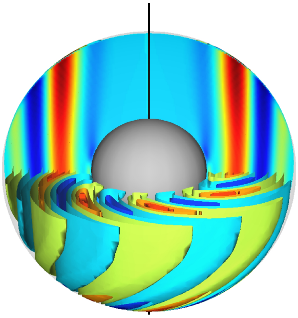
Dynamo, Dynamical Systems and Topology

NORDITA, Stockholm



Convection-driven spherical dynamos

- Applications to the Sun

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Modelling approach

- ***Global dynamo models in rotating spherical shells and with the Boussinesq approximation capture the basic first order physics of the convective-dynamo problem***

    Thermally/chemically driven convection

    Magnetic field generation

    Lorentz force acts on fluid

- ***Such models have been relatively successful in modelling the Geodynamo because compressibility is not important.***

    Westward drift of magnetic structures

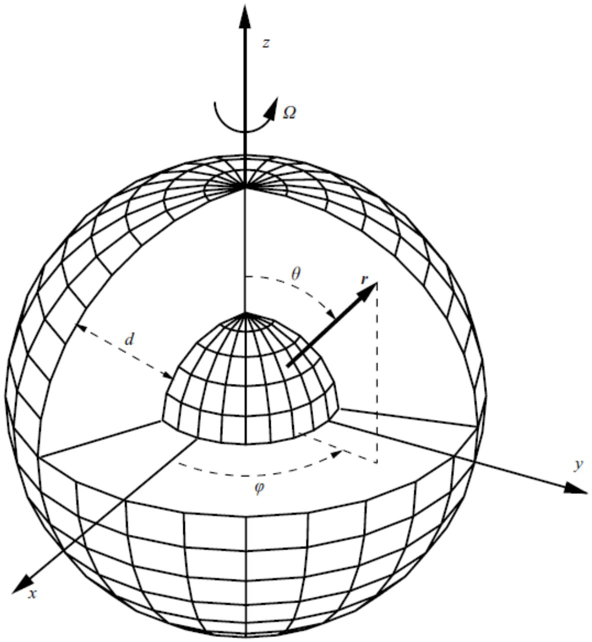
    Reversals and excursions

    Scaling properties

- ***We propose to use such models to capture the regular solar magnetic cycle and the solar differential rotation.***

    Note: Anelastic spherical shell models are not significantly better

Convective spherical shell dynamos



Model equations & parameters

Boussinesq approximation

$$\nabla \cdot \boldsymbol{u} = 0, \quad \nabla \cdot \boldsymbol{B} = 0,$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla \pi - \tau \boldsymbol{k} \times \boldsymbol{u} + \Theta \boldsymbol{r} + \nabla^2 \boldsymbol{u} + \boldsymbol{B} \cdot \nabla \boldsymbol{B},$$

$$P\,(\partial_t \Theta + \boldsymbol{u} \cdot \nabla \Theta) = R\,\boldsymbol{r} \cdot \boldsymbol{u} + \nabla^2 \Theta,$$

$$P_m(\partial_t \boldsymbol{B} + \boldsymbol{u} \cdot \nabla \boldsymbol{B}) = P_m\,\boldsymbol{B} \cdot \nabla \boldsymbol{u} + \nabla^2 \boldsymbol{B}.$$

$$R = \frac{\alpha\gamma\beta d^6}{\nu\kappa}, \quad \tau = \frac{2\Omega d^2}{\nu}, \quad P = \frac{\nu}{\kappa}, \quad P_m = \frac{\nu}{\lambda}$$

Simitev & Busse, JFM, 2005

Basic state & scaling

$$T_S = T_0 - \beta d^2 r^2/2$$

$$\boldsymbol{g} = -d\gamma \boldsymbol{r}$$

Length scale:	d
Time scale:	d^2/ν
Temp. scale:	$\nu^2/\gamma\alpha d^4$
Magn. flux density:	$\nu(\mu\varrho)^{1/2}/d$

Boundary Conditions

$$\boldsymbol{r} \cdot \boldsymbol{u} = \boldsymbol{r} \cdot \nabla \boldsymbol{r} \times \boldsymbol{u}/r^2 = 0,$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{r}} \cdot \boldsymbol{B}_{\text{int}} = \hat{\boldsymbol{e}}_{\boldsymbol{r}} \cdot \boldsymbol{B}_{\text{ext}},$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{r}} \times \boldsymbol{B}_{\text{int}} = \hat{\boldsymbol{e}}_{\boldsymbol{r}} \times \boldsymbol{B}_{\text{ext}},$$

$$\Theta = 0, \text{ at } r = r_i \equiv 2/3 \text{ and } r_o \equiv 5/3$$

Numerical Methods

3D non-linear problem:

Toroidal-poloidal representation

$$\boldsymbol{u} = \nabla \times (\nabla v \times \boldsymbol{r}) + \nabla w \times \boldsymbol{r} \;, \qquad \boldsymbol{B} = \nabla \times (\nabla h \times \boldsymbol{r}) + \nabla g \times \boldsymbol{r}$$

Spectral decomposition in spherical harmonics and Chebyshev polynomials

$$x = \sum_{l,m,n} X_{l,n}^m(t)\, T_n(r)\, P_l^m(\cos\theta) e^{im\varphi} \qquad \text{where } x = (v, w, \Theta, g, h)^T$$

Scalar equations

$$\partial_t X_{l,n}^m = \hat{\mathcal{L}} X_{l,n}^m + N_{l,n}^m(X) \qquad \text{where } \hat{\mathcal{L}} X_{l,n}^m\text{: linear}, N_{l,n}^m(X)\text{: non-linear}$$

Pseudo-spectral method. Time-stepping: Crank-Nicolson & Adams-Bashforth

$$[X_{l,n}^m]^{k+1} = \left(1 - \frac{\Delta t}{2}\hat{\mathcal{L}}\right)^{-1} \left\{ \left(1 + \frac{\Delta t}{2}\hat{\mathcal{L}}\right) [F_{l,n}^m]^k + \frac{\Delta t}{2}\left(3[N_{l,n}^m]^k - [N_{l,n}^m]^{k-1}\right) \right\}$$

Resolution: radial=41, latitudinal=193, azimuthal=96.

Linear problem: *Galerkin spectral method for the linearised equations leading to an eigenvalue problem for the critical parameters.*

Tilgner, IJNMF, 1999

Oscillations of dipolar dynamos

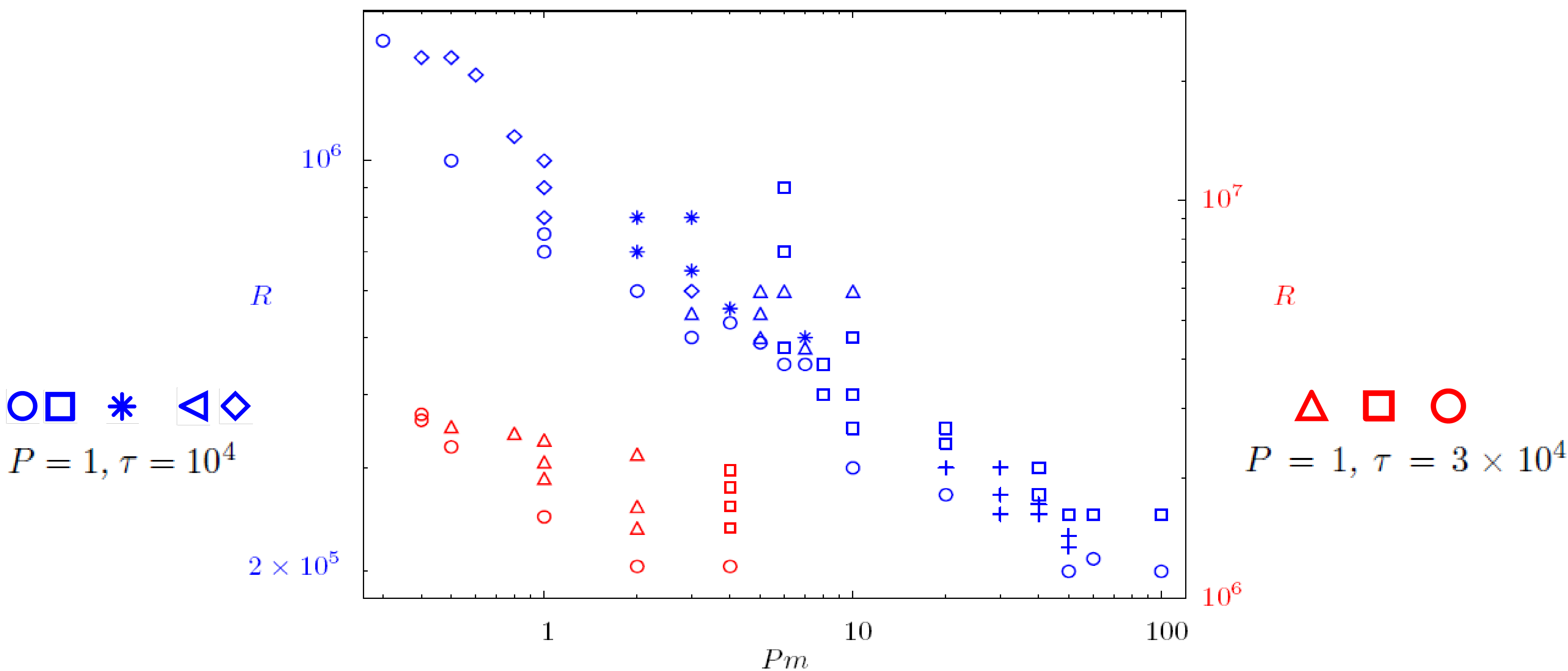
*Ref: Busse F.H. & Simitev R.D., **Phys. Earth Planet. Inter.**, 168, 237, 2008.*

*Busse F.H. & Simitev R.D., **Geophys. Astrophys. Fluid Dyn.**, 100, 2006.*

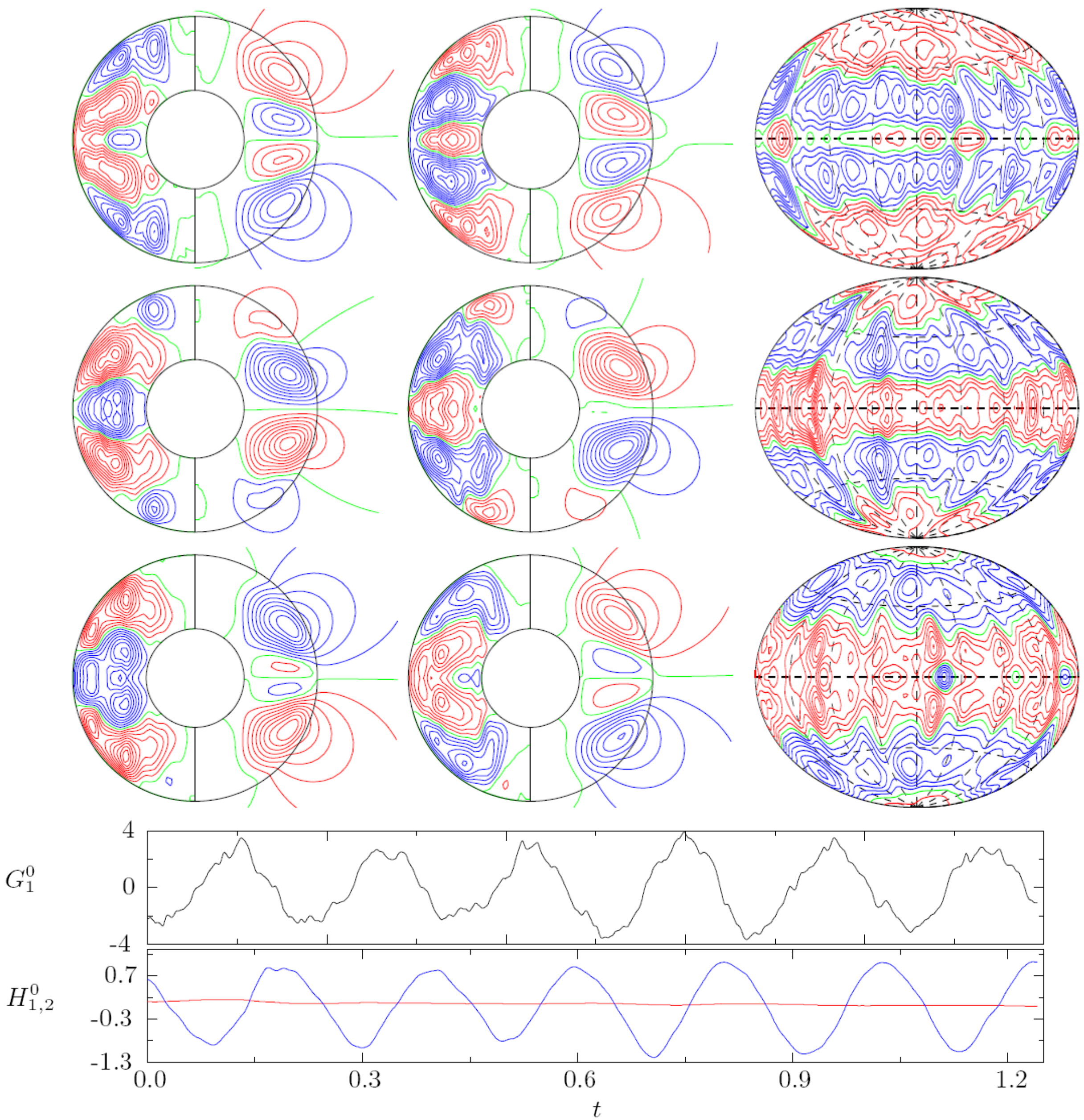
- *Examples of regular oscillations*
- *Parker's plane layer theory of dynamo wave*

Types of dynamos in the parameter space

- *No dynamo*
- *Regular and chaotic non-oscillatory dipolar dynamos (at large Pm/P and not far above dynamo onset)*
- *Oscillatory dipolar dynamos (at values of R larger than those of non-oscillatory dipoles)*
- *Hemispherical dynamos – always oscillatory*
- *Quadrupolar dynamos – always oscillatory*



Example of a quadrupolar oscillation



$P = 5, \tau = 5 \times 10^3$

$R = 8 \cdot 10^5, \ P_m = 3$

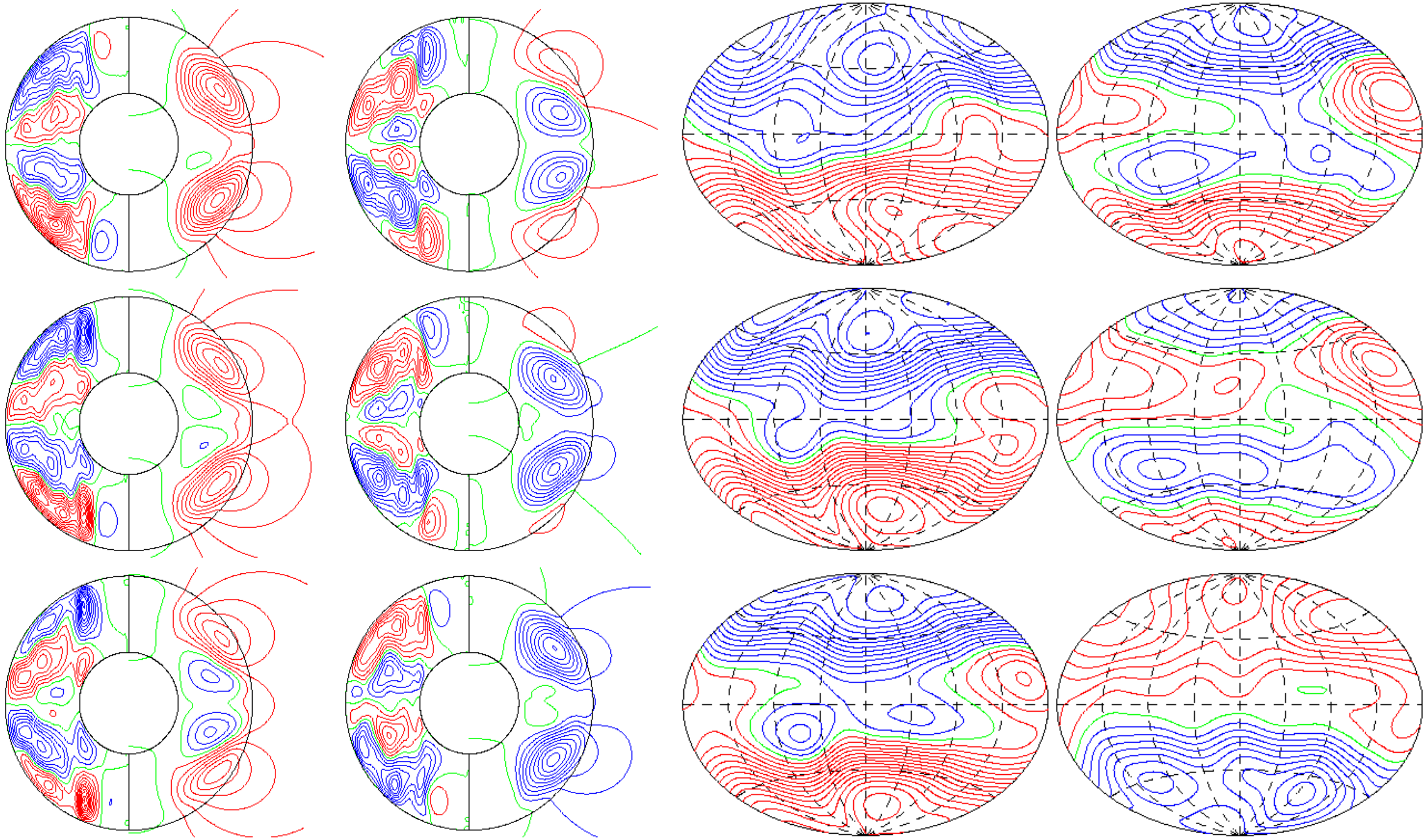
One period (first then second column)

Mean meridional filedlines of constant $\overline{B_\varphi}$ *(left),* $r \sin\vartheta \partial_\theta \overline{h}$ *(right) and radial magn. field.*

Time series of toroidal G_1^0 *and poloidal* H_1^0, H_2^0 *magn. coefficients.*

An example of a dipolar oscillation

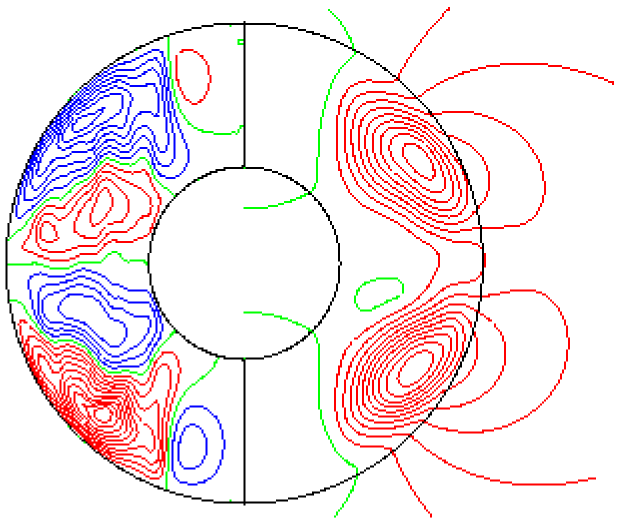
$$R = 3.5 \cdot 10^6,\ \tau = 3 \cdot 10^4,\ P = 0.75 \text{ and } P_m = 0.65$$

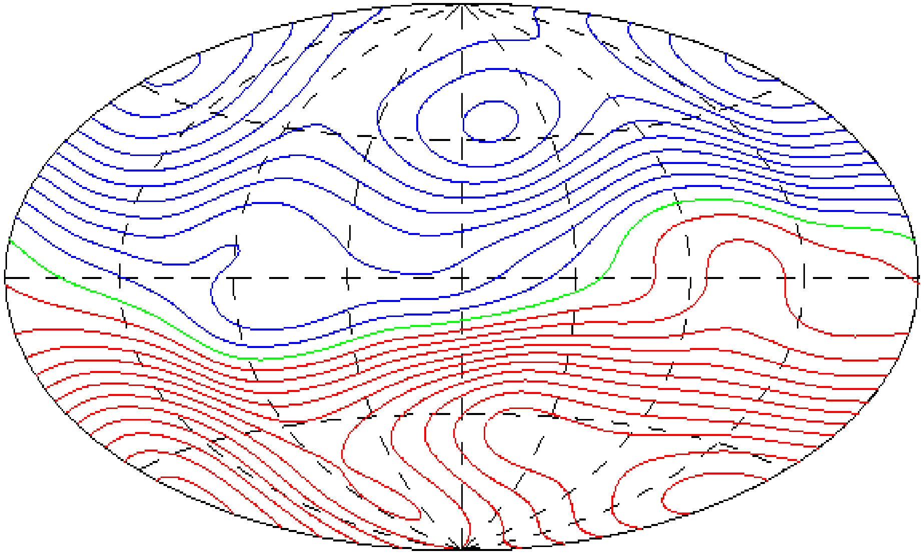


Half-period of oscillation (column-by-column)

An example of a dipolar oscillation

$$R = 3.5 \cdot 10^6,\ \tau = 3 \cdot 10^4,\ P = 0.75 \text{ and } P_m = 0.65$$





http://www.maths.gla.ac.uk/~rs/res/B/anim.bm.gif

http://www.maths.gla.ac.uk/~rs/res/B/anim.radmagn_2.gif

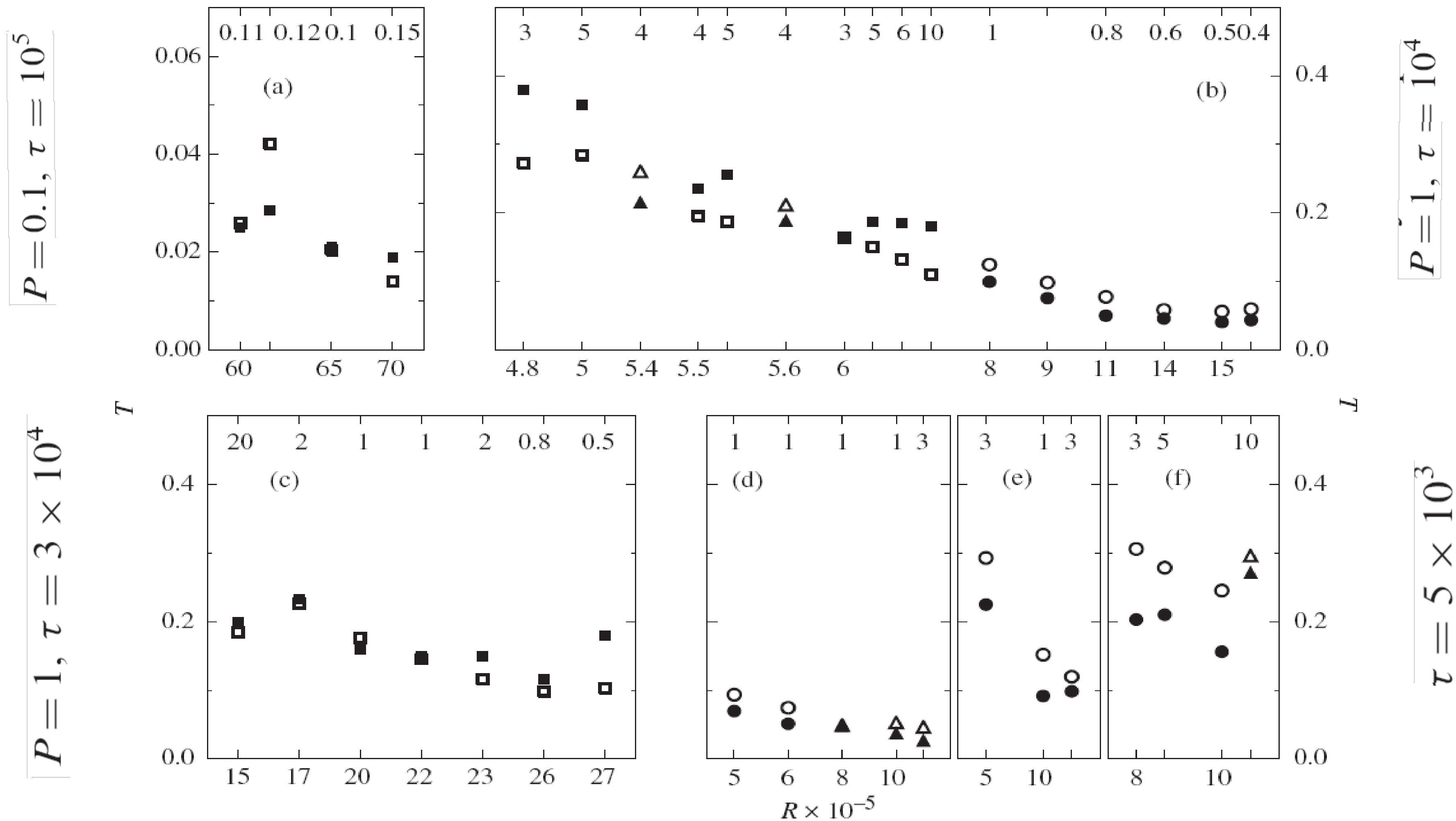
A period of oscillation (column-by-column)

Fit to a mean-field dynamo wave model

Parker's mean field dynamo wave model:

$$T \approx 4\pi^2 \left(P_m \frac{\pi}{3} \langle \check{\boldsymbol{v}} \cdot \nabla \times \check{\boldsymbol{v}} \rangle \sqrt{2\,\overline{E_t}} \right)^{-1/2}$$

Parker, Astrophys. J., 1955.   *Busse & Simitev, GAFD 2006*



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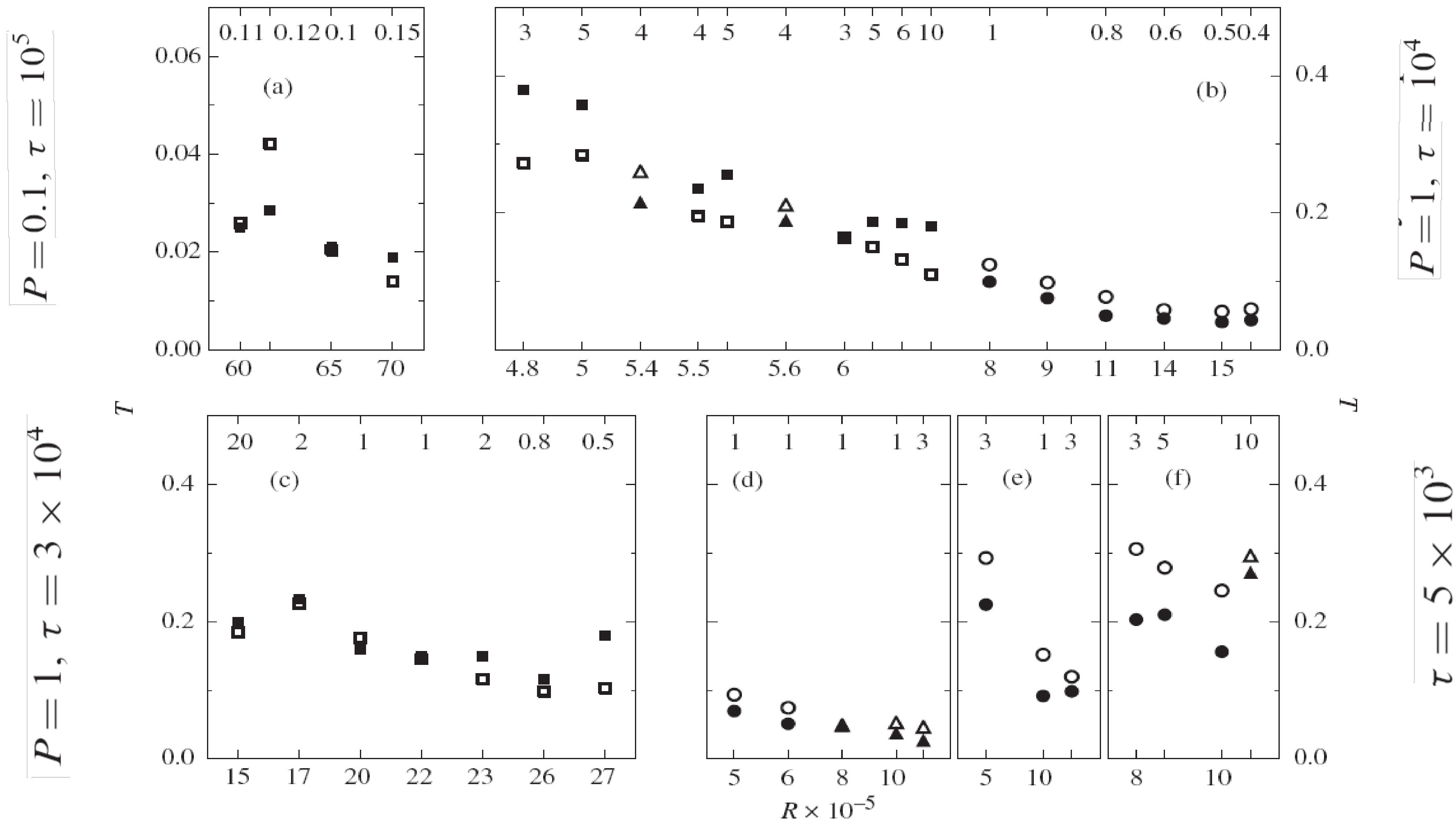
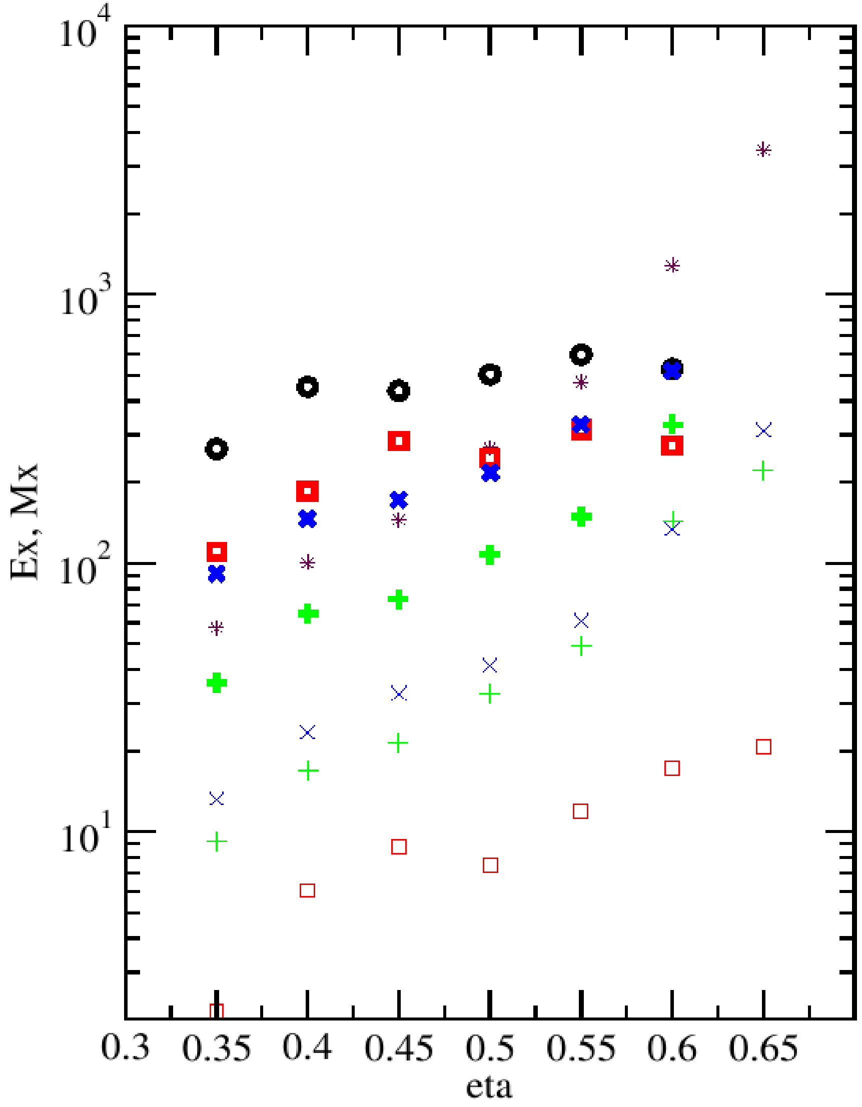


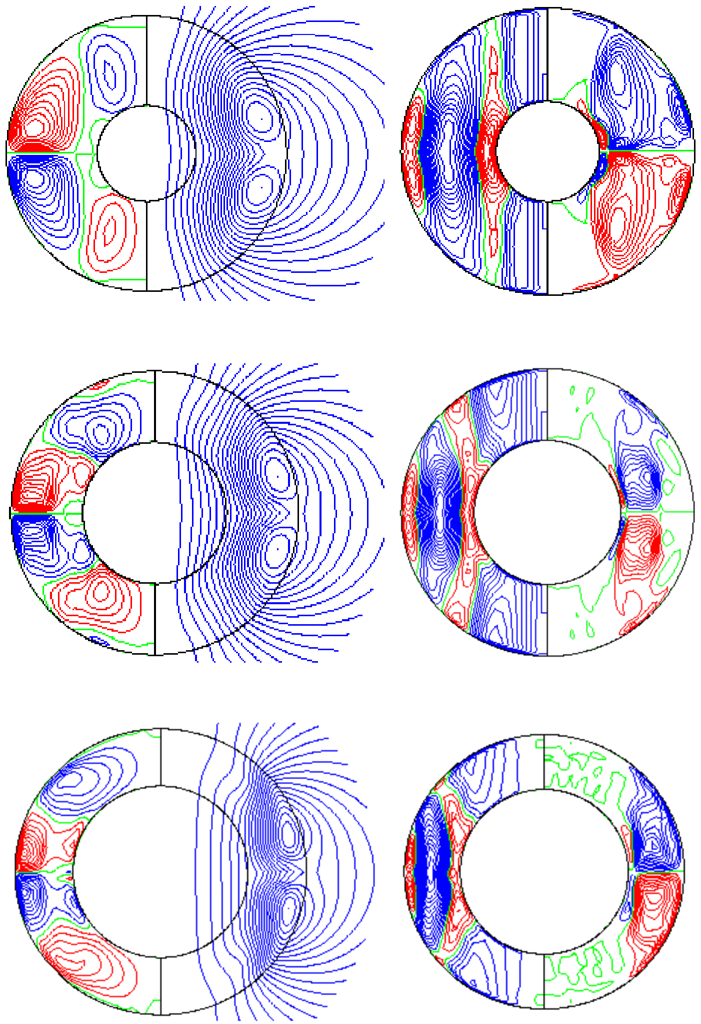
Applications to the Sun

- *Oscillations in thin-shell dynamos*
- *Effects of velocity boundary conditions*

Dependence on the shell thickness – motivated by Dormy, (EPL 2010)

$P = 1, \tau = 2000, R = 10^5, P_m = 5$ *no-slip boundary conditions*



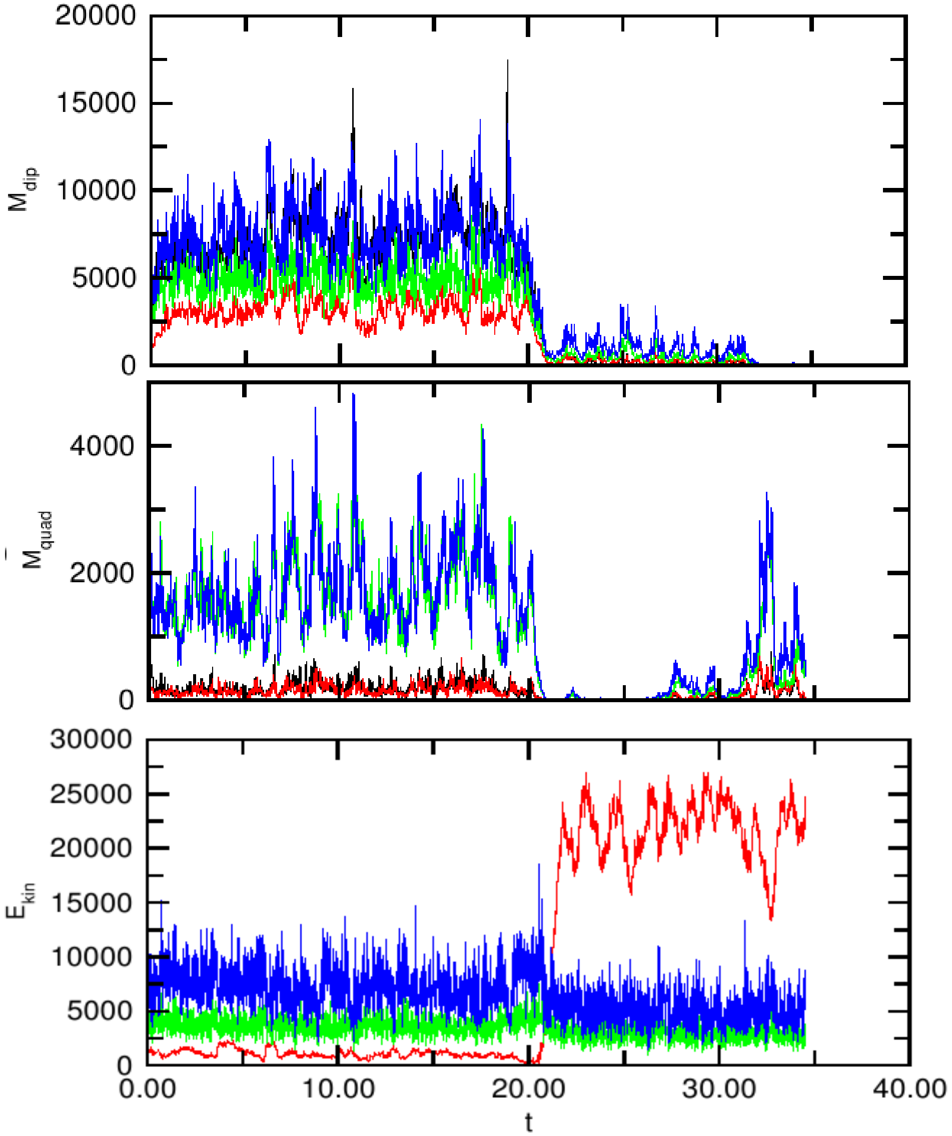


0.35

0.5

0.6

Transition to fluctuating dynamos in thin shells

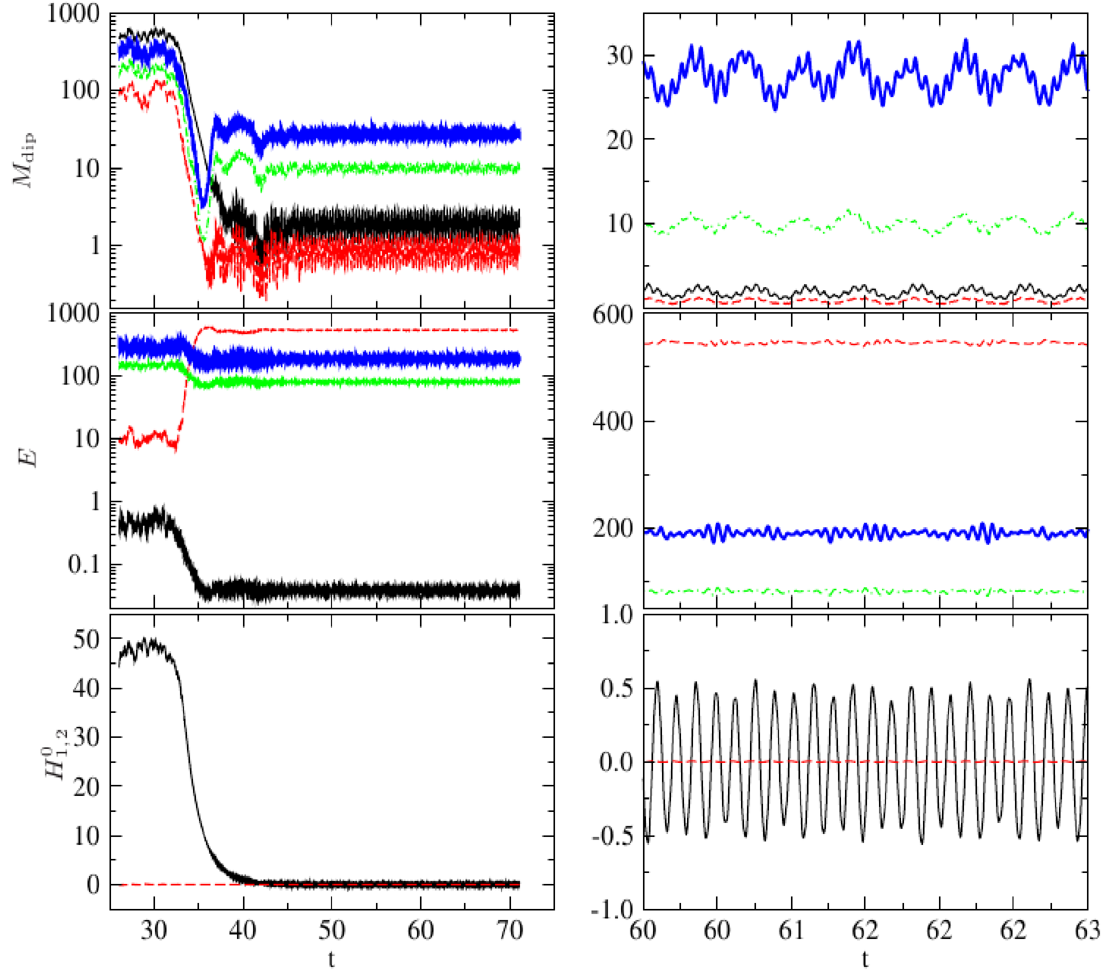


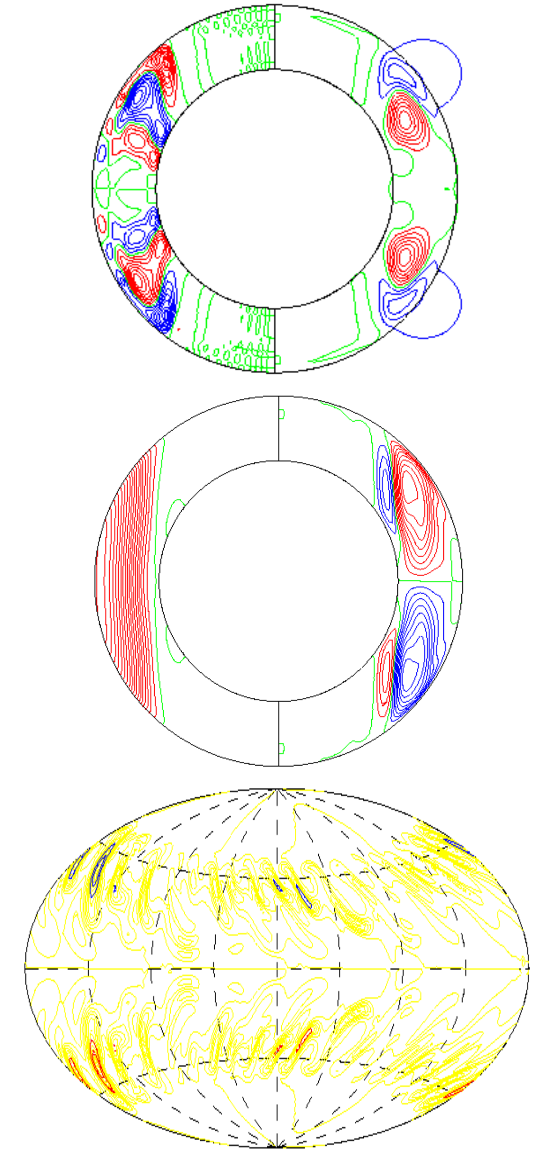
$$\eta = 0.65, P = 1, \tau = 2000,$$

$$R = 1.5 \times 10^5, P_m = 5$$

No-slip at inner boundary
Stress free at outer boundary

Regular dipolar oscillations in thin shells

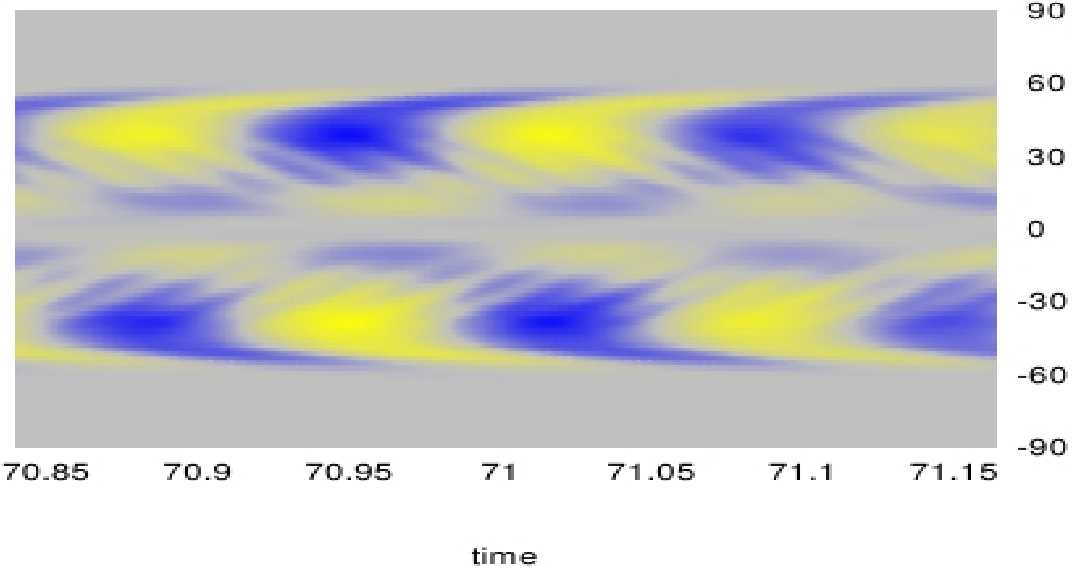




$$\eta = 0.65 \quad \mathrm{P} = 1,\ \mathrm{R} = 10^5,\ \tau = 2000,\ \mathrm{P_m} = 4.5,$$

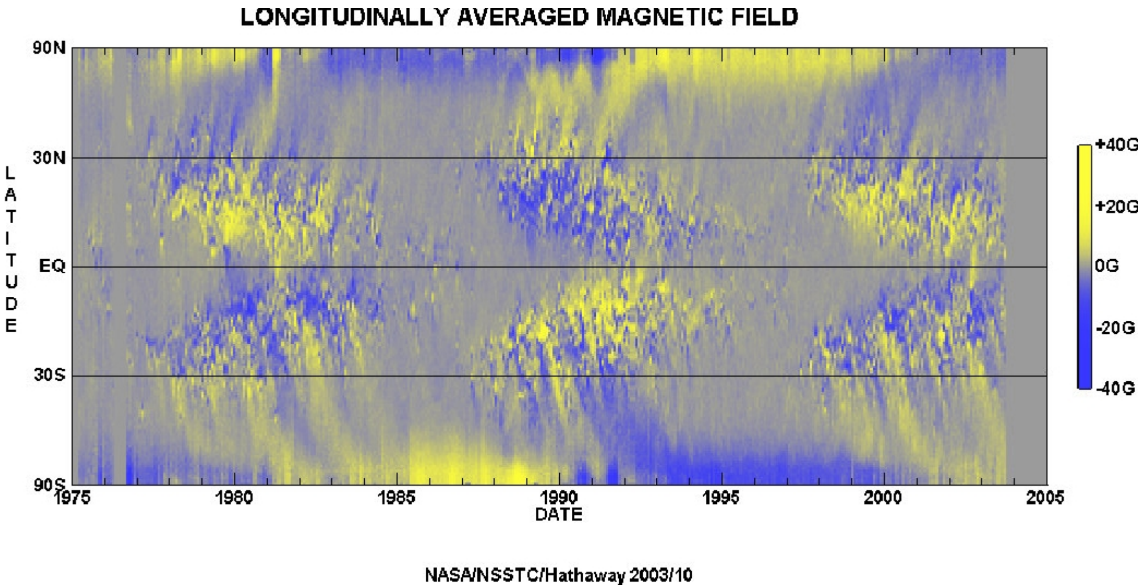
No-slip at inner boundary, stress free at outer boundary

Butterfly diagram



$$P = 1, \ R = 10^5, \ \tau = 2000,$$
$$P_m = 4.5, \quad \eta = 0.65$$

No-slip at inner boundary, stress free at outer boundary



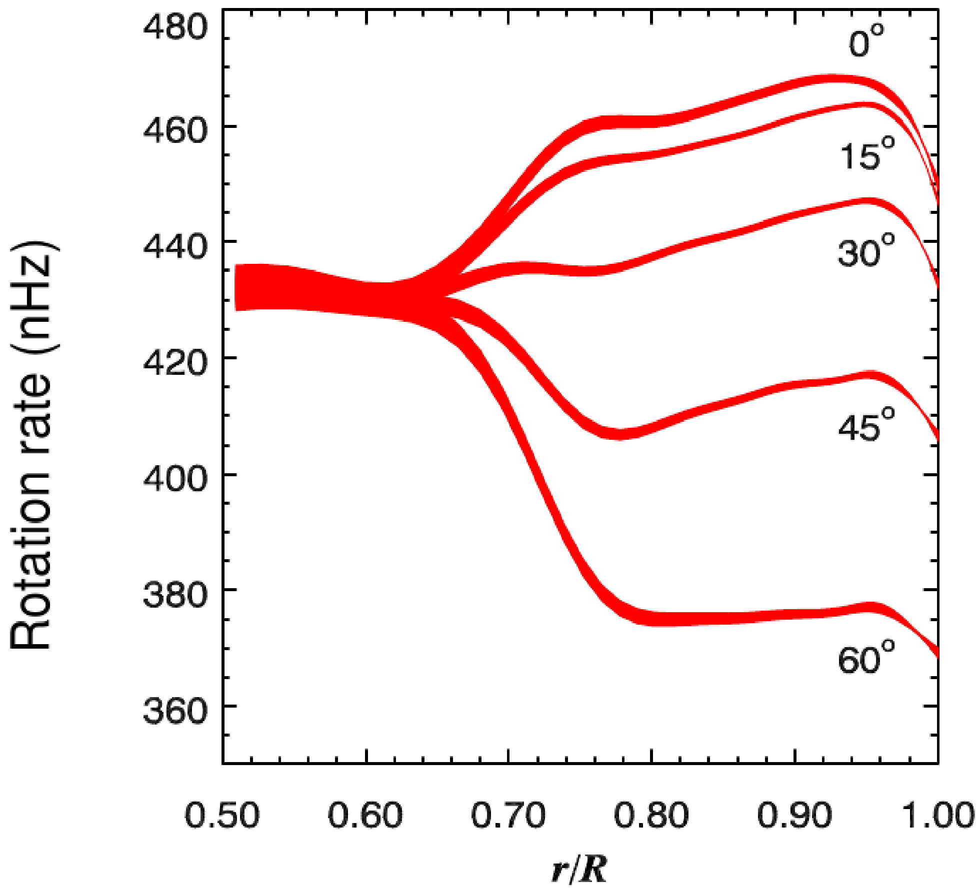
Hathaway, D.H., et al, 2003, Astrophys. J., 589, 665-670

Radial profiles of the differential rotation and $\beta \neq 0$

PLOT HERE

$$\mathrm{P} = 1,\ \mathrm{R} = 10^5,\ \tau = 2000,$$
$$\mathrm{P_m} = 4.5,\quad \eta = 0.65$$

No-slip at inner boundary,
stress free at outer boundary



Internal Rotation of the Sun
as found by helioseismology,
NSF's National Solar Observatory

A modified boundary condition

Along with solar observations, a recent analytical analysis of a simplified model problem (Busse, Solar Physics 2007) indicates that a suitable way to reproduce the decrease of differential rotation with radius near the surface is to employ the following boundary condition

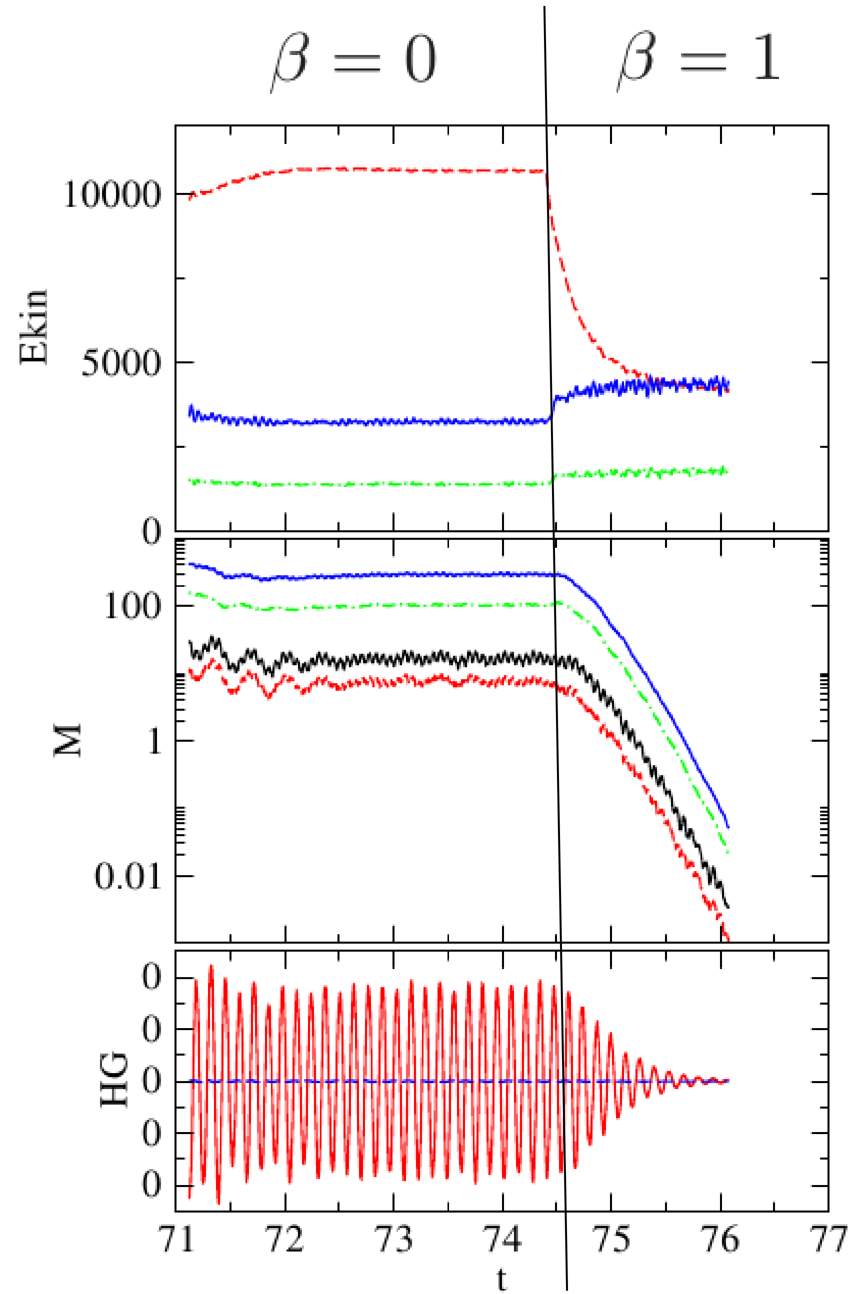
$$(\partial_r + \beta) W_{l=1}^{m=0} = 0$$

$$\text{at} \quad r = r_o$$

Beta is a fitting parameter.

$$\beta = 1,\ \eta = 0.65,\ P = 1$$

$$\tau = 2000,\ R = 10^5,\ P_m = 4$$



Regularly oscillating dipoles with $\beta \neq 0$

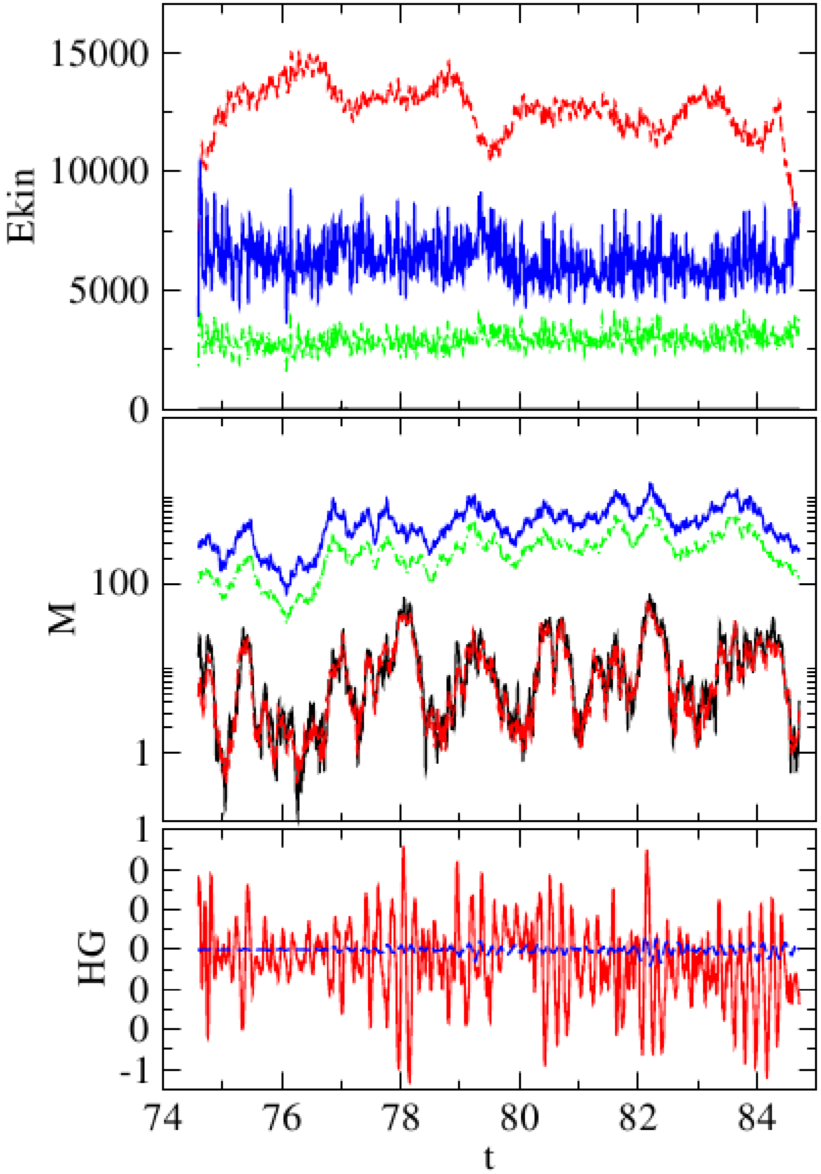
After systematic variation of parameter values cases with regular oscillations have been found.

$$\beta = 0.5,\ \eta = 0.65,\ P = 1$$

$$\tau = 2000,\ R = 1.2 \times 10^5$$

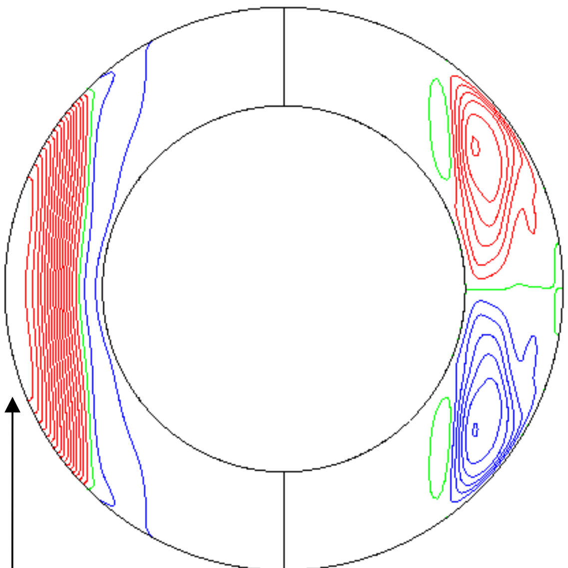
$$P_m = 4.5$$

Fairly regular dipolar oscillations →

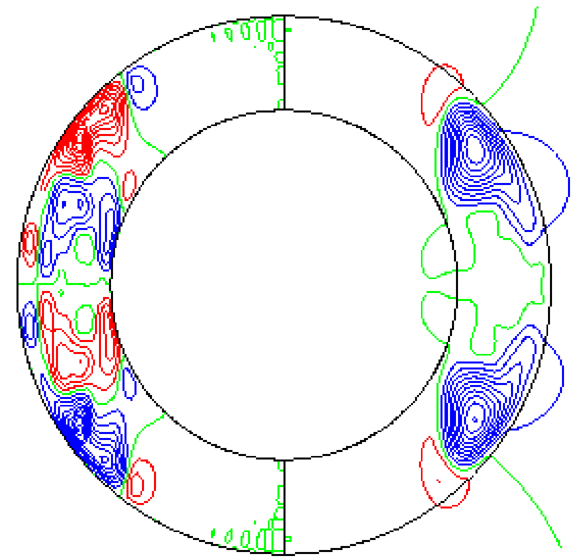


Sunspot drift in dipoles with $\beta \neq 0$

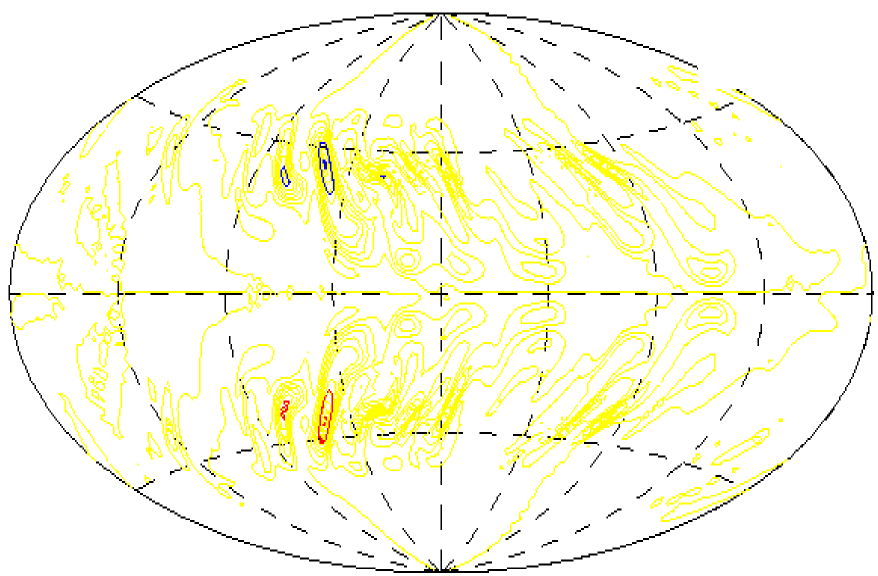
$\beta = 1.5,\ \eta = 0.65,\ P = 1 \quad \tau = 2000,\ R = 1.2 \times 10^5 \quad P_m = 4.5$



Maxima of Bphi propagate from higher latitudes powards the equator



Diff rot decreases near surface



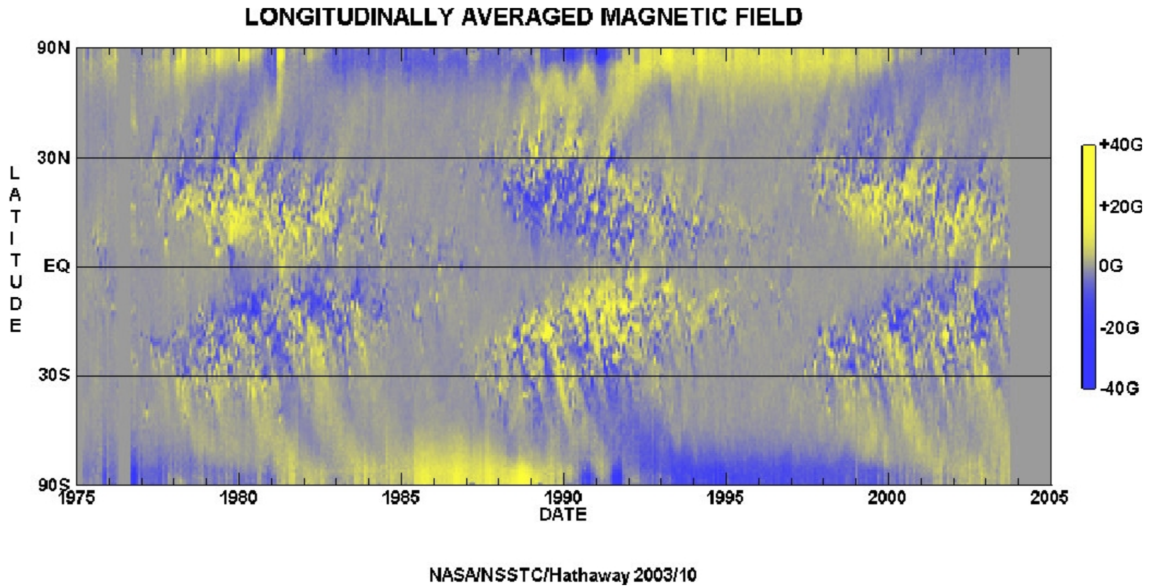
Butterfly diagram

$$\beta = 0.5,\ \eta = 0.65,\ P = 1$$

$$\tau = 2000,\ R = 1.2 \times 10^5$$

$$P_m = 4.5$$

No-slip at inner boundary, stress free at outer boundary



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Radial profiles of the differential rotation and $\beta \neq 0$

PLOT HERE

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No-slip at inner boundary,
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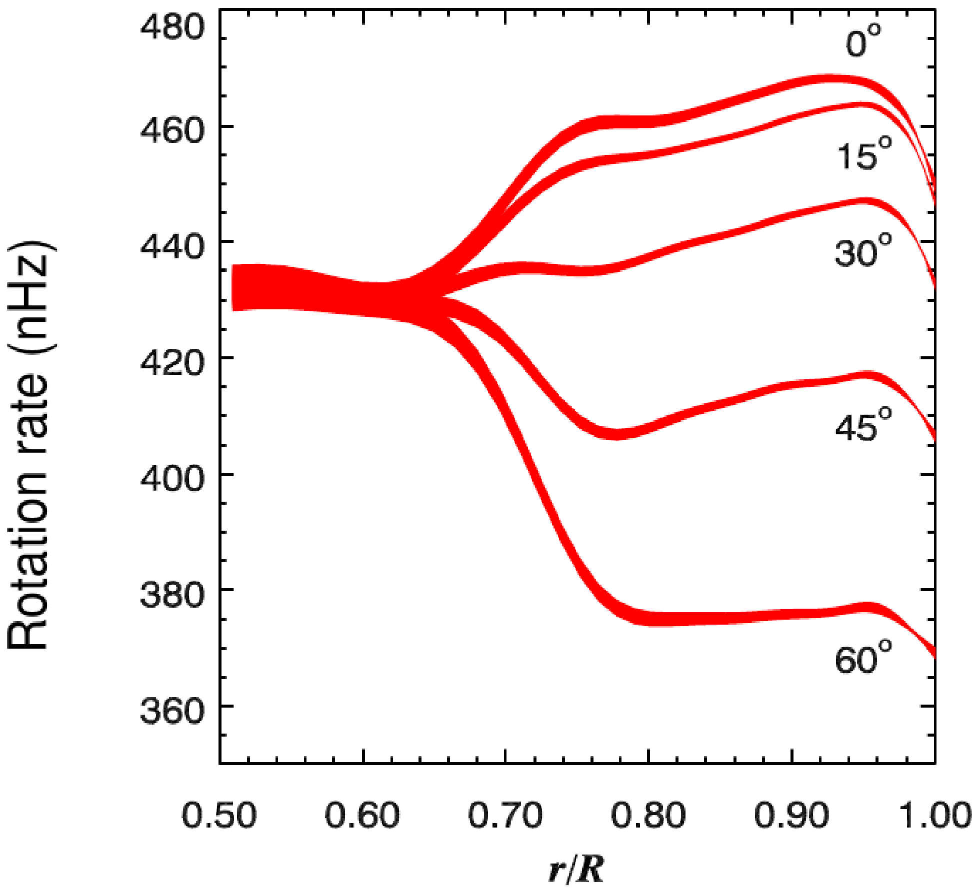

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Conclusion