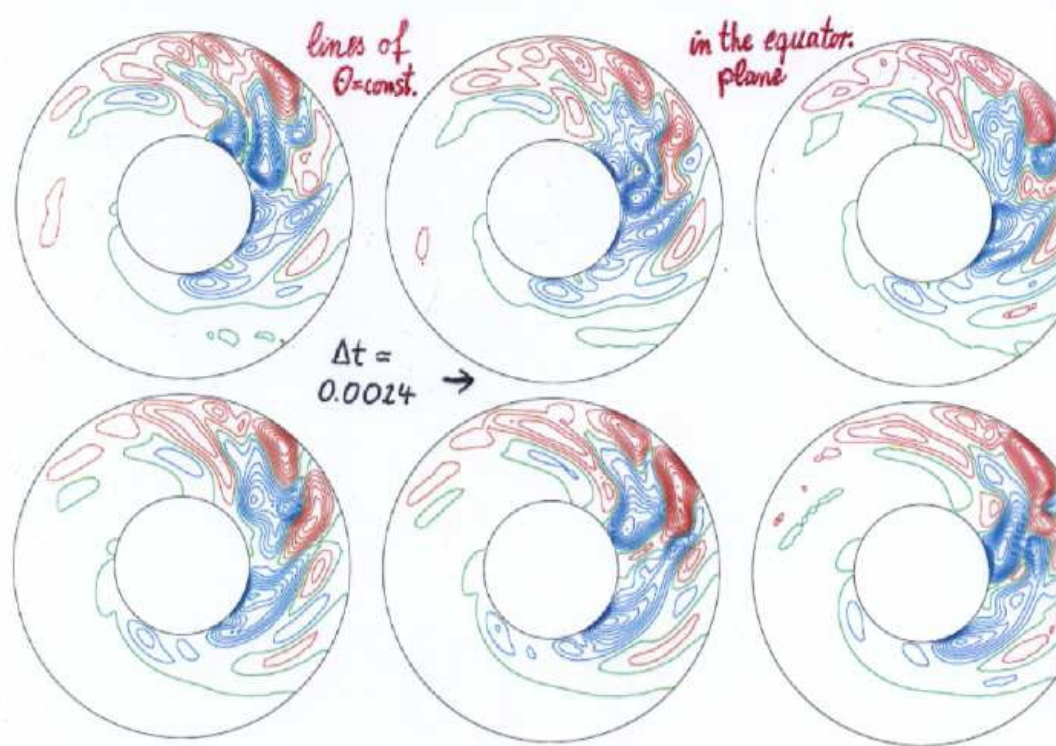
lines of constant u_r at $r/r_1 = \frac{1}{2}$



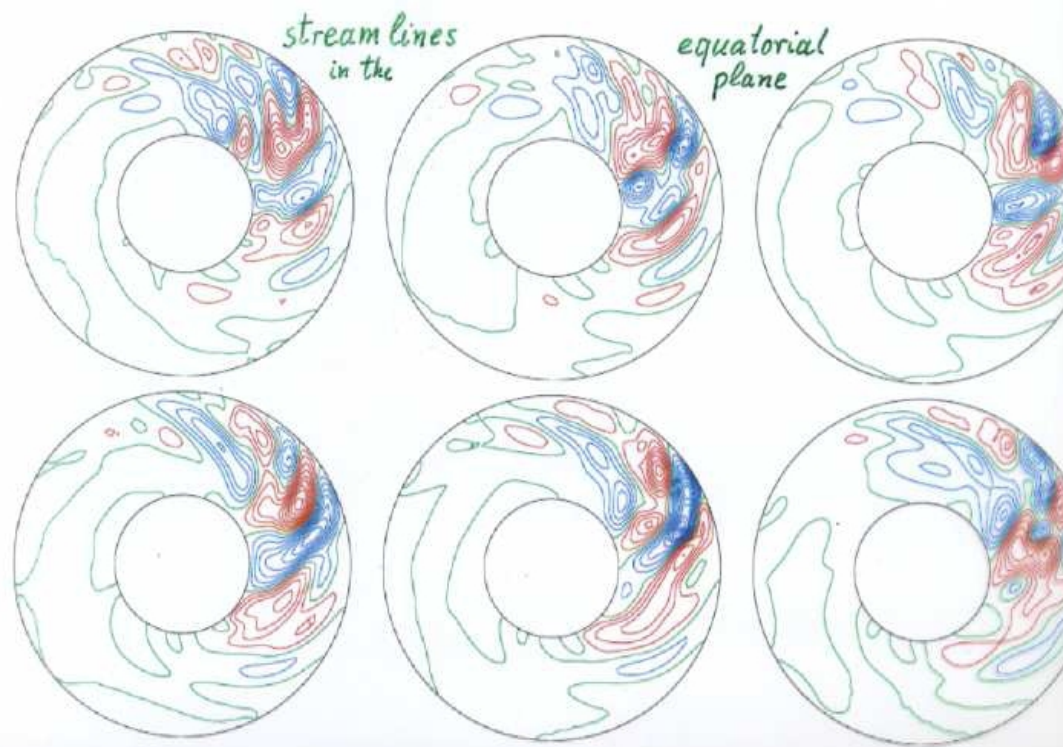


Time Series for Convection with $\tau=10^4$, $P=1$, $\eta=0.4$

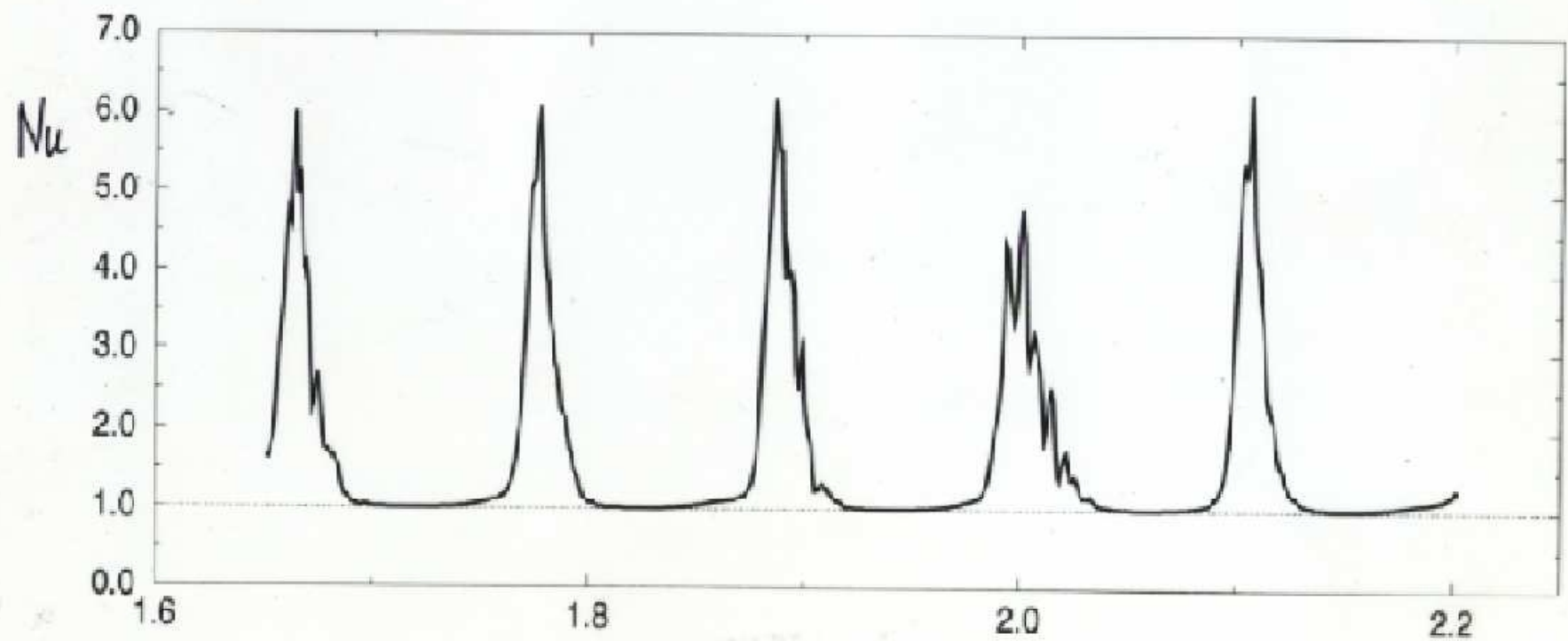
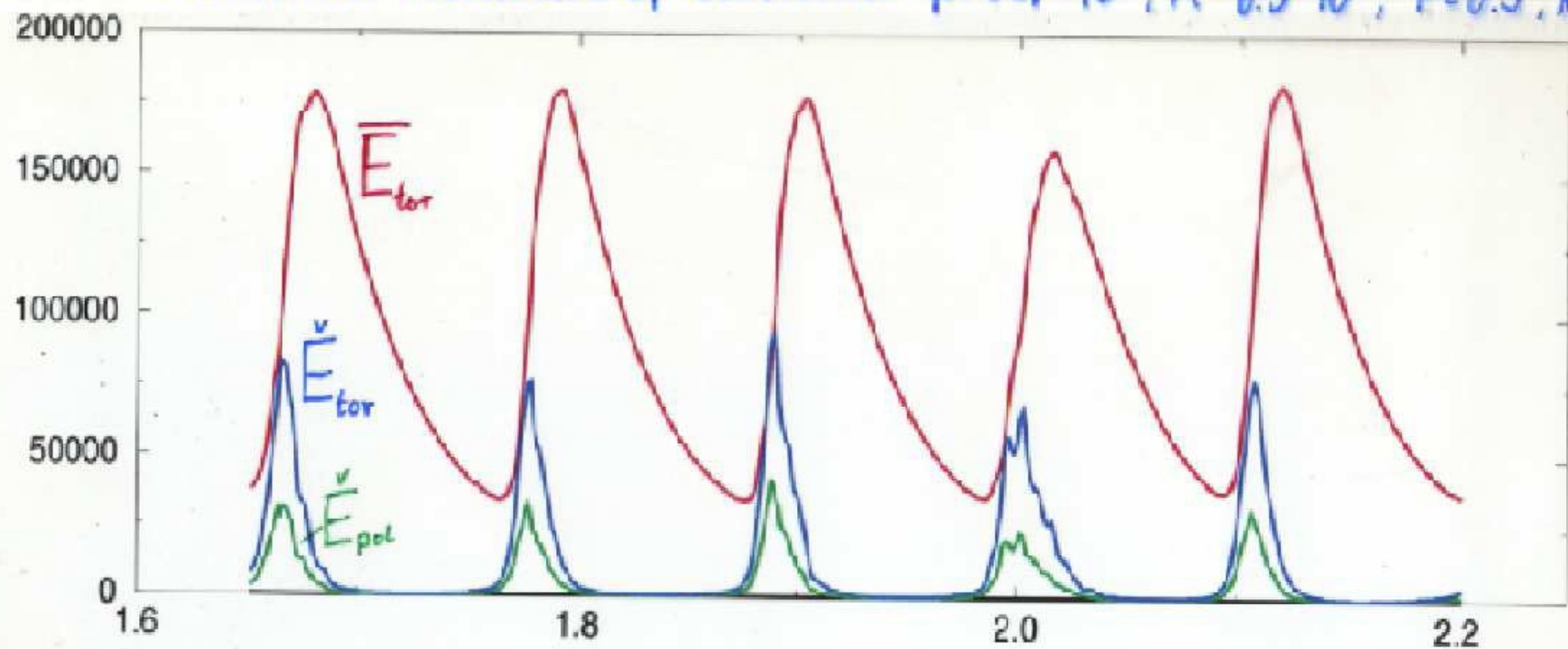


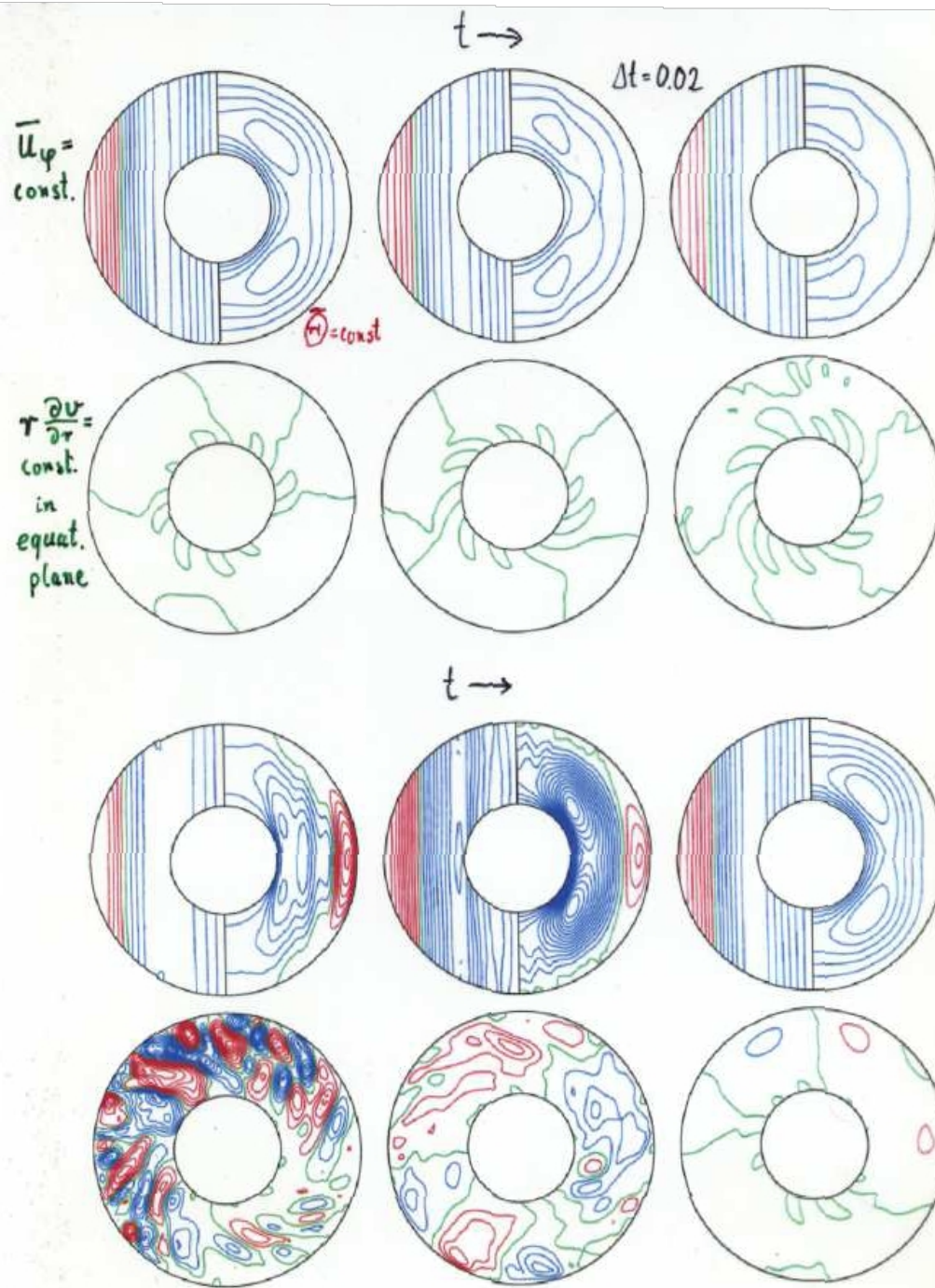


Localized convection for $\tau = 10^4$, $R = 7.5 \cdot 10^5$, $P = 1$



Relaxation Oscillation of Convection for $\tau_s T = 10^8$, $R = 6.5 \cdot 10^5$, $P = 0.5$, $m_s = 1$



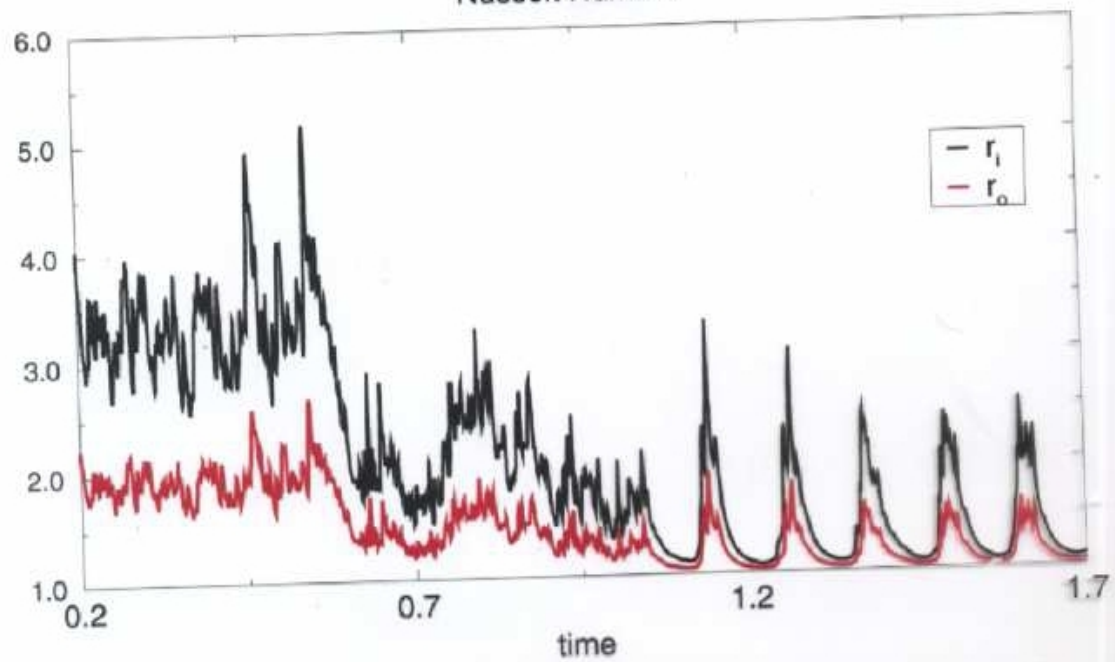
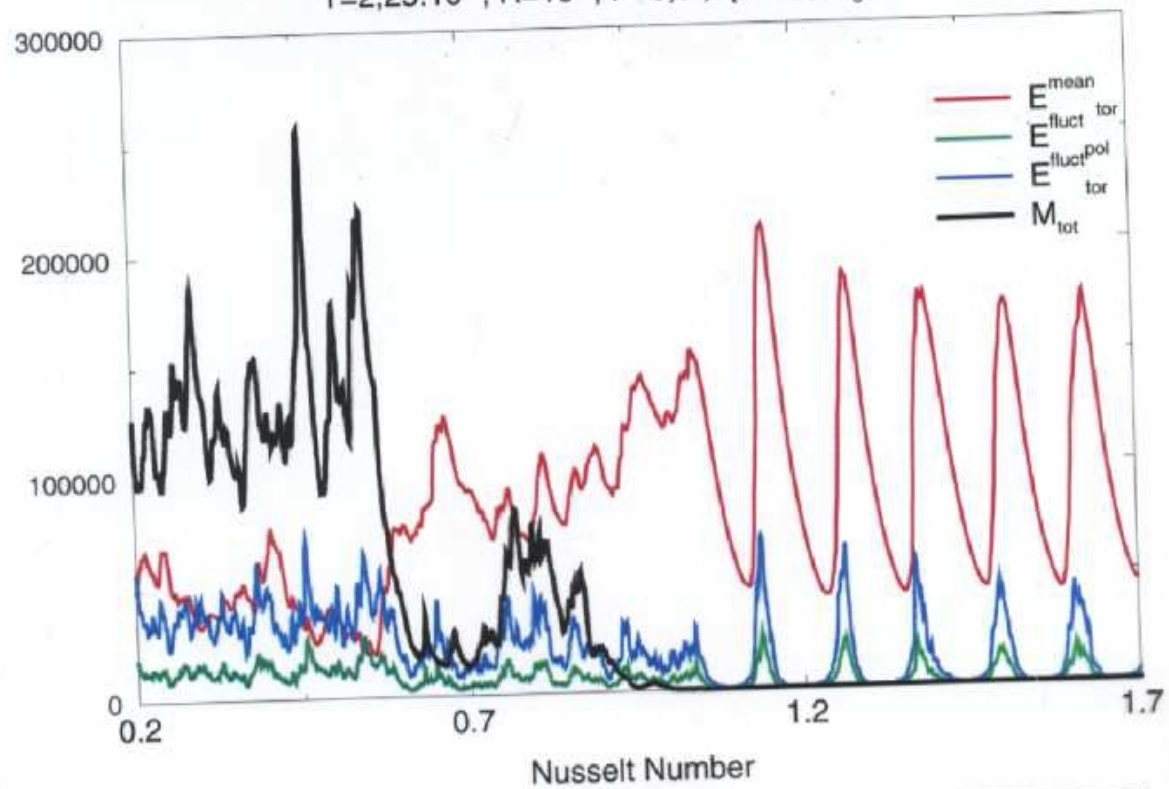


Relaxation Oscillation of Convection

$$T = 2.25 \cdot 10^8, R = 1.2 \cdot 10^6, P = 0.5$$

Resumption of Relaxation Oscillation after Decay of Magnetic Field

$T=2,25 \cdot 10^6$; $R=10^6$; $P=0,5$; $\eta=0.4$; $m_0=1$



Generation of Magnetic Fields by Convection in Rotating Spherical Fluid Shells

Basic Equations: Equation of Motion:

$$\left(\frac{\partial}{\partial t} + \underline{u} \cdot \nabla\right) \underline{u} + 2 \underline{\Omega} \times \underline{u} = -\nabla \pi - \alpha_t \Theta \underline{g} + \nu \nabla^2 \underline{u} + \frac{1}{\rho_0} (\nabla \times \underline{B}) \times \underline{B}$$

$$\nabla \cdot \underline{u} = 0 \Rightarrow \underline{u} = \nabla \times (\nabla \times \underline{r} v) + \nabla \times \underline{r} w$$

Energy equation:

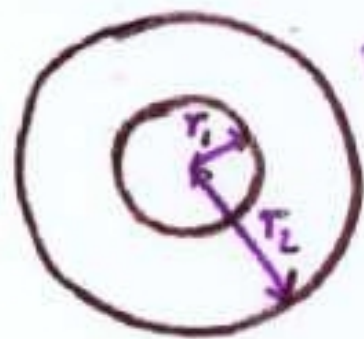
$$\left(\frac{\partial}{\partial t} + \underline{u} \cdot \nabla\right) \Theta + \underline{u} \cdot \nabla (T_s - T_{is}) = \kappa \nabla^2 \Theta$$

Induction equation:

$$\left(\frac{\partial}{\partial t} + \underline{u} \cdot \nabla\right) \underline{B} - \lambda \nabla^2 \underline{B} = \underline{B} \cdot \nabla \underline{u}$$

$$\nabla \cdot \underline{B} = 0 \Rightarrow \underline{B} = \nabla \times (\nabla \times \underline{r} h) + \nabla \times \underline{r} g$$

Basic State : $T_s - T_{is} = T_0 - \frac{1}{2} \beta r^2$



$$\eta \equiv \frac{r_i}{r_o} \quad g = -\hat{r} \cdot \mathbf{r} / r_o$$

Dimensionless Description with
 $[r_o - r_i], [\frac{(r_o - r_i)^2}{\nu}], [\beta r_o^2], [\sqrt{g \mu} \frac{r_o - r_i}{\nu}]$

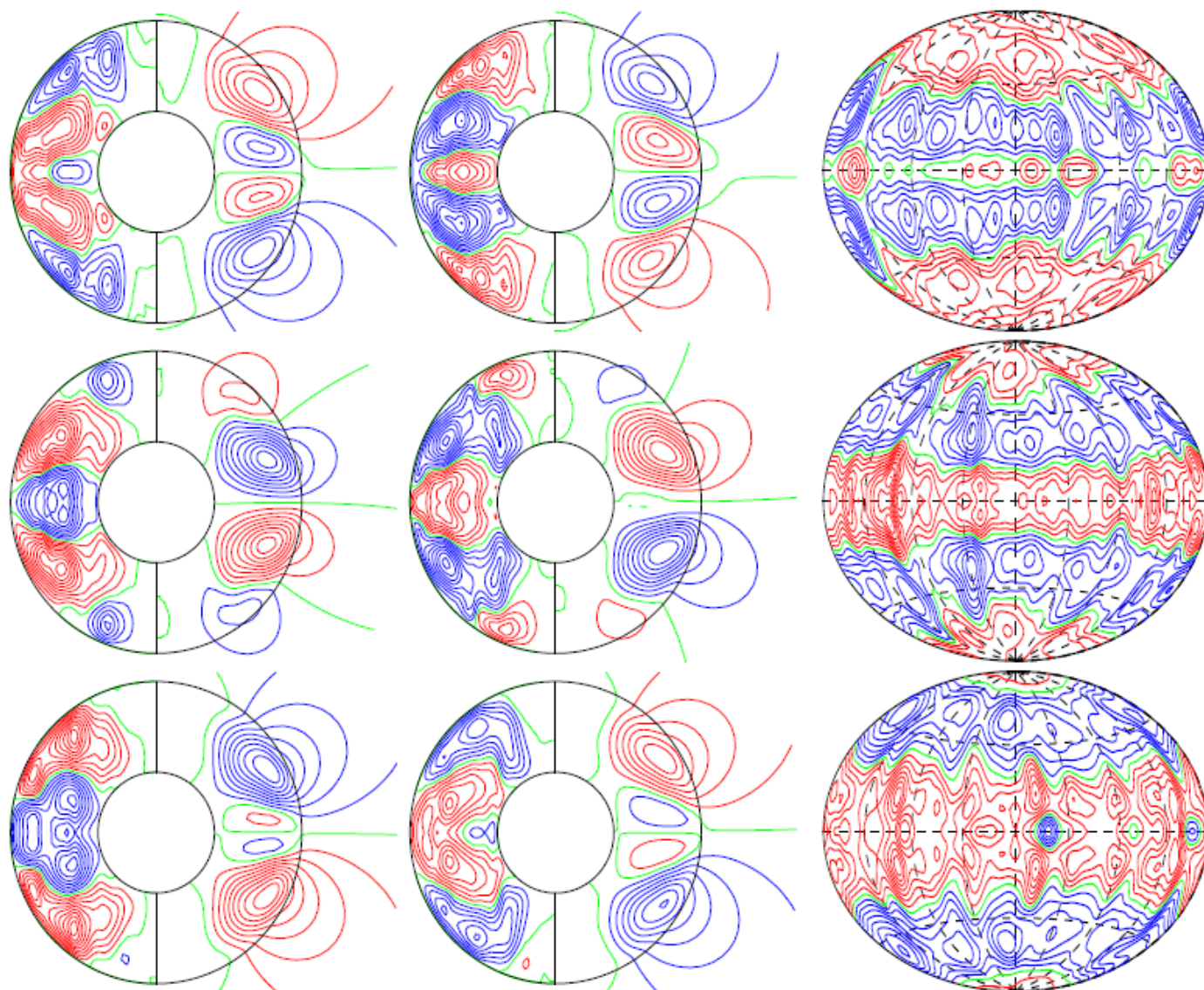
Boundary Conditions : $\frac{\partial \psi}{\partial r} = v = \frac{\partial}{\partial r} \frac{w}{r} = \Omega = g = 0$ at $r = r_i$

$$h = \sum_l h_l Y_l(\vartheta, \varphi) : \quad \frac{\partial}{\partial r} h_l = \begin{cases} -(l+1)/r_o \cdot h_l & \text{at } r = r_o \equiv \frac{1}{1-\eta} \\ l/r_i \cdot h_l & \text{at } r = r_i \equiv \frac{\eta}{1-\eta} \end{cases}$$

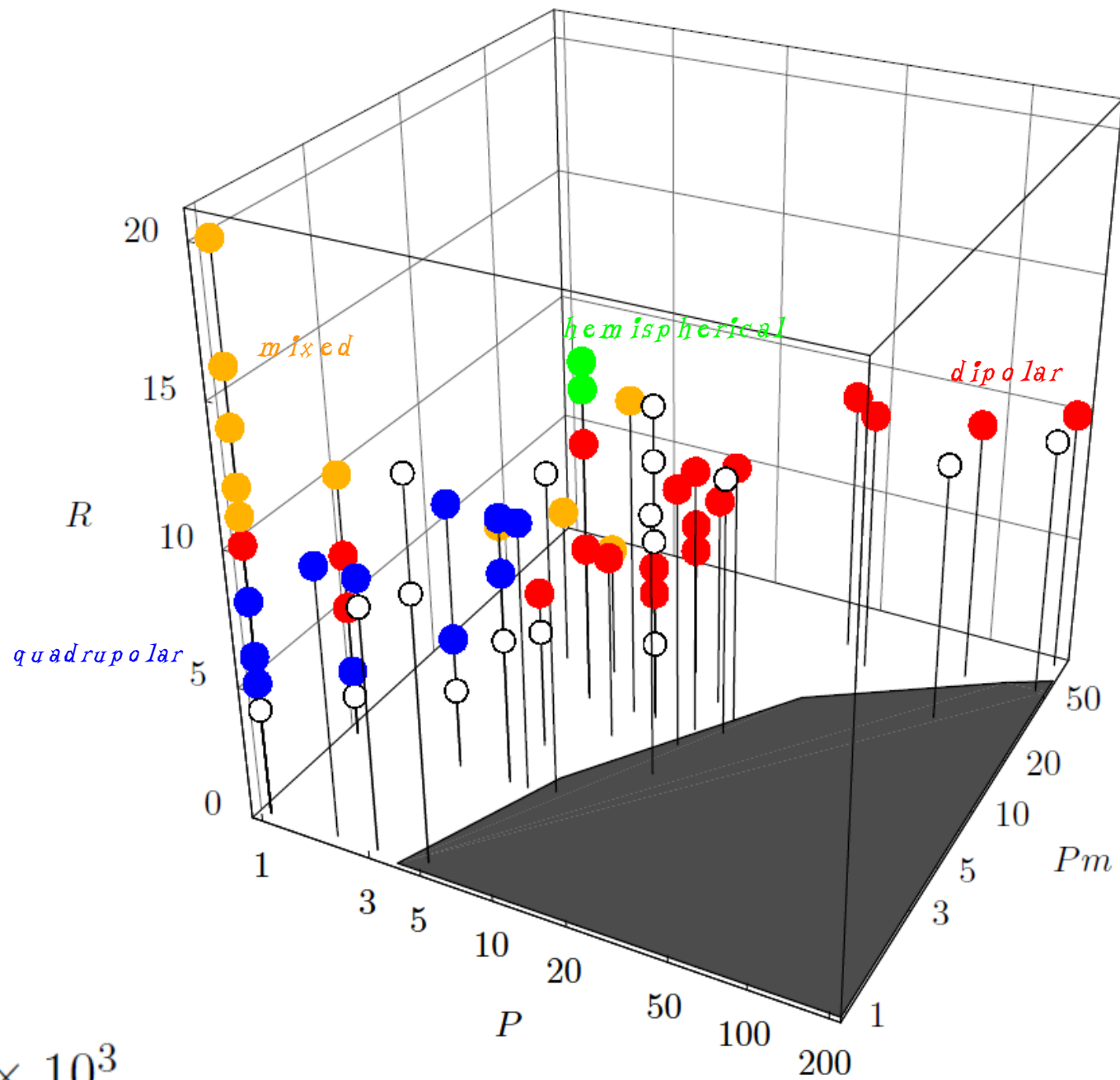
Dimensionless Parameters : $R \equiv \frac{\alpha_i \hat{r} \beta (r_o - r_i)^5}{\nu \chi}$, $\tau \equiv \frac{2 \Omega (r_o - r_i)^2}{\nu} \equiv T$
 $P \equiv \frac{\nu}{\chi}$, $P_m \equiv \frac{\nu}{\lambda}$

A period of quadrupolar dynamo oscillations

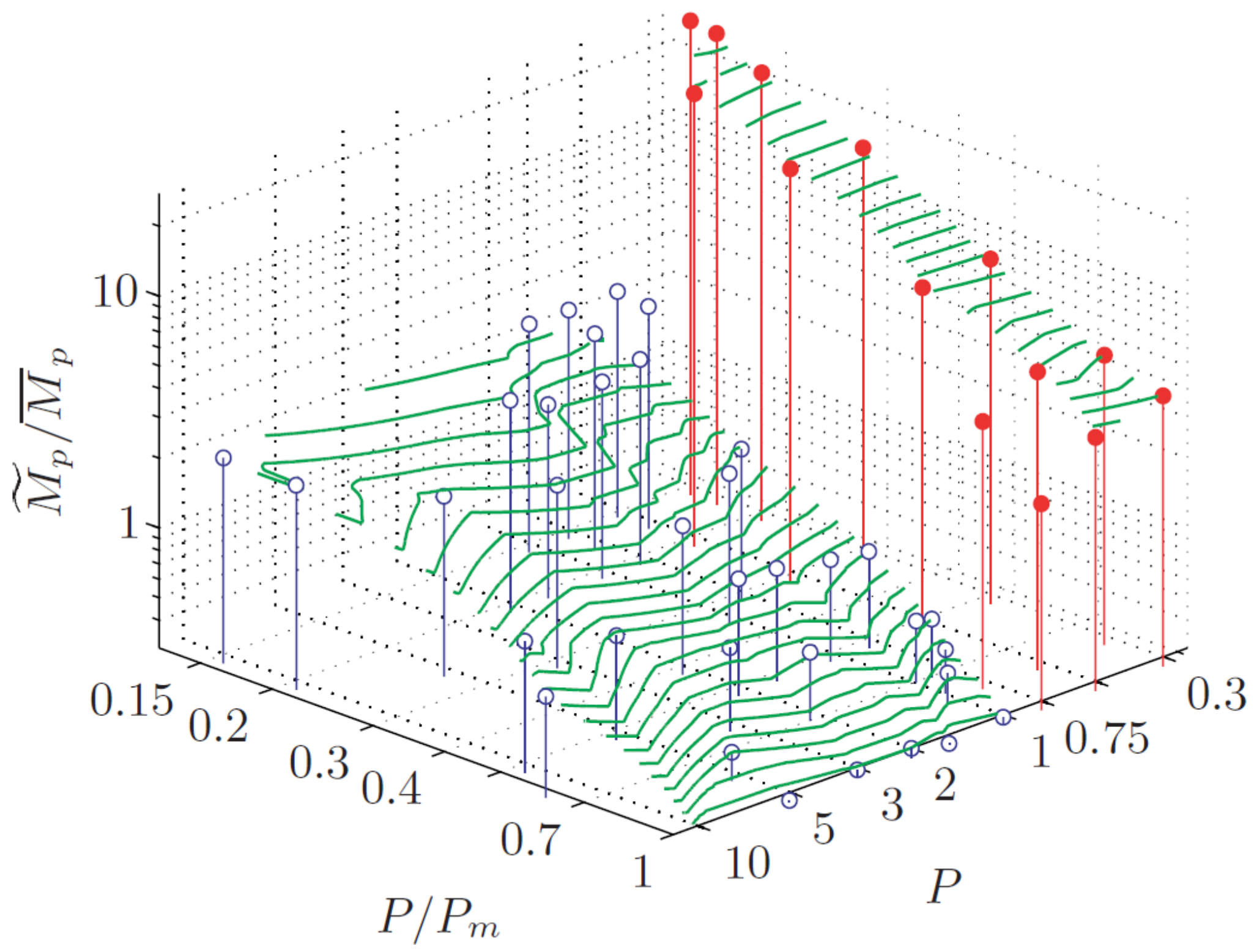
$$P = 5, \tau = 5 \times 10^3, R = 8 \cdot 10^5, P_m = 3.$$



Where do convection-driven dynamos exist?



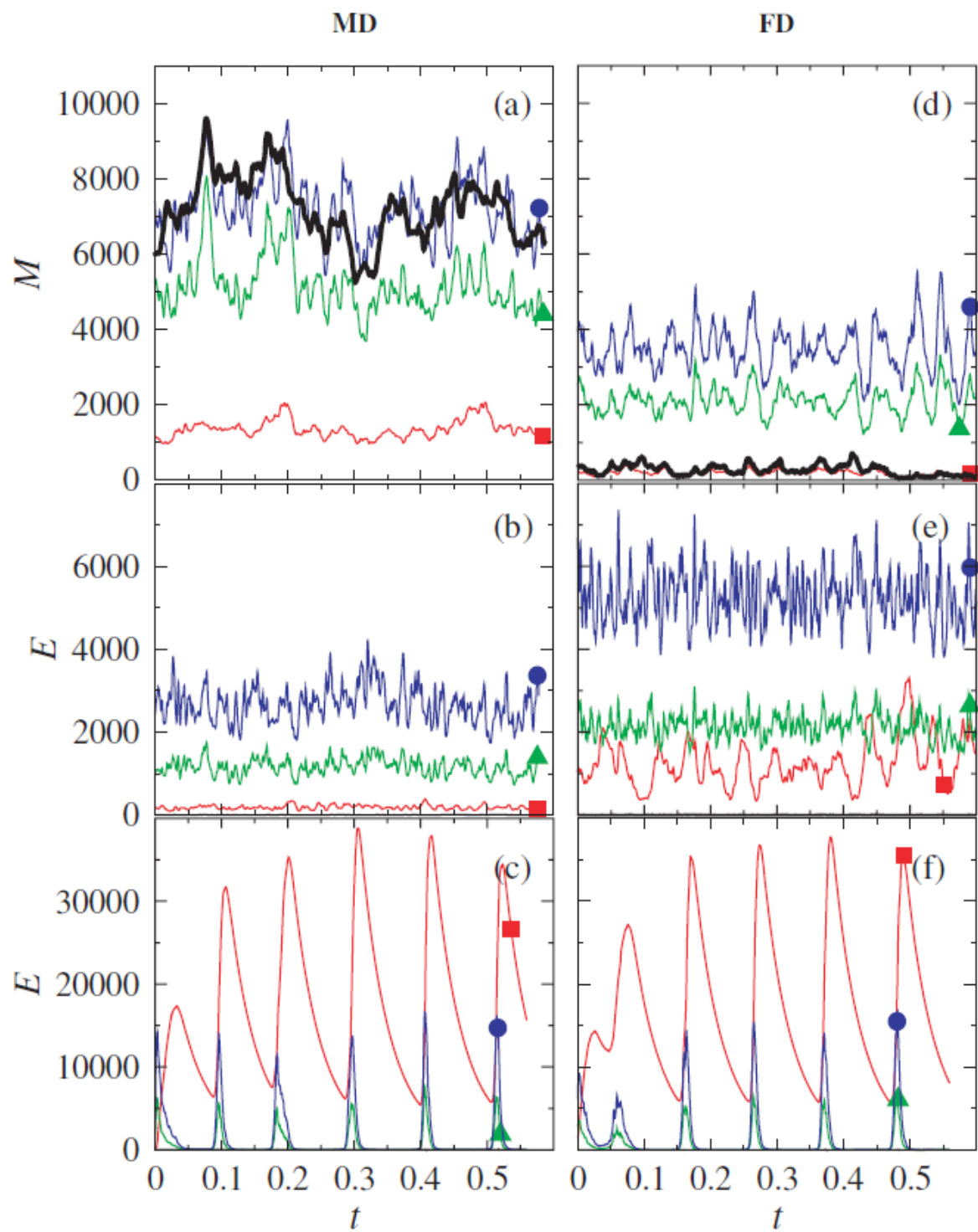
$$\tau = 5 \times 10^3$$



$$R = 3.5 \times 10^6$$

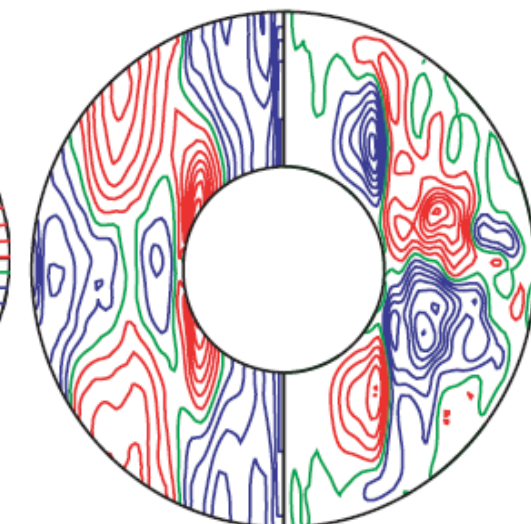
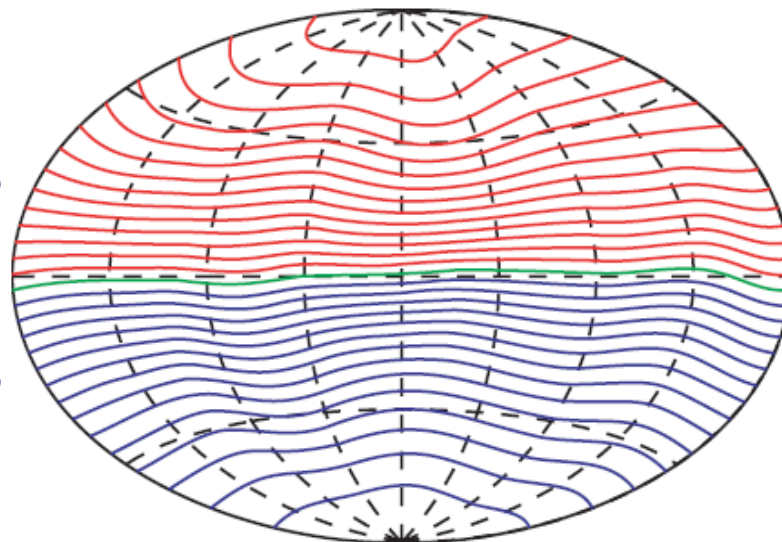
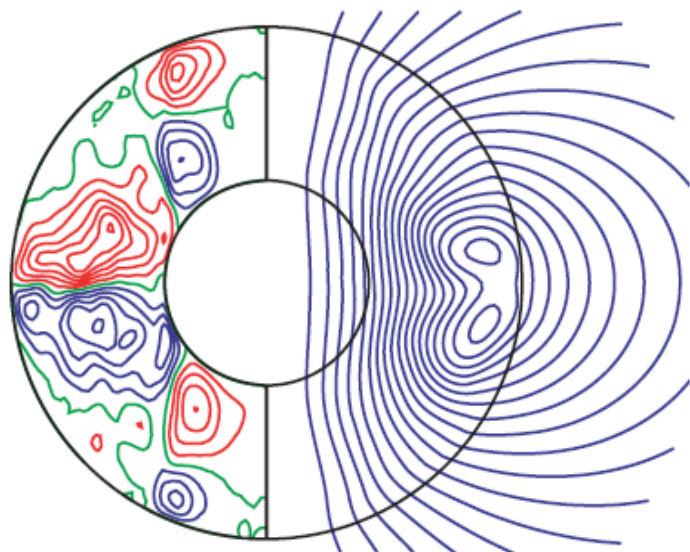
$$P_m = 1.5$$

$$\tau = 3 \times 10^4, P = 0.75$$

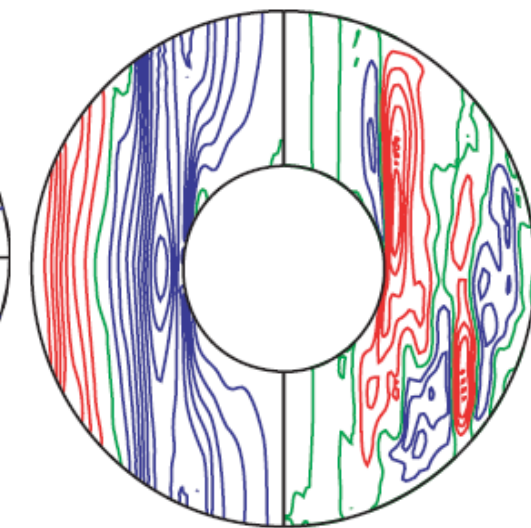
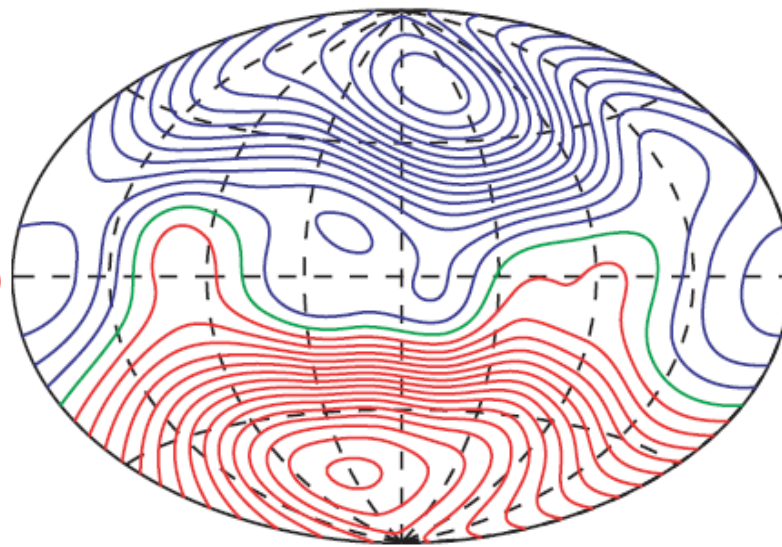
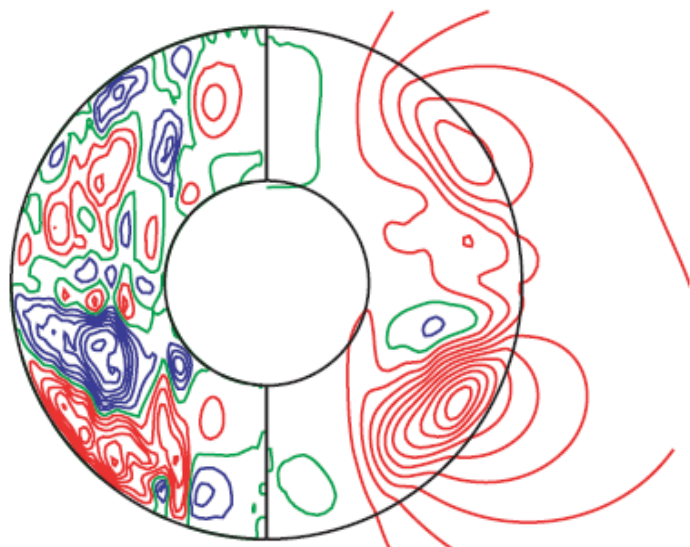


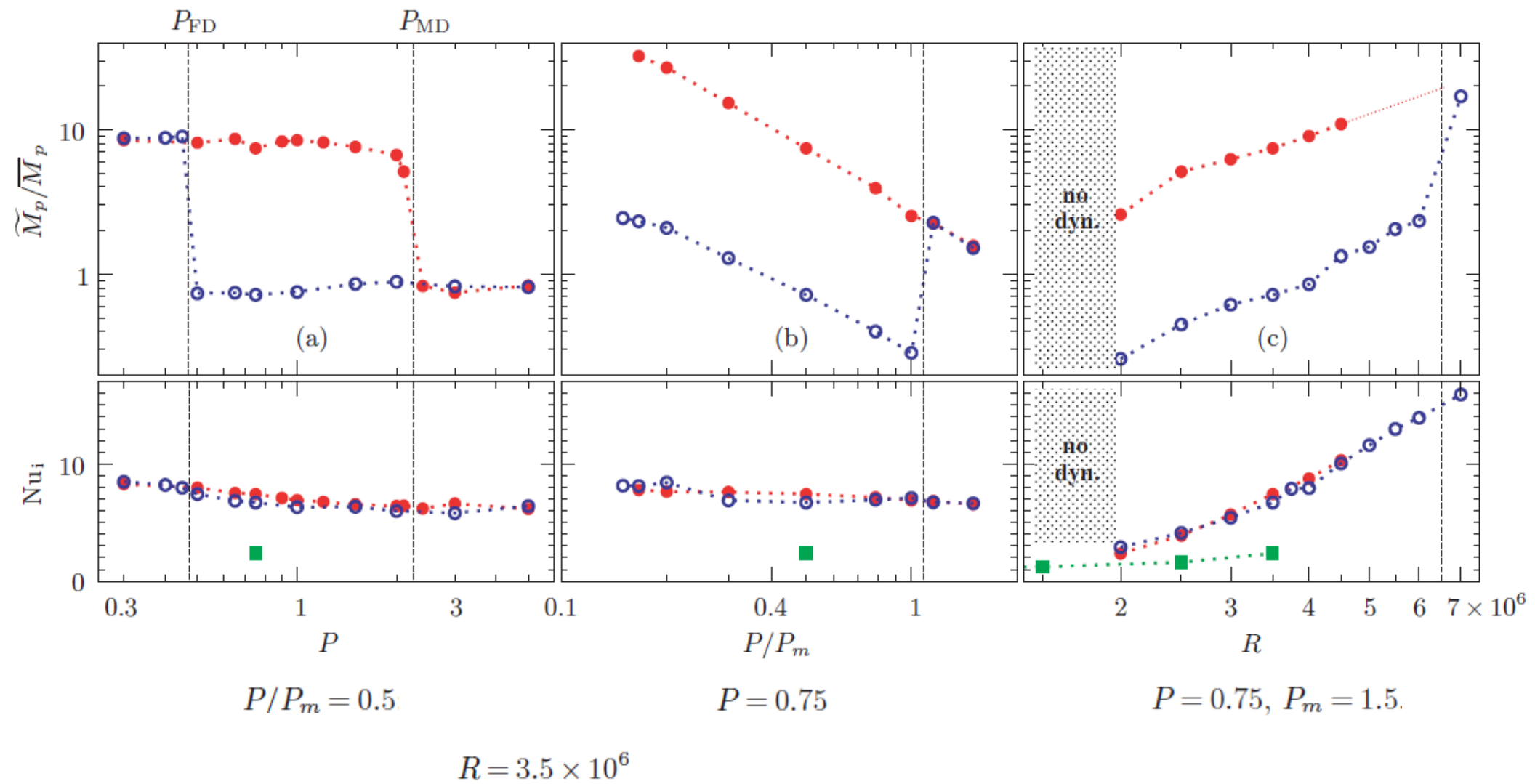
$$R = 3.5 \times 10^6, \tau = 3 \times 10^4, P = 0.75 \text{ and } P_m = 1.5$$

MD

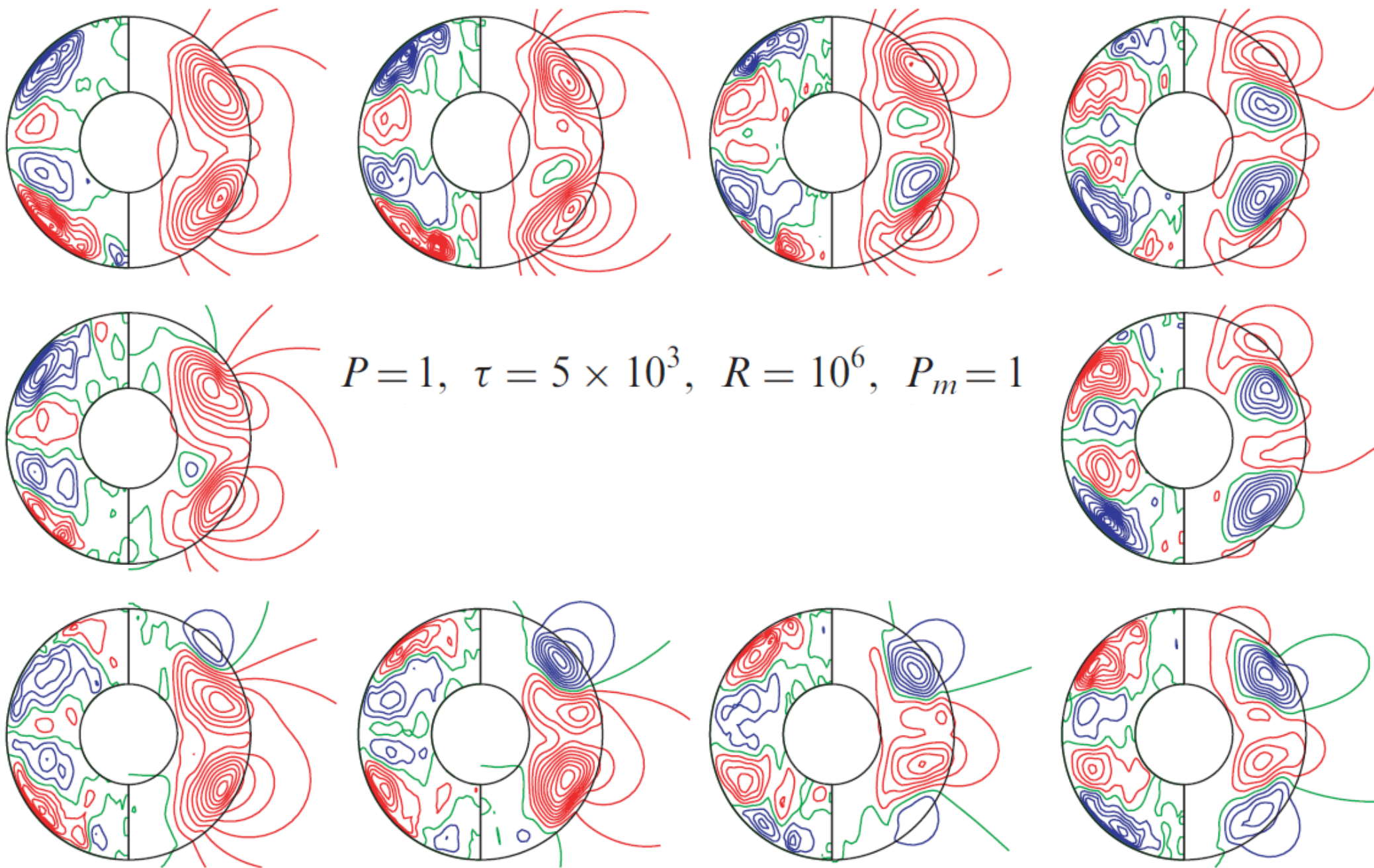


FD





A period of dipolar oscillations



Dynamo Oscillations, Dynamo Waves (Parker, 1955)

$$\underline{B} = \underline{B}_p + \underline{i} B, \quad \underline{B}_p = \nabla \times \underline{i} A, \quad \underline{u} = \underline{\dot{u}} + \underline{i} U$$

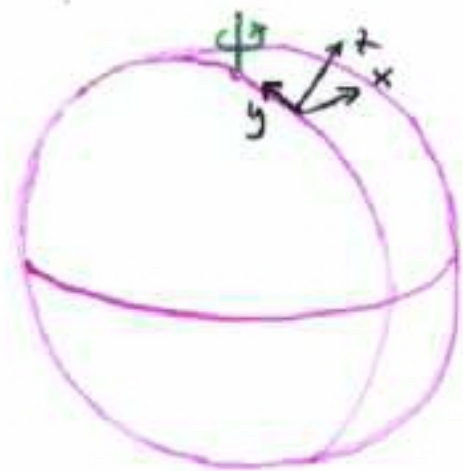
$$\frac{\partial}{\partial t} A = \alpha B + \lambda \nabla^2 A$$

$$\frac{\partial}{\partial t} B = \underline{B}_p \cdot \nabla U + \lambda \nabla^2 B$$

$$(A, B) = (\hat{A}, \hat{B}) \exp\{i \underline{q} \cdot \underline{x} + \sigma t\}$$

$$p \hat{A} = \alpha \hat{B}, \quad p \hat{B} = -i (\underline{q} \times \nabla U)_x \hat{A}$$

$$p^2 = 2i\gamma = 2i(-\frac{1}{2}\alpha(\underline{q} \times \nabla U)_x)$$



$$\text{with } p \equiv \sigma + \lambda |\underline{q}|^2$$

for $\gamma > 0$: $p = \pm(1+i)\sqrt{\gamma} \Rightarrow G = -\lambda q^2 \pm \sqrt{\gamma} + i(\pm\sqrt{\gamma})$

growth for $-\alpha(q \times \nabla U)_x > 2\lambda^2 q^4$, wave propagates in $-q$ -direction

for $\gamma < 0$: $p = \pm(1-i)\sqrt{|\gamma|} \Rightarrow G = -\lambda q^2 \pm \sqrt{|\gamma|} \mp i\sqrt{|\gamma|}$

growth for $\alpha(q \times \nabla U)_x > 2\lambda^2 q^4$, wave propagates in q -direction

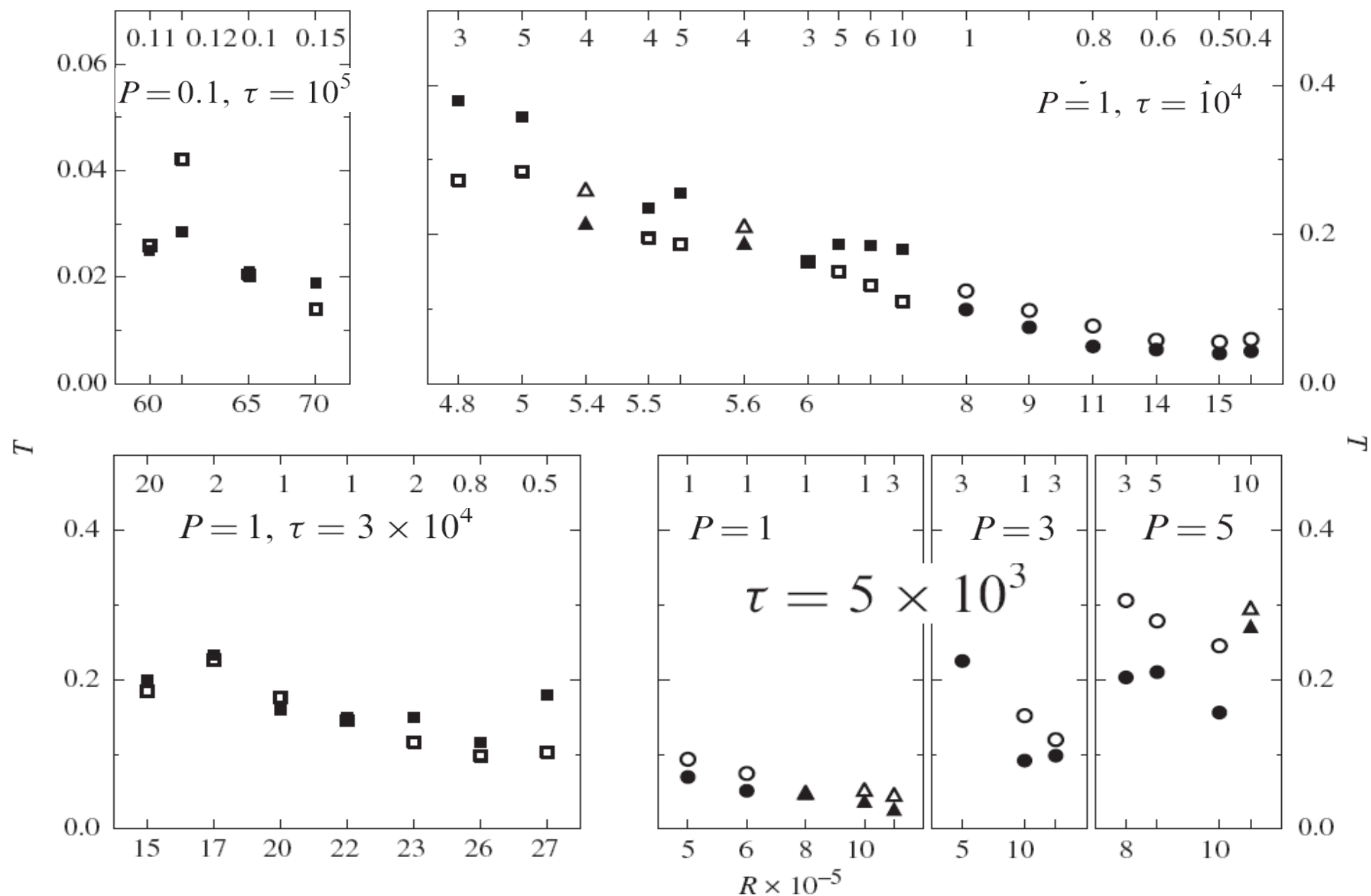
$$\alpha \equiv -\frac{\lambda}{3} \iint \frac{q^2 F(q, \omega)}{\omega^2 + \lambda^2 q^4} dq d\omega \approx -\frac{1}{3\lambda} \int q^2 \bar{F}(q) dq \text{ with } \bar{F}(q) = \int F(q, \omega) d\omega$$

Helicity: $H = \langle \underline{u} \cdot \nabla \times \underline{u} \rangle = \iint F(q, \omega) dq d\omega$ Helicity
Spectrum
Function F

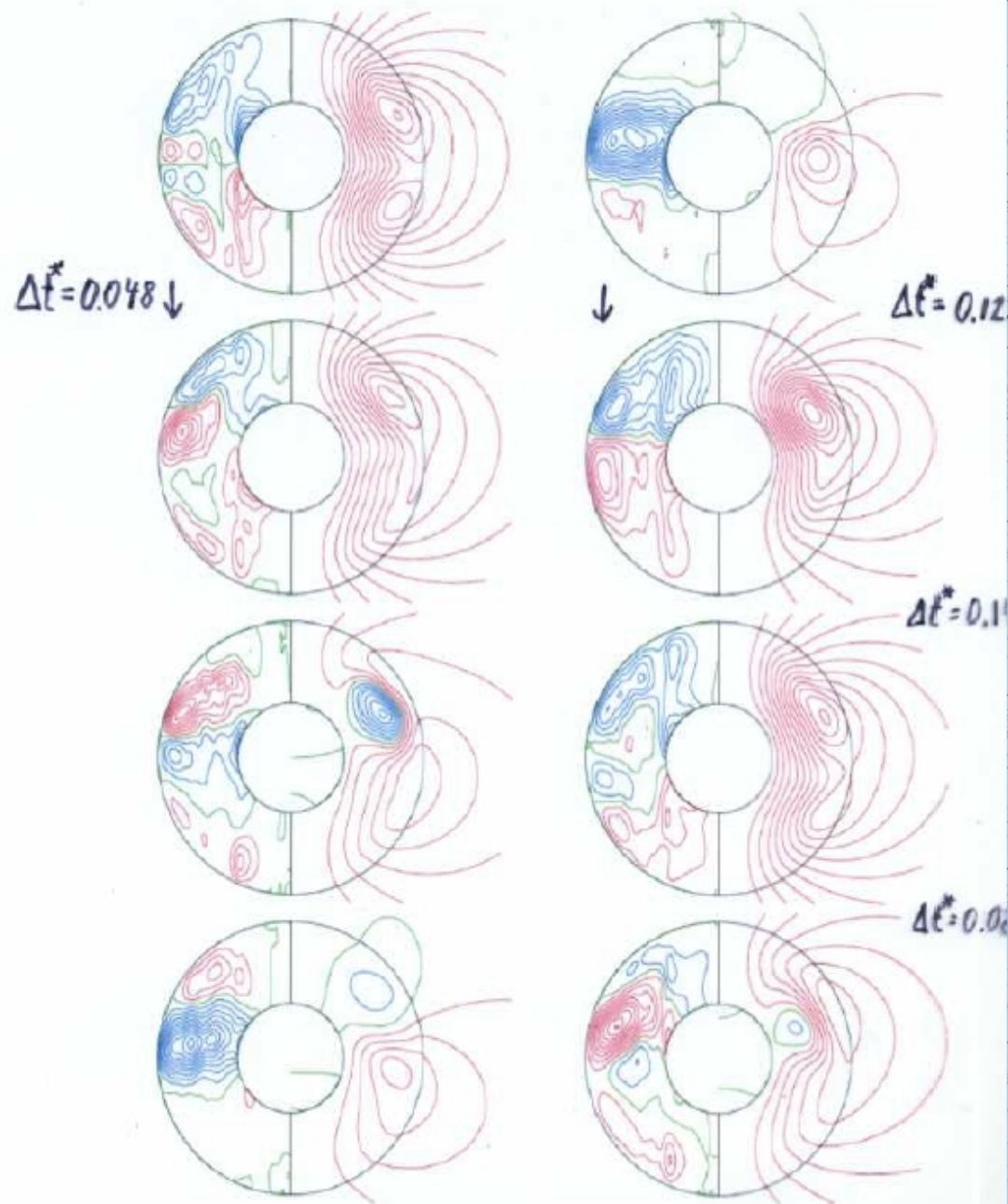
$$q \approx \frac{2\pi}{d}, \quad \frac{\partial U}{\partial z} \approx \frac{v}{d} \sqrt{2E_{\text{tor}}} / d, \quad \gamma = \omega^2 = \frac{\text{Helicity}}{3\lambda q^2} \frac{1}{2} q \frac{\partial U}{\partial z}$$

$$\tilde{\omega} = \frac{d^2}{v} \omega = \frac{1}{2\pi} \left(P_m \frac{\text{Helicity}}{3} \pi \sqrt{2E_{\text{tor}}} \right)^{\frac{1}{2}}$$

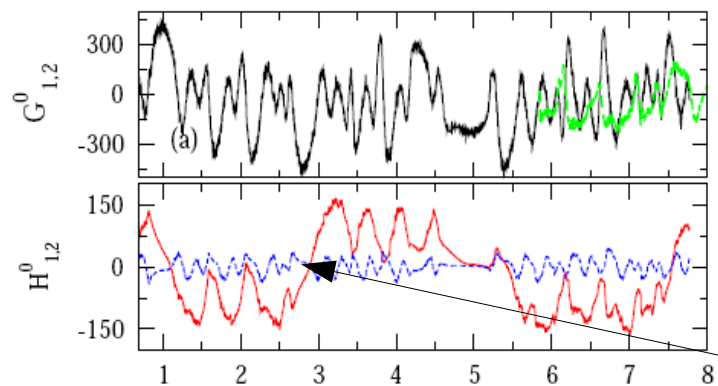
Period of oscillations: model vs. numerics



$$\tau = 10^5, \quad R = 4 \cdot 10^6, \quad P = 0.1, \quad P_m = 0.5$$

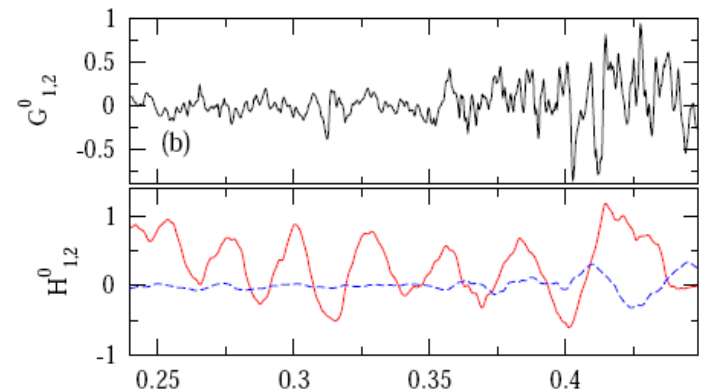


Reversals cased by toroidal flux oscillations

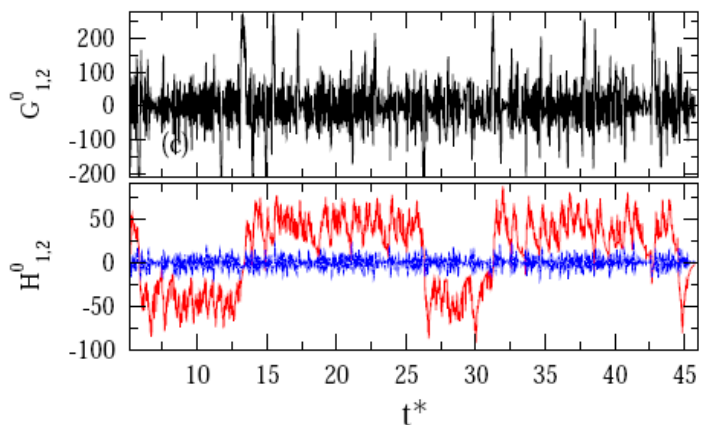


$$P = 0.1, \tau = 10^5$$

$$R = 4 \times 10^6, P_m = 0.5$$



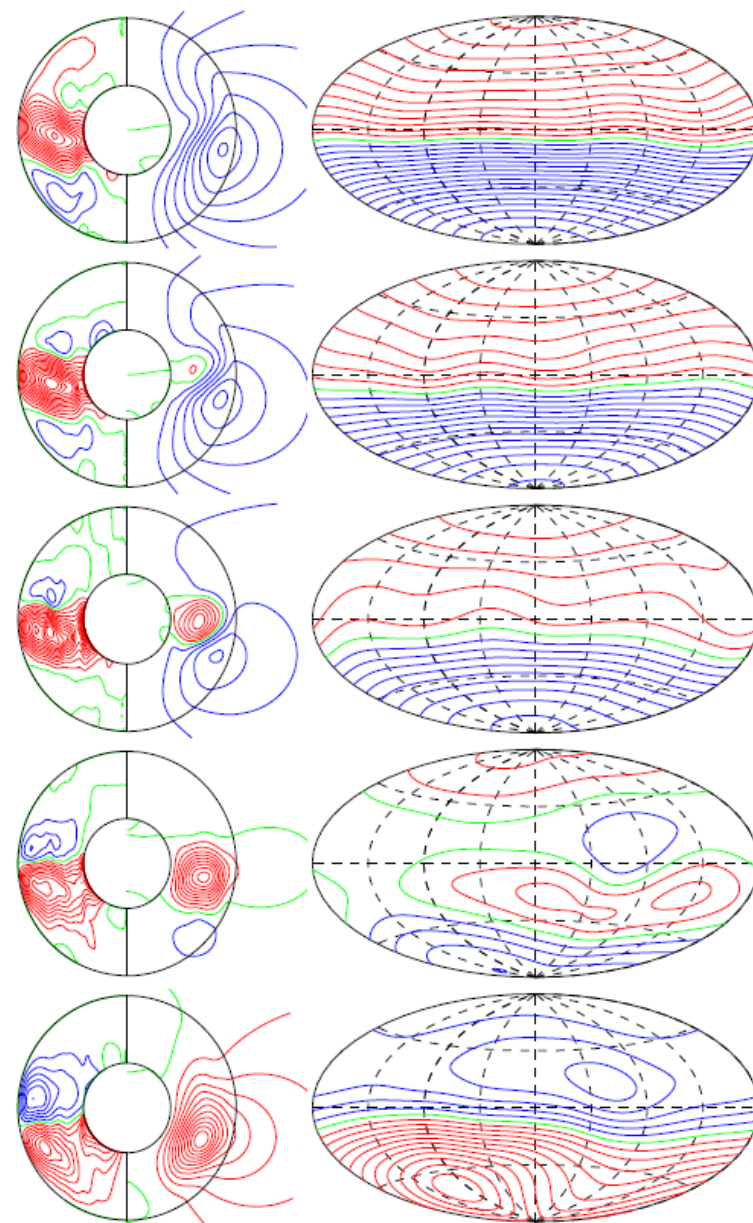
*Oscillating FD
dynamo*



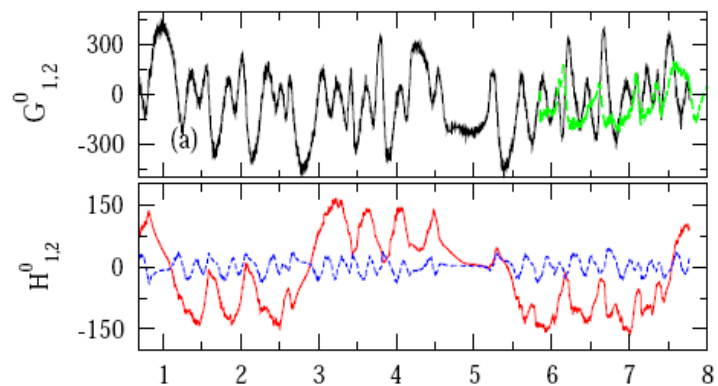
$$P = 0.1, \tau = 3 \times 10^4$$

$$R = 850000, P_m = 1$$

Busse & Simitev, PEPI, 2008

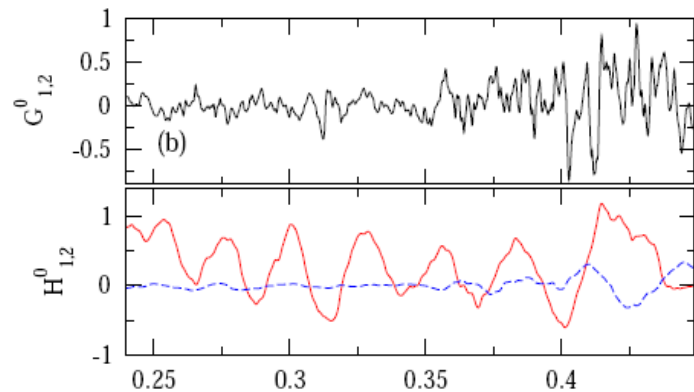


Reversals cased by toroidal flux oscillations

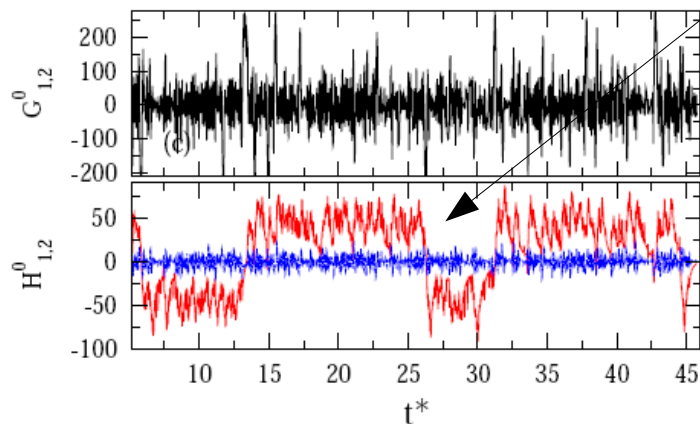


$$P = 0.1, \tau = 10^5$$

$$R = 4 \times 10^6, P_m = 0.5$$



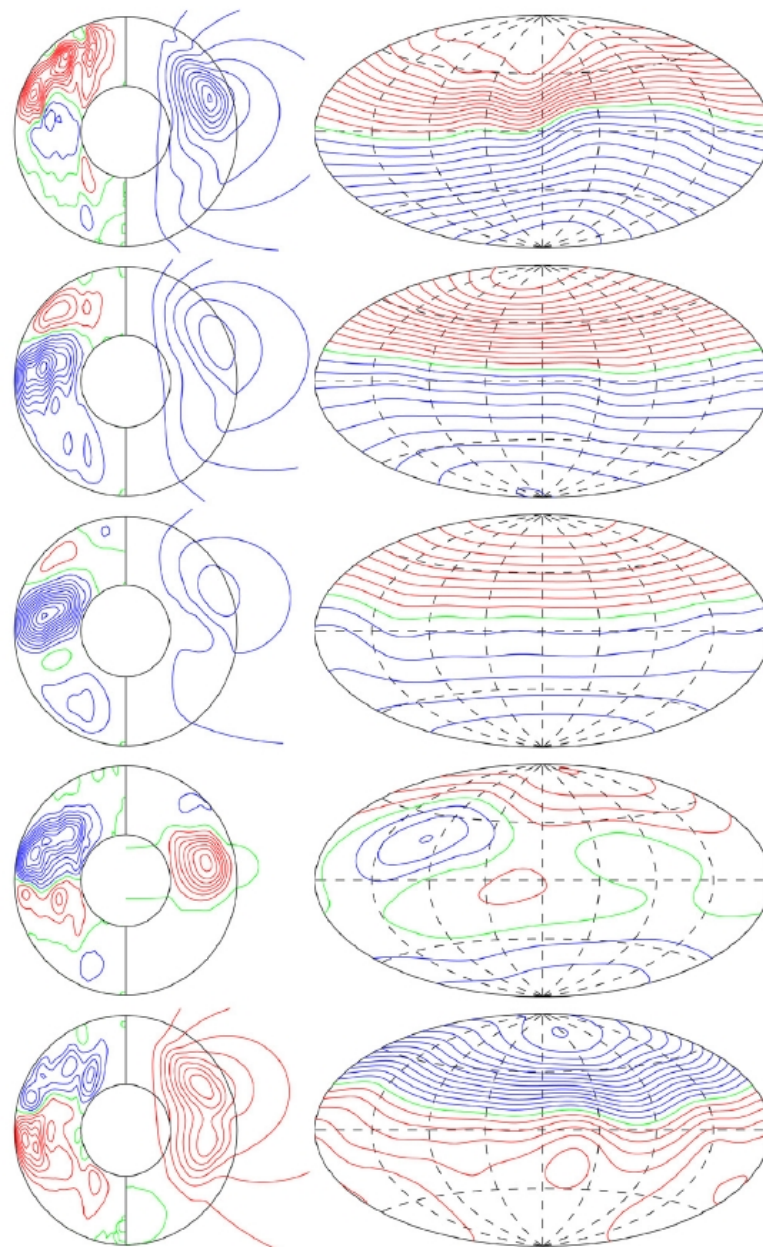
Oscillating FD dynamo



$$P = 0.1, \tau = 3 \times 10^4$$

$$R = 850000 \quad P_m = 1$$

Busse & Simitev, PEPI, 2008



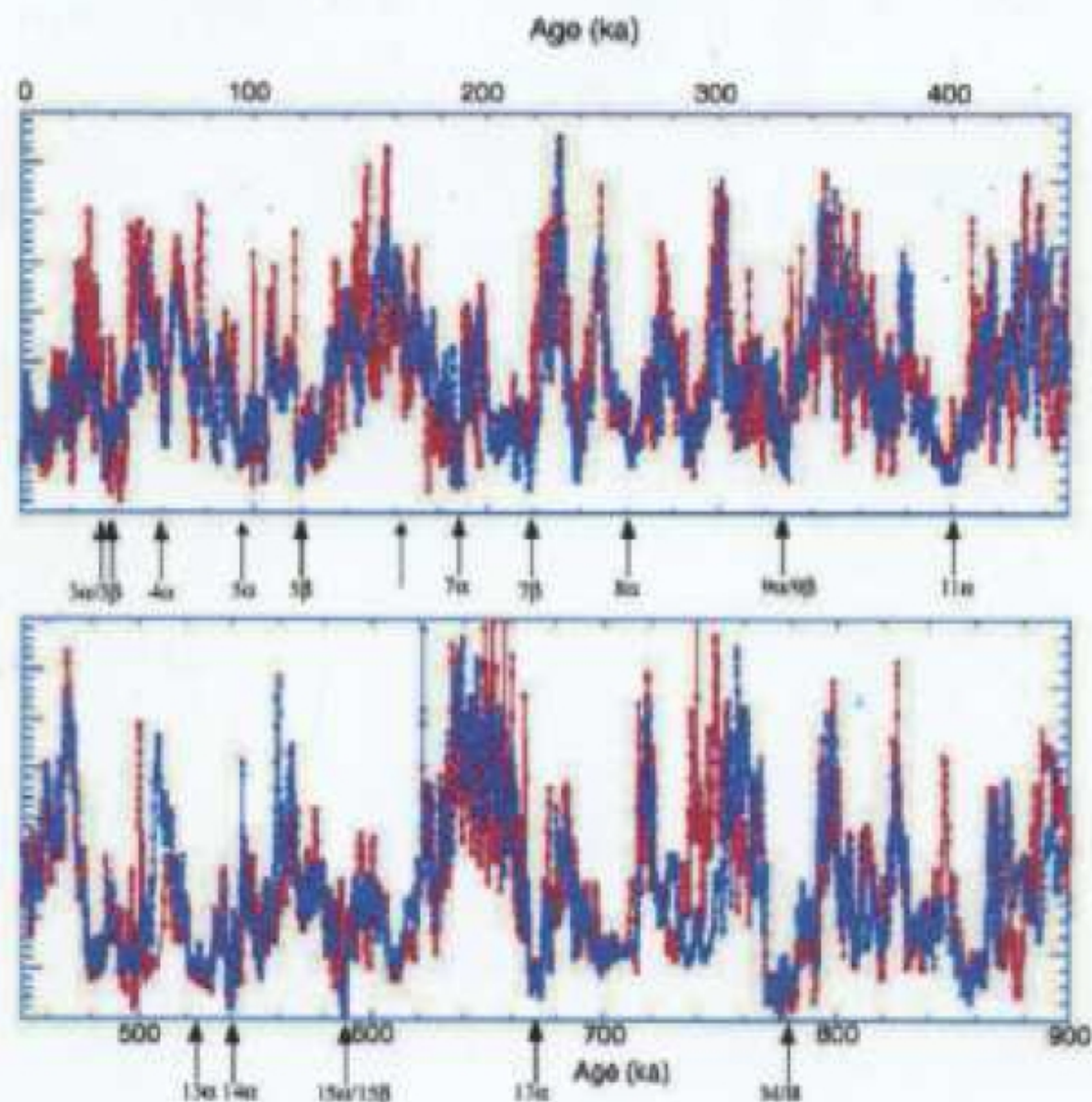
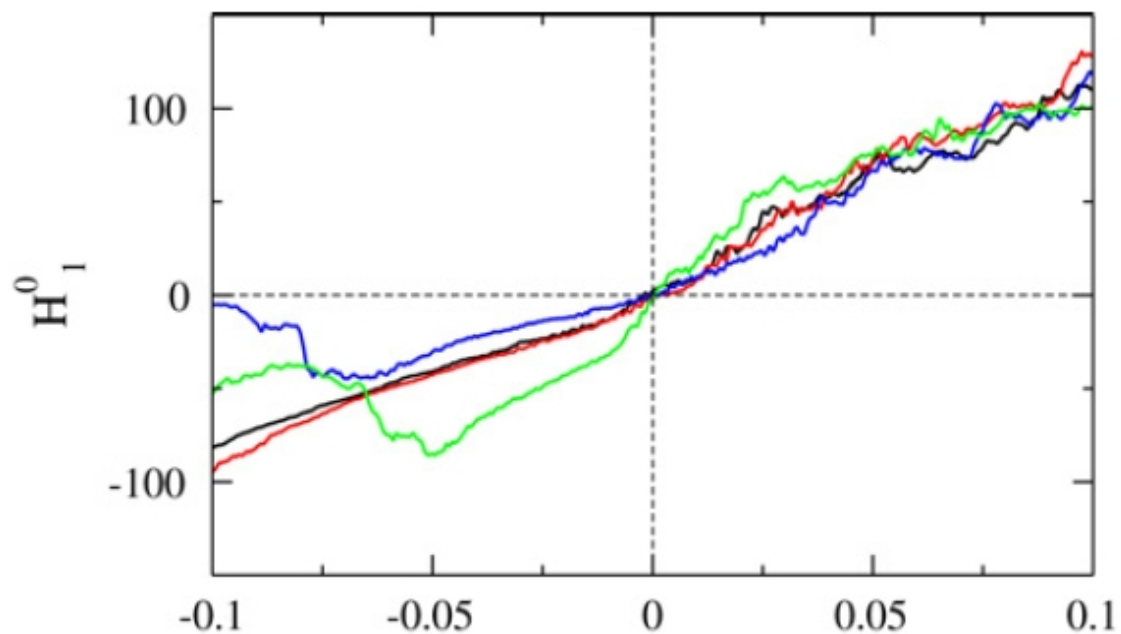


Fig. 6. Brunhes paleointensity record from ODP Sites 983/984 (Channell, 1999; Channell et al., 2004). Age models are independent and based on oxygen isotope data. Note the strong correspondence in the relative paleointensity estimates from two sites about 100 km apart. These records represent our most high-resolution evidence for overall Brunhes chron paleointensity variability. The ages of known Brunhes chron excursions (Table 1) are indicated by arrows. Note that all excursions occur in distinctive intervals of low paleointensity.

$$P = 0.1, \tau = 10^5$$

$$R = 4 \times 10^6, P_m = 0.5$$



$$P = 0.1, \tau = 3 \times 10^4$$

$$R = 850,000, P_m = 1$$

