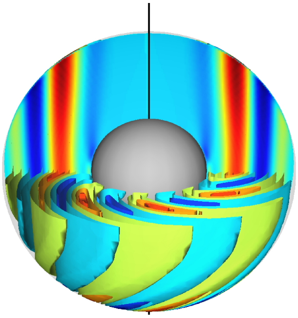
At
Stanford



Minimal models of the Solar dynamo

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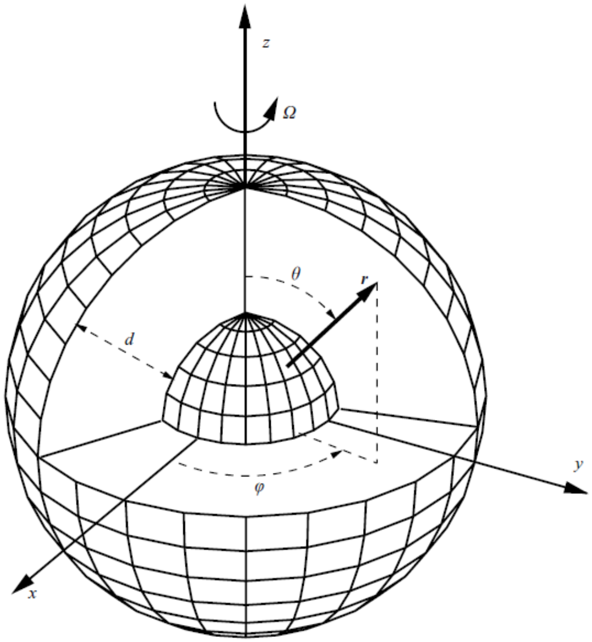
Convective spherical shell dynamos



Model equations & parameters

Boussinesq approximation

$$\nabla \cdot \boldsymbol{u} = 0, \quad \nabla \cdot \boldsymbol{B} = 0,$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} =$$
$$-\nabla \pi - \tau \boldsymbol{k} \times \boldsymbol{u} + \Theta \boldsymbol{r} + \nabla^2 \boldsymbol{u} + \boldsymbol{B} \cdot \nabla \boldsymbol{B},$$

$$P\left(\partial_t \Theta + \boldsymbol{u} \cdot \nabla \Theta\right) = R\,\boldsymbol{r} \cdot \boldsymbol{u} + \nabla^2 \Theta,$$

$$P_m(\partial_t \boldsymbol{B} + \boldsymbol{u} \cdot \nabla \boldsymbol{B}) = P_m\,\boldsymbol{B} \cdot \nabla \boldsymbol{u} + \nabla^2 \boldsymbol{B}.$$

$$R = \frac{\alpha\gamma\beta d^6}{\nu\kappa}, \quad \tau = \frac{2\Omega d^2}{\nu}, \quad P = \frac{\nu}{\kappa}, \quad P_m = \frac{\nu}{\lambda}$$

Simitev & Busse, JFM, 2005

Basic state & scaling

$$T_S = T_0 - \beta d^2 r^2 / 2$$

$$\boldsymbol{g} = -d\gamma \boldsymbol{r}$$

Length scale:	d
Time scale:	d^2/ν
Temp. scale:	$\nu^2/\gamma\alpha d^4$
Magn. flux density:	$\nu(\mu\varrho)^{1/2}/d$

Boundary Conditions

$$\boldsymbol{r} \cdot \boldsymbol{u} = \boldsymbol{r} \cdot \nabla \boldsymbol{r} \times \boldsymbol{u} / r^2 = 0,$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{r}} \cdot \boldsymbol{B}_{\mathrm{int}} = \hat{\boldsymbol{e}}_{\boldsymbol{r}} \cdot \boldsymbol{B}_{\mathrm{ext}},$$

$$\hat{\boldsymbol{e}}_{\boldsymbol{r}} \times \boldsymbol{B}_{\mathrm{int}} = \hat{\boldsymbol{e}}_{\boldsymbol{r}} \times \boldsymbol{B}_{\mathrm{ext}},$$

$$\Theta = 0, \ \text{at } r = r_i \equiv 2/3 \text{ and } r_o \equiv 5/3$$

Numerical Methods

3D non-linear problem:

Toroidal-poloidal representation

$$\boldsymbol{u} = \nabla \times (\nabla v \times \boldsymbol{r}) + \nabla w \times \boldsymbol{r} \;, \qquad \boldsymbol{B} = \nabla \times (\nabla h \times \boldsymbol{r}) + \nabla g \times \boldsymbol{r}$$

Spectral decomposition in spherical harmonics and Chebyshev polynomials

$$x = \sum_{l,m,n} X_{l,n}^m(t)\, T_n(r)\, P_l^m(\cos\theta) e^{im\varphi} \qquad \text{where } x = (v, w, \Theta, g, h)^T$$

Scalar equations

$$\partial_t X_{l,n}^m = \hat{\mathcal{L}} X_{l,n}^m + N_{l,n}^m(X) \qquad \text{where } \hat{\mathcal{L}} X_{l,n}^m\text{: linear}, N_{l,n}^m(X)\text{: non-linear}$$

Pseudo-spectral method. Time-stepping: Crank-Nicolson & Adams-Bashforth

$$[X_{l,n}^m]^{k+1} = \left(1 - \frac{\Delta t}{2}\hat{\mathcal{L}}\right)^{-1} \left\{ \left(1 + \frac{\Delta t}{2}\hat{\mathcal{L}}\right) [F_{l,n}^m]^k + \frac{\Delta t}{2}\left(3[N_{l,n}^m]^k - [N_{l,n}^m]^{k-1}\right) \right\}$$

Resolution: radial=41, latitudinal=193, azimuthal=96.

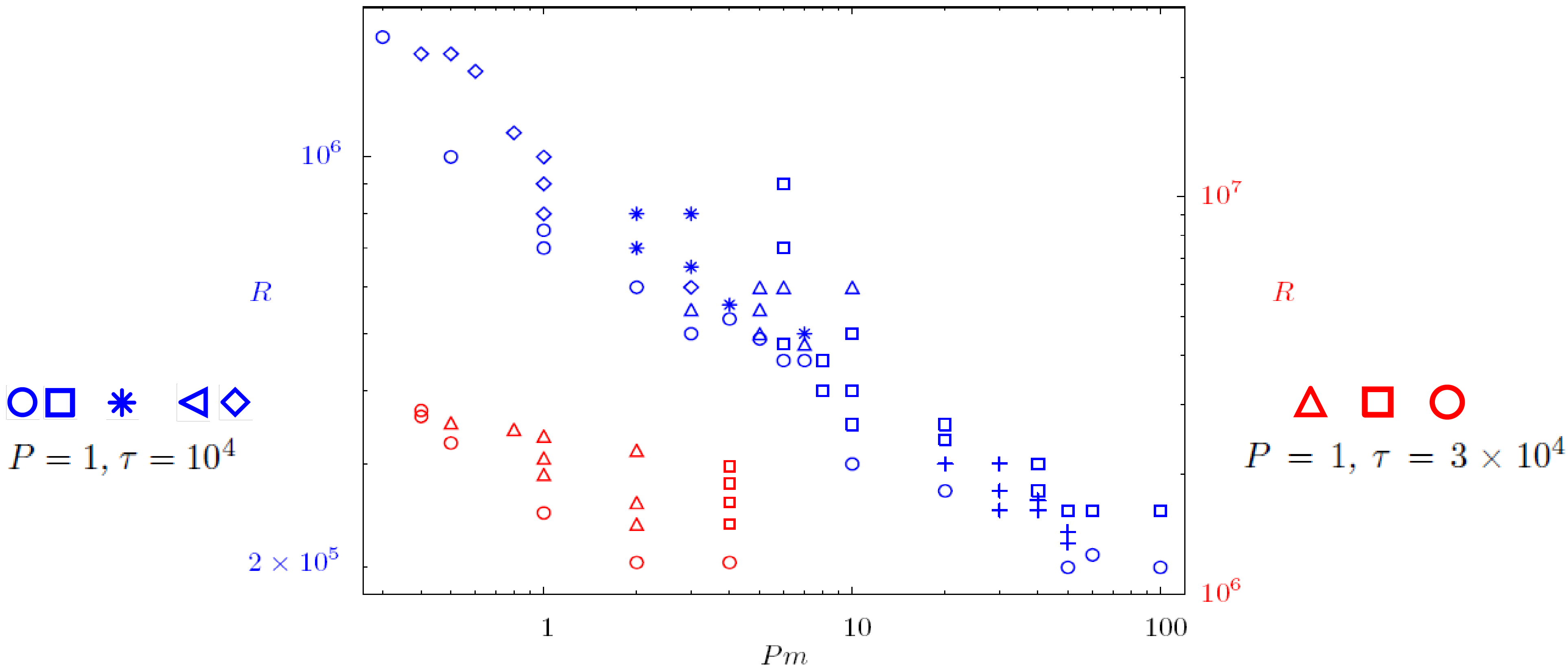
Linear problem:

Galerkin spectral method for the linearised equations leading to an eigenvalue problem for the critical parameters.

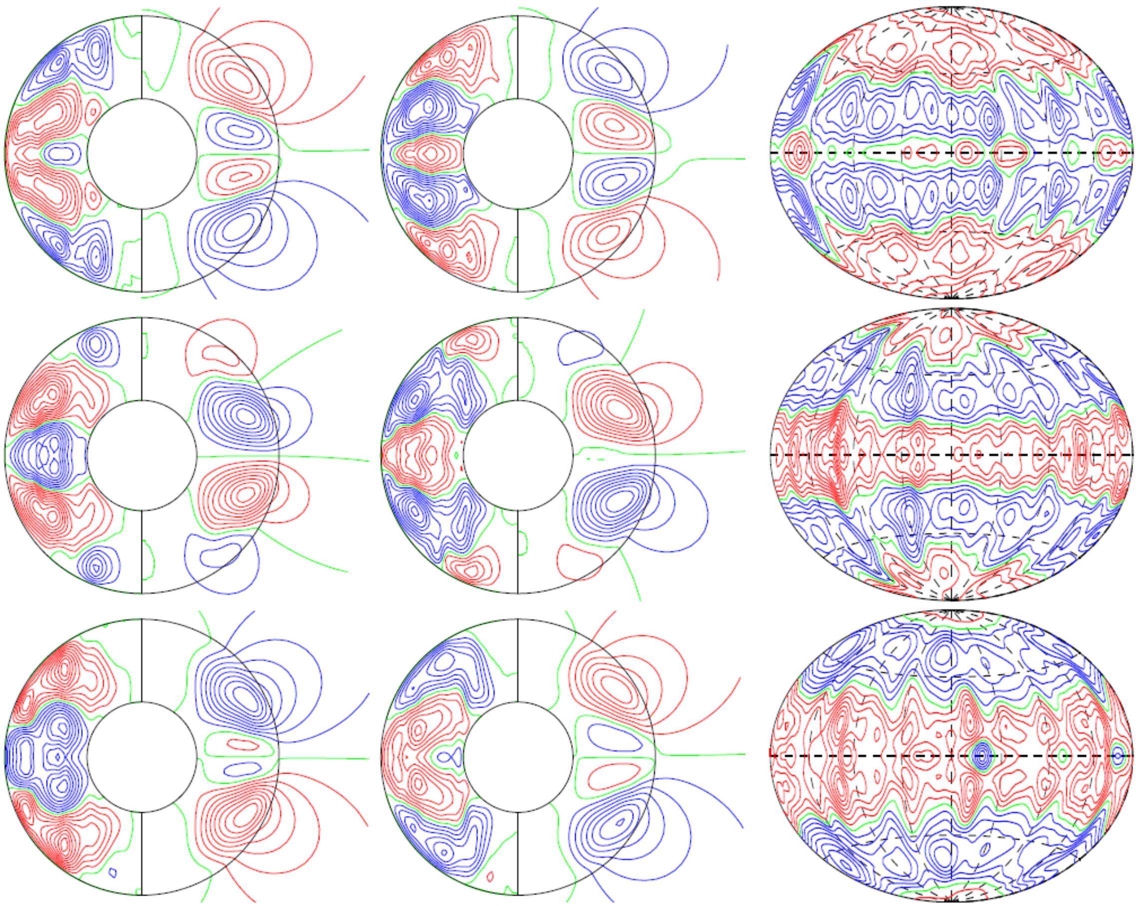
Tilgner, IJNMF, 1999

Types of dynamos in the parameter space (thick shells)

- ○ *No dynamo*
- + □ *Regular and chaotic non-oscillatory dipolar dynamos (at large Pm/P and not far above dynamo onset)*
- ✱ *Oscillatory dipolar dynamos (at values of R larger than those of non-oscillatory dipoles)*
- △ *Hemispherical dynamos – always oscillatory*
- ◇ *Quadrupolar dynamos – always oscillatory*



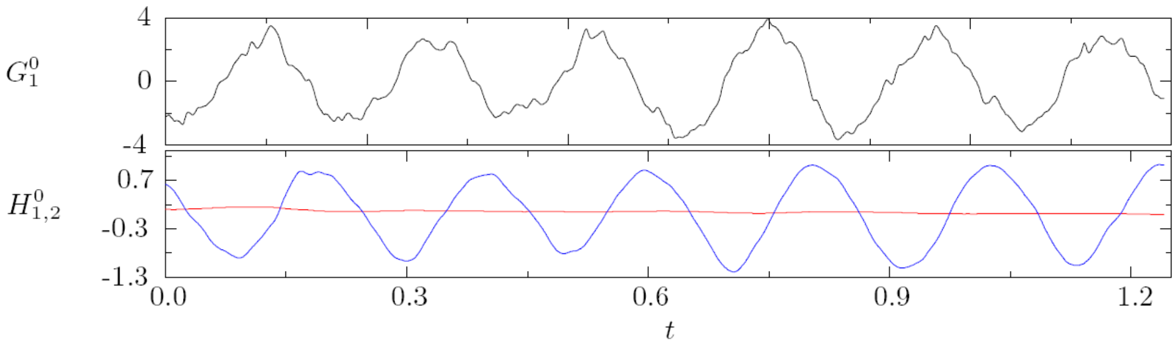
Example of a quadrupolar oscillation (thick shells)



$P = 5, \tau = 5 \times 10^3$
$R = 8 \cdot 10^5, \; P_m = 3$

One period (first then second column)

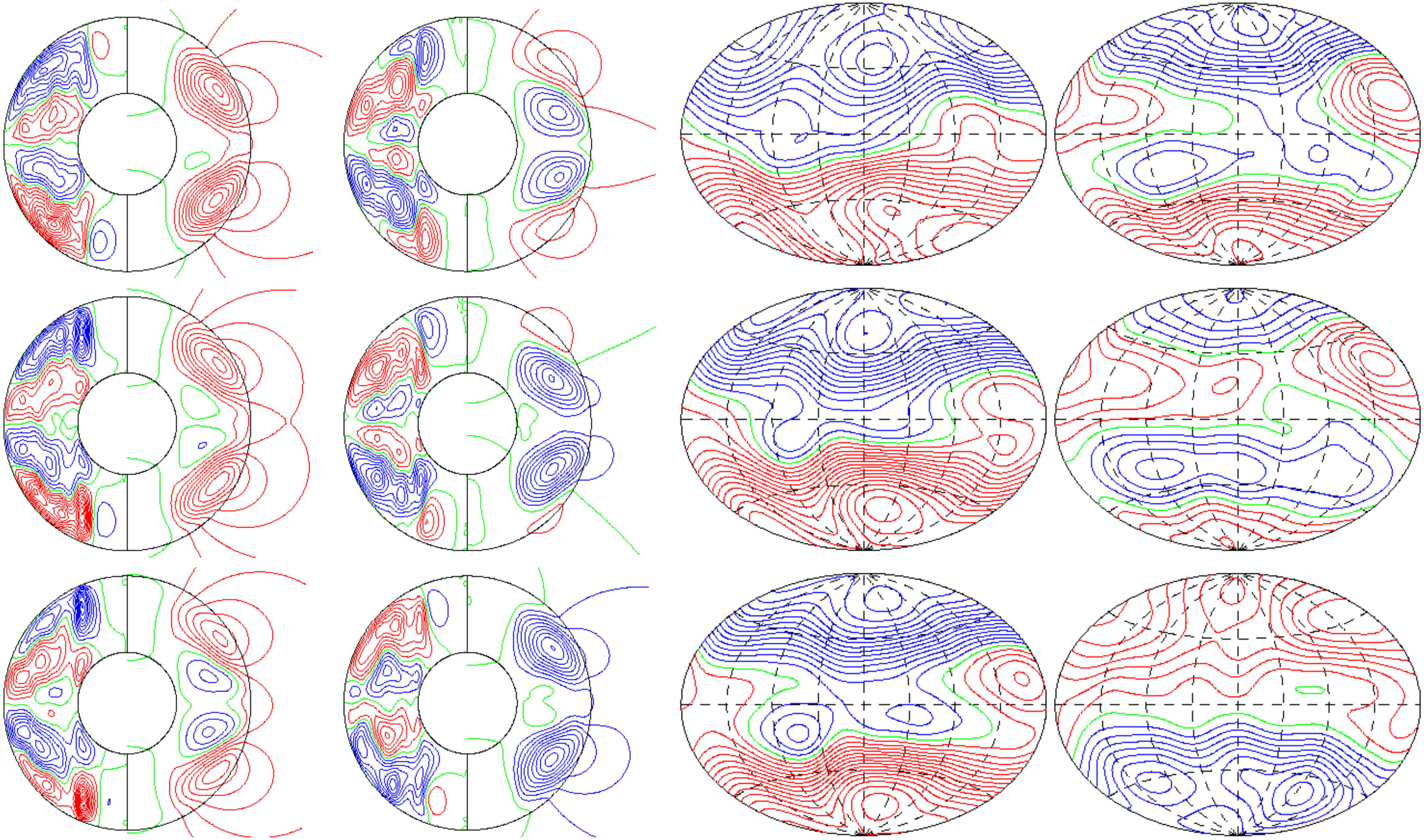
Mean meridional filedlines of constant $\overline{B_\varphi}$ *(left),* $r \sin\vartheta \partial_\theta \overline{h}$ *(right) and radial magn. field.*



Time series of toroidal G_1^0 *and poloidal* H_1^0, H_2^0 *magn. coefficients.*

An example of a dipolar oscillation (thick shells)

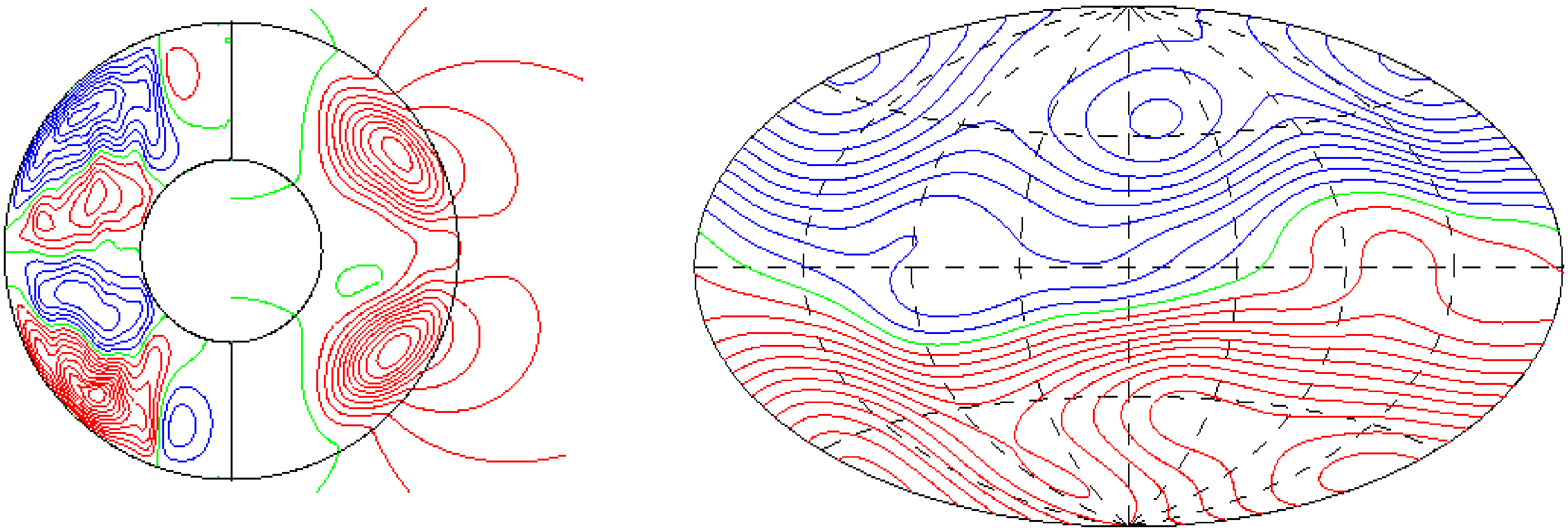
$$R = 3.5 \cdot 10^6,\ \tau = 3 \cdot 10^4,\ P = 0.75 \text{ and } P_m = 0.65$$



Half-period of oscillation (column-by-column)

An example of a dipolar oscillation (thick shells)

$$R = 3.5 \cdot 10^6,\ \tau = 3 \cdot 10^4,\ P = 0.75 \text{ and } P_m = 0.65$$

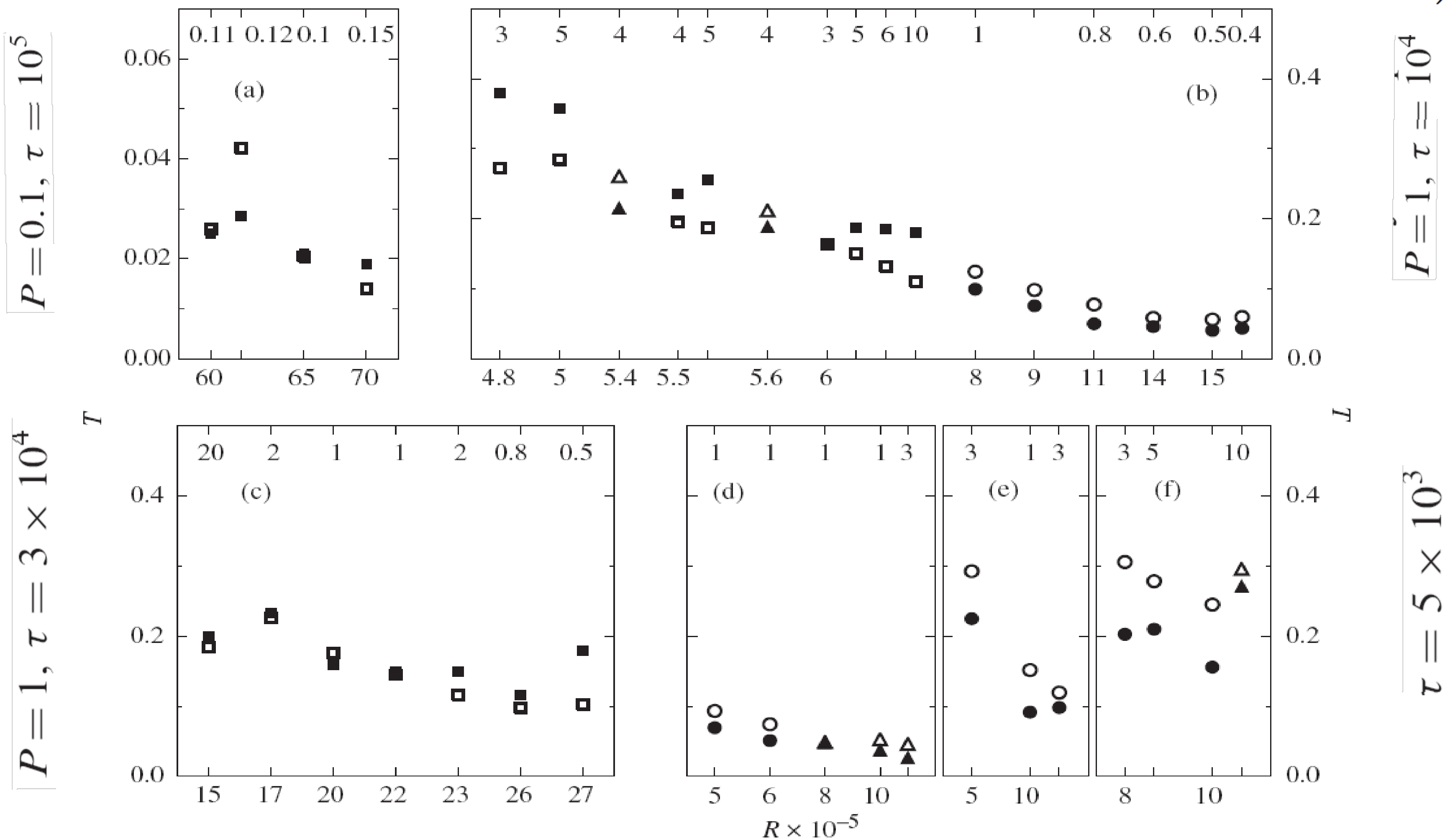


http://www.maths.gla.ac.uk/~rs/res/B/anim.bm.gif

http://www.maths.gla.ac.uk/~rs/res/B/anim.radmagn_2.gif

Fit to a mean-field dynamo wave model (thick shells)

Parker's mean field dynamo wave model: $$T \approx 4\pi^2 \left(P_m \frac{\pi}{3} \langle \check{\boldsymbol{v}} \cdot \nabla \times \check{\boldsymbol{v}} \rangle \sqrt{2\ \overline{E_t}} \right)^{-1/2}$$



Parker, Astrophys. J., 1955. Busse & Simitev, GAFD 2006

***Solid symbols**: numerical data*

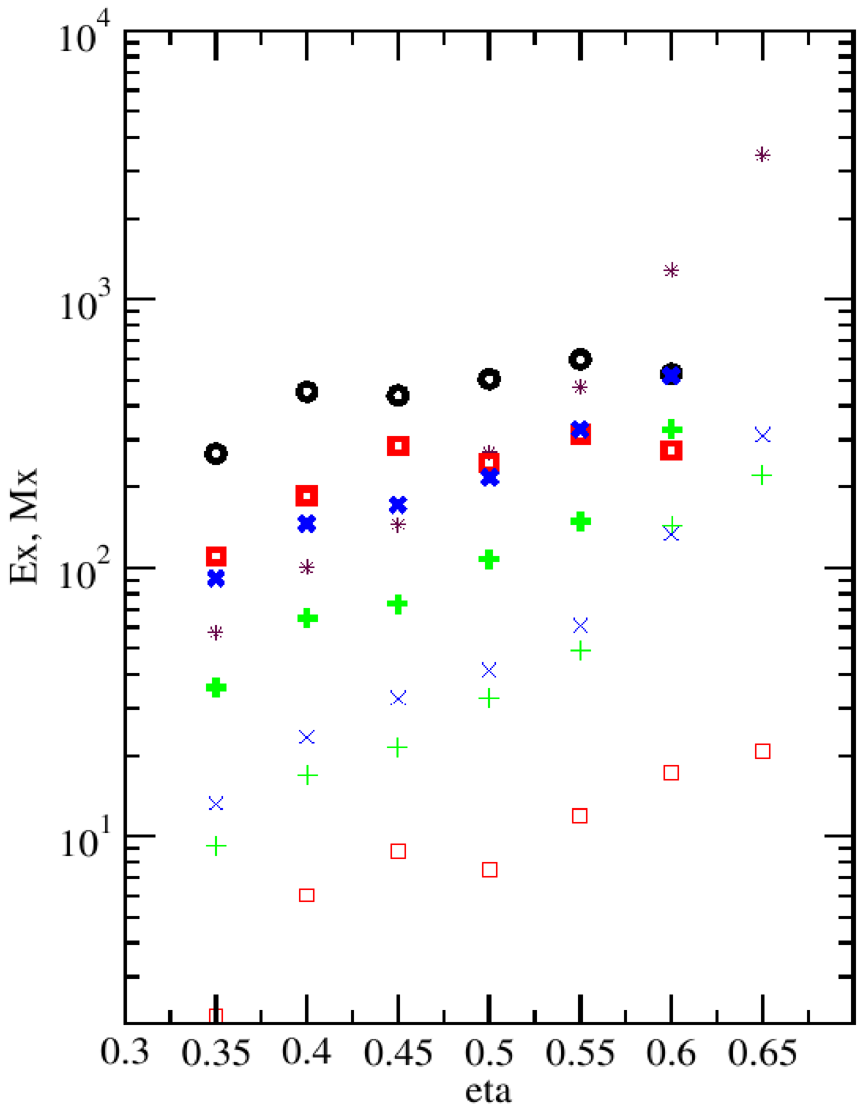
***Empty symbols**: theory*

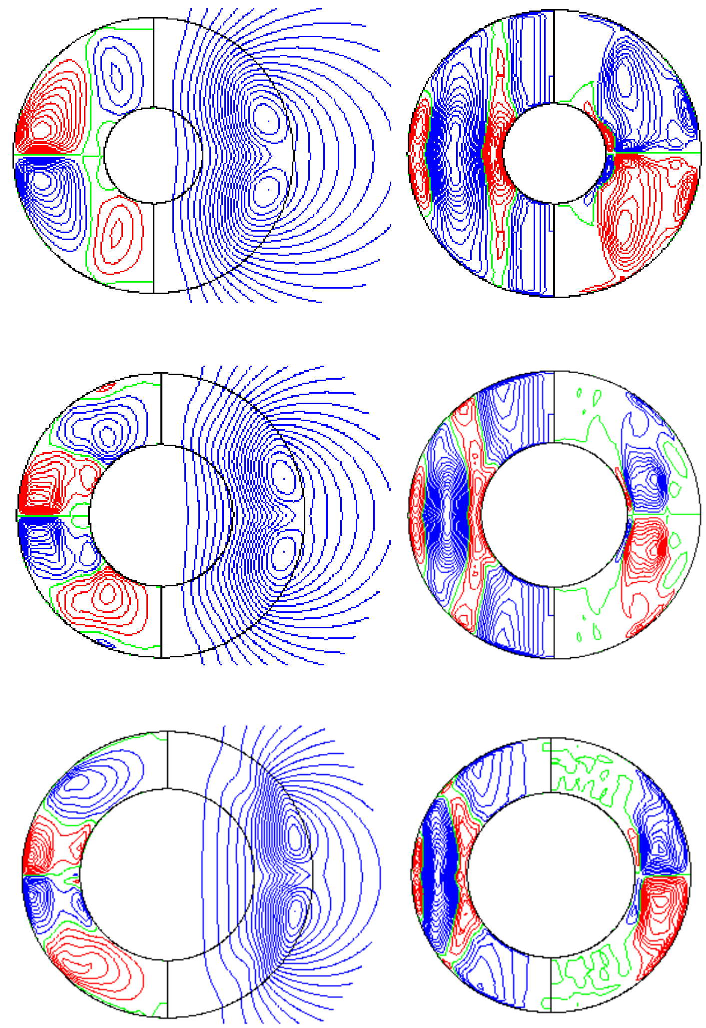
Dependence on the shell thickness

Motivation: *Goudard & Dormy, (EPL 2010)*

$P = 1, \tau = 2000, R = 10^5, P_m = 5$

no-slip boundary conditions





0.35

0.5

0.6

Black *– mean poloidal,* ***Red*** *– mean toroidal,* ***Green*** *– fluct poloidal,* ***Blue*** *– fluct toroidal*

Solid – magnetic energy components; Empty – kinetic energy components

Types of dynamos in the parameter space (thin shells)

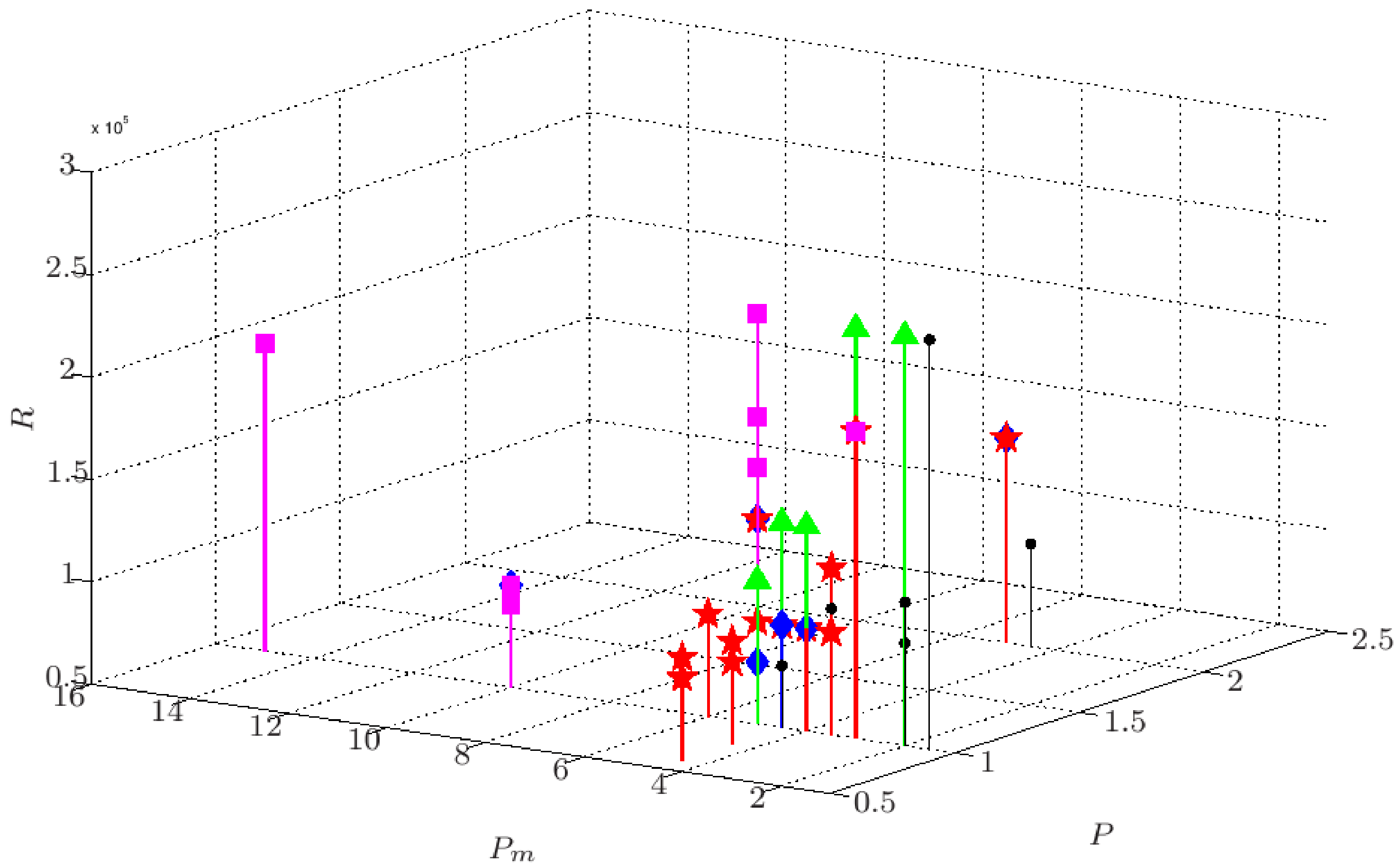
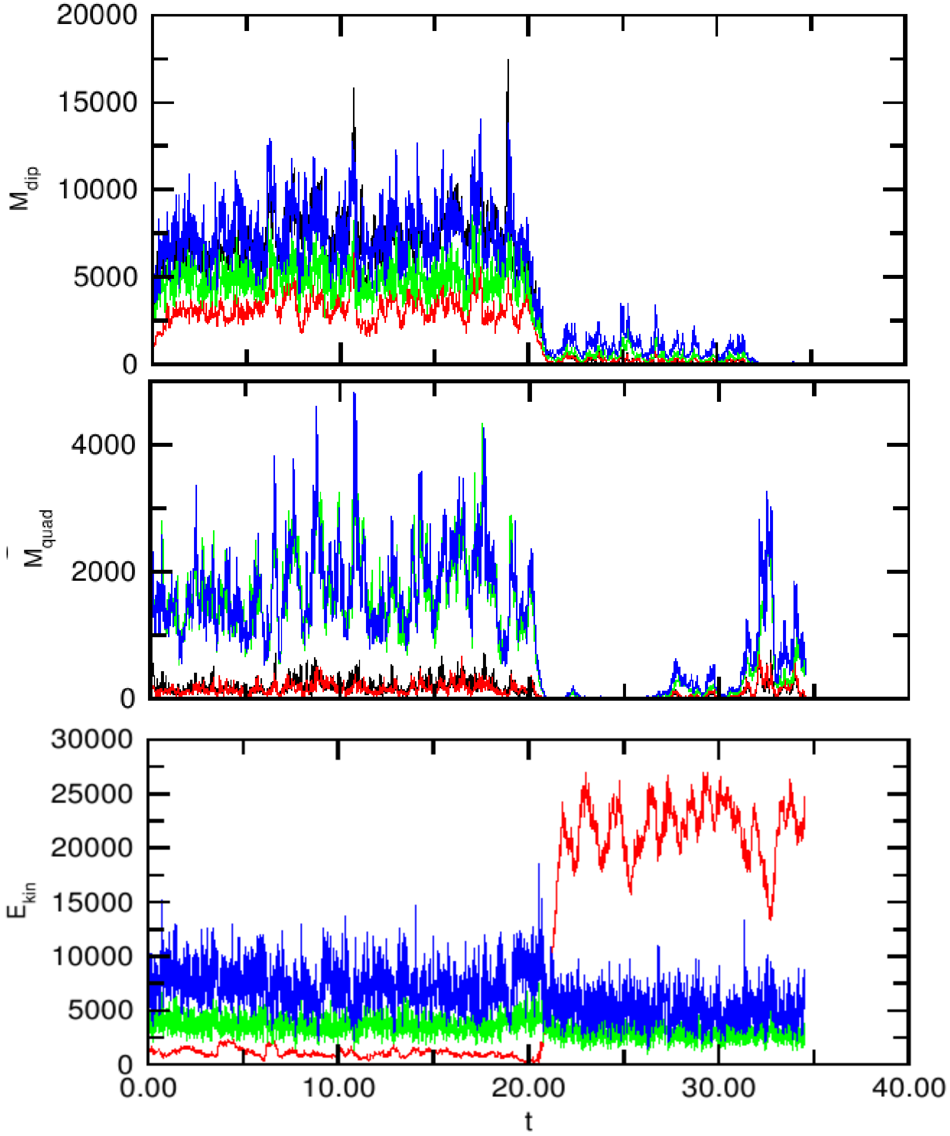


Figure 1: Convection-driven dynamos as a function of R, P and P_m for $\tau = 2000$. Decaying dynamos are indicated by solid black dots, MD dynamos are indicated by blue diamonds, FD dynamos are indicated by red stars, mixed dynamos are indicated by pink squares, and quadrupolar dynamos are indicated by green triangles.

Transition to fluctuating dynamos in thin shells

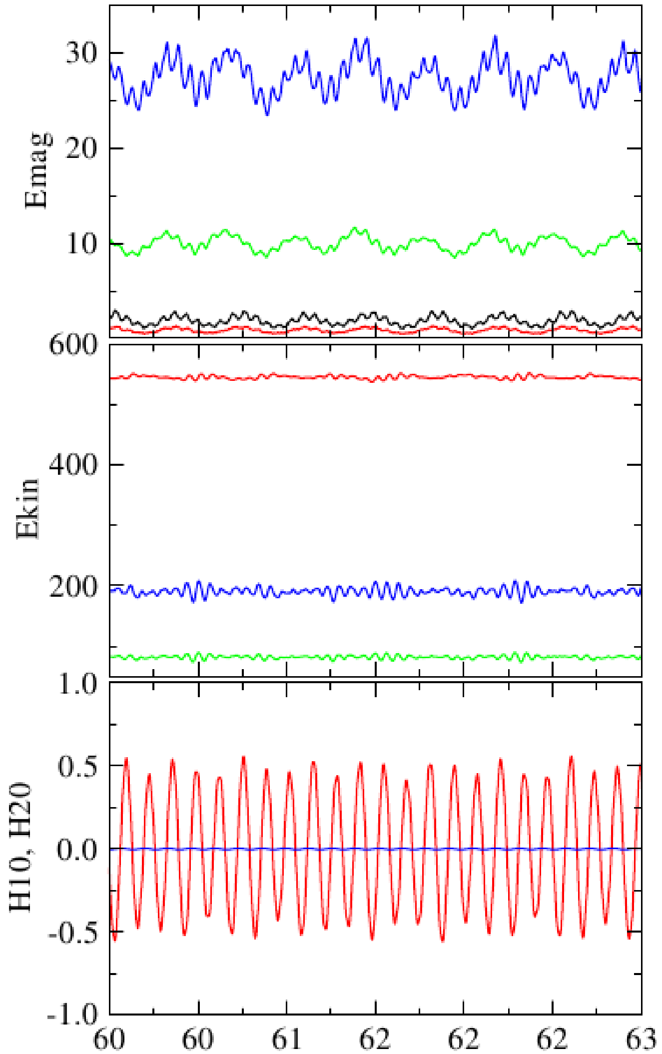


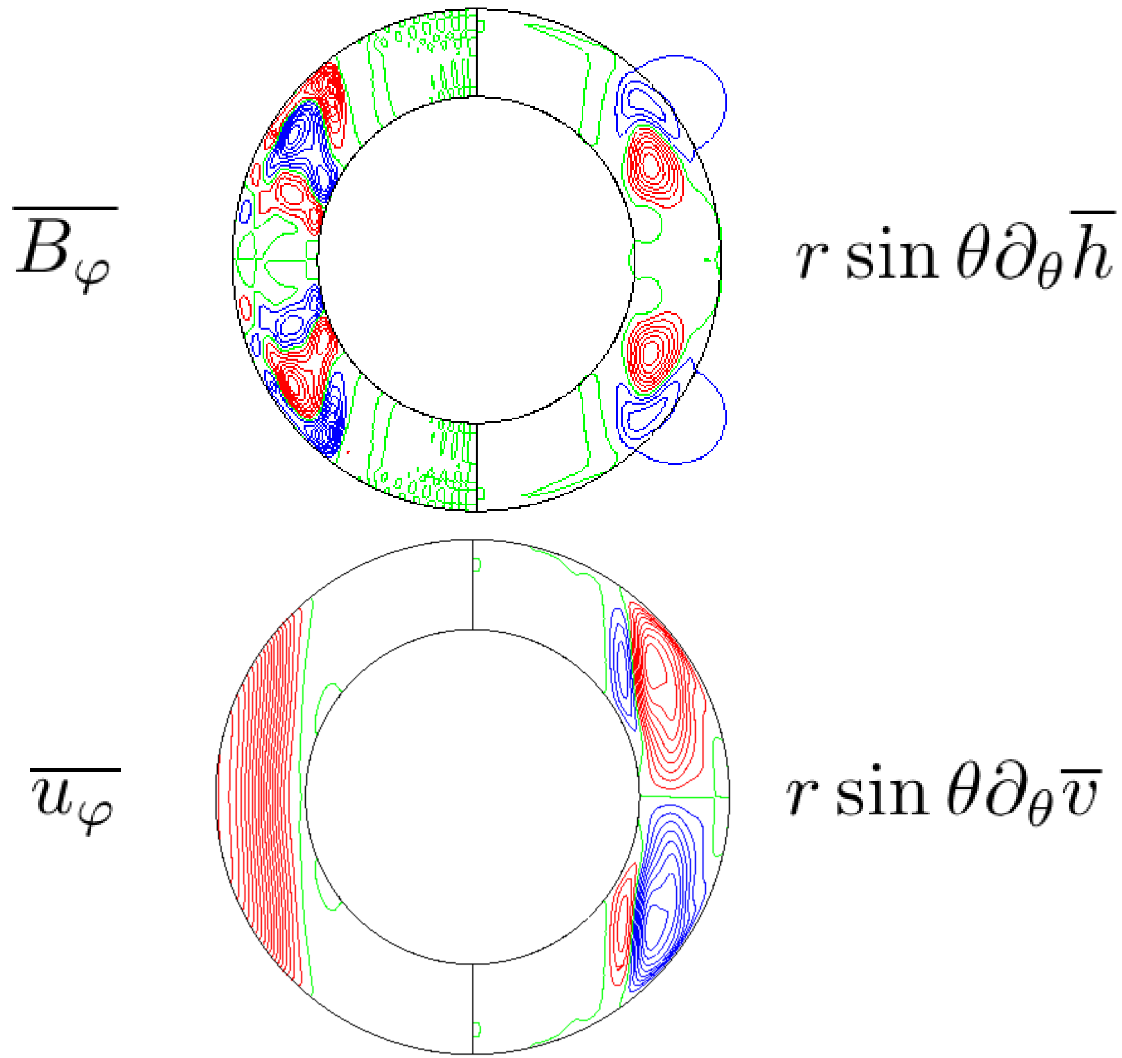
Crucial assumption:

Stress free at outer boundary
No-slip at inner boundary

$$\eta = 0.65, P = 1, \tau = 2000,$$
$$R = 1.5 \times 10^5, P_m = 5$$

Regular dipolar oscillations in thin shells

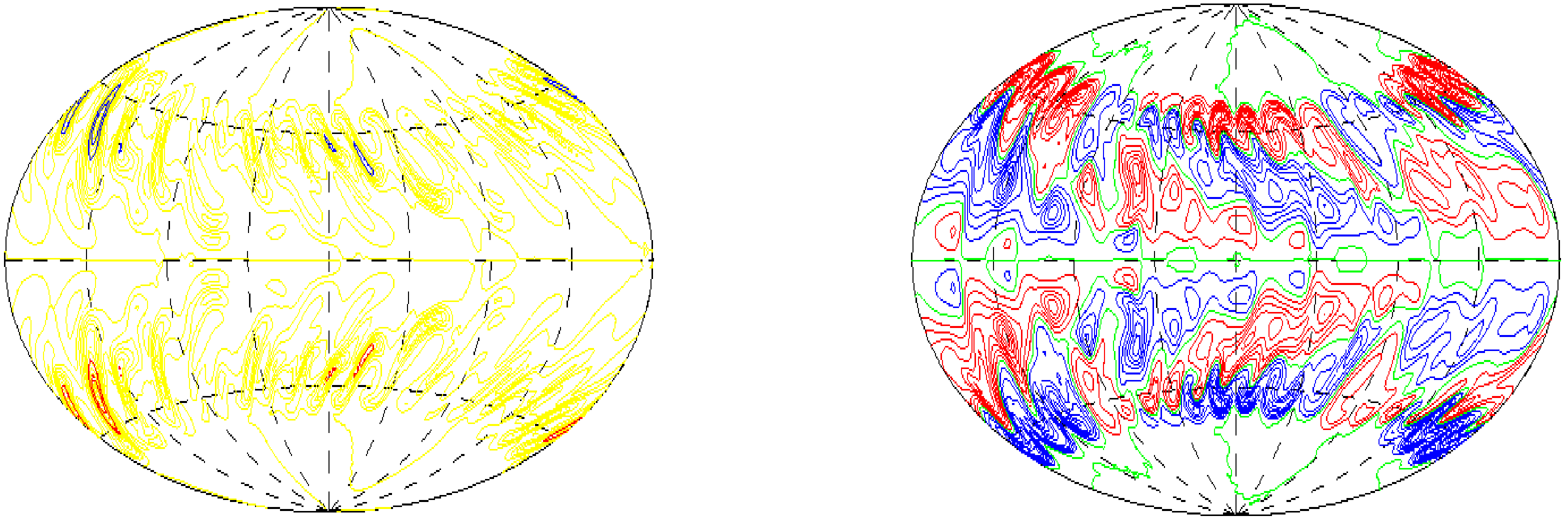




$\eta = 0.65 \quad \mathrm{P} = 1,\ \mathrm{R} = 10^5,\ \tau = 2000,\ \mathrm{P_m} = 4.5,$

No-slip at inner boundary, stress free at outer boundary

Regular dipolar oscillations in thin shells

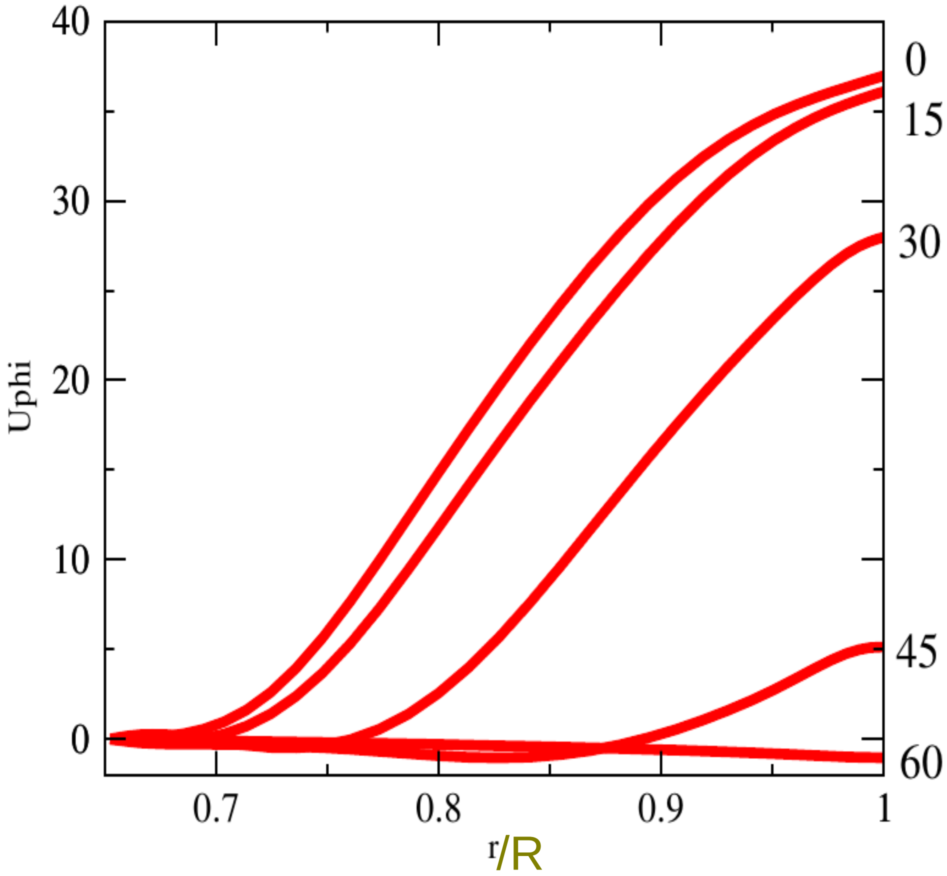


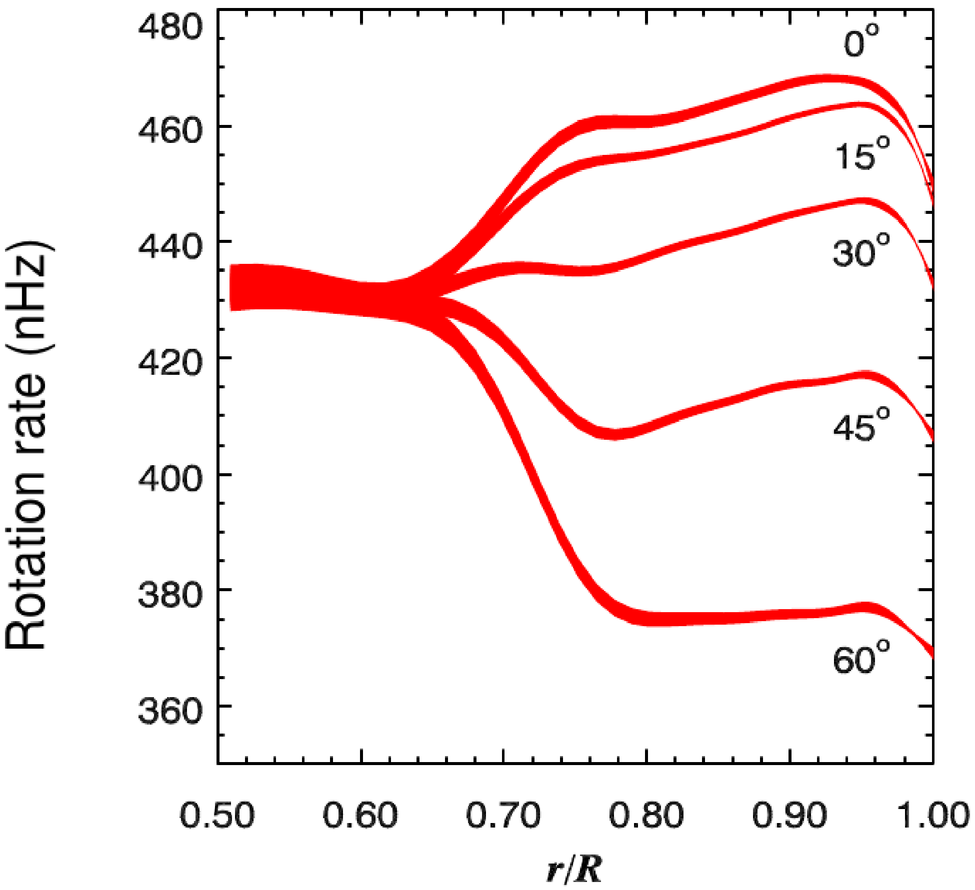
$$\frac{\partial g}{\partial \theta} \quad \text{at } r = r_i + 0.98 \qquad\qquad B_r \quad \text{at } r = r_o$$

$\eta = 0.65 \quad \mathrm{P} = 1,\ \mathrm{R} = 10^5,\ \tau = 2000,\ \mathrm{P_m} = 4.5,$

No-slip at inner boundary, stress free at outer boundary

Radial profiles of the differential rotation





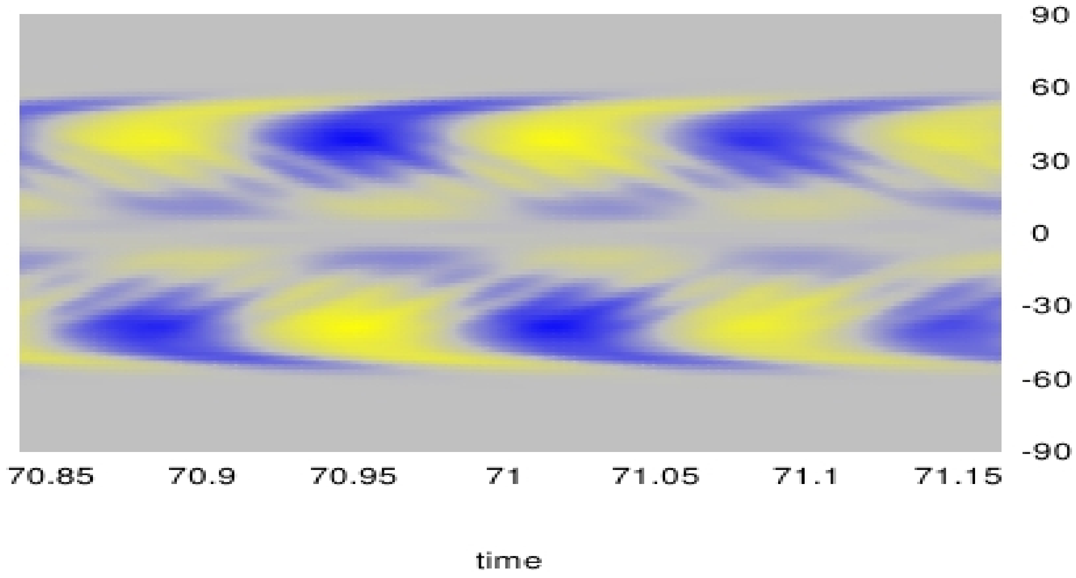
$$\mathrm{P} = 1,\ \mathrm{R} = 10^5,\ \tau = 2000,$$

$$P_m = 4 \qquad \eta = 0.65$$

No-slip at inner boundary,
stress free at outer boundary

Internal Rotation of the Sun
as found by helioseismology,
NSF's National Solar Observatory

Butterfly diagram

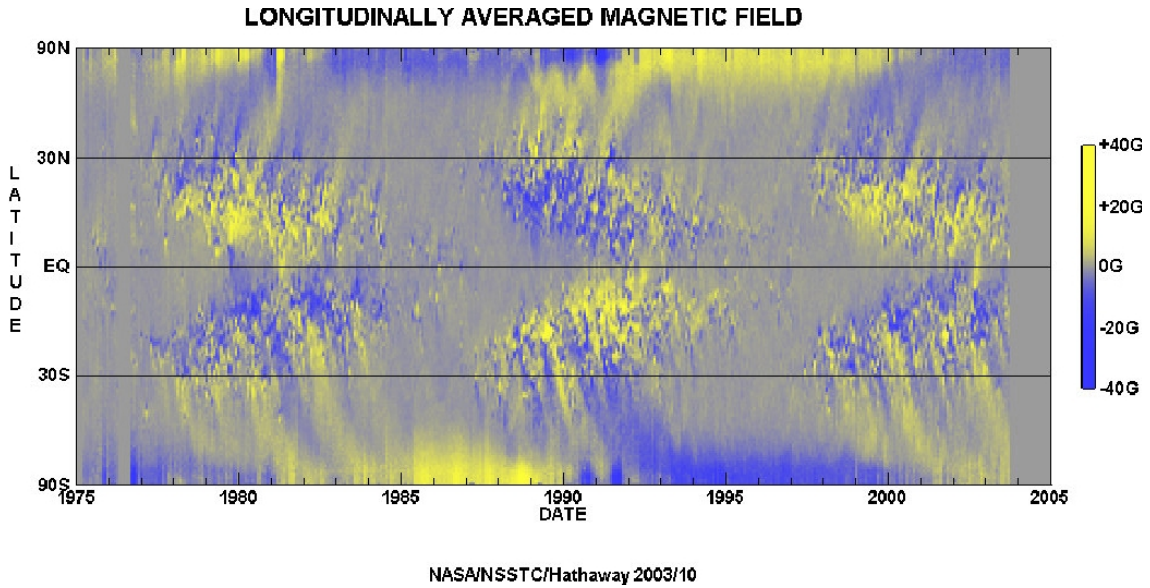


$$\mathrm{P} = 1,\ \mathrm{R} = 10^5,\ \tau = 2000,$$

$$P_m = 4 \qquad \eta = 0.65$$

No-slip at inner boundary,
stress free at outer boundary

$$r\partial_\theta \overline{g} \qquad \text{at } r = r_i + 0.95$$

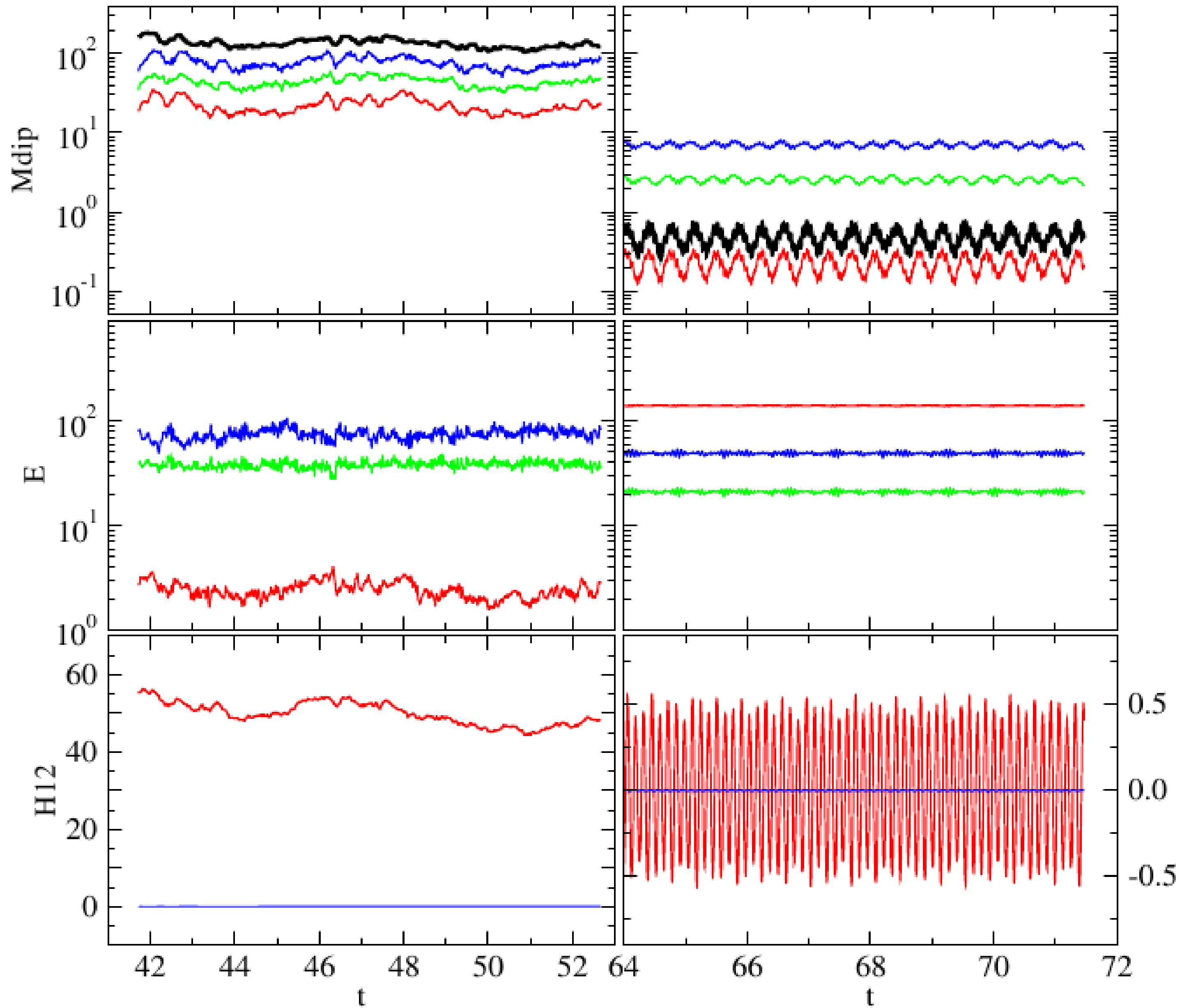


Hathaway, D.H., et al, 2003,
Astrophys. J., 589, 665-670

Bistability of dynamo solutions $\eta = 0.65$

$\mathrm{P} = 1,\ \mathrm{R} = 10^5,\ \tau = 2000,\ \mathrm{P_m} = 4.5,$

No-slip at inner boundary, stress free at outer boundary



Bistability and hysteresis in the MD <==> FD transition (thin shells)

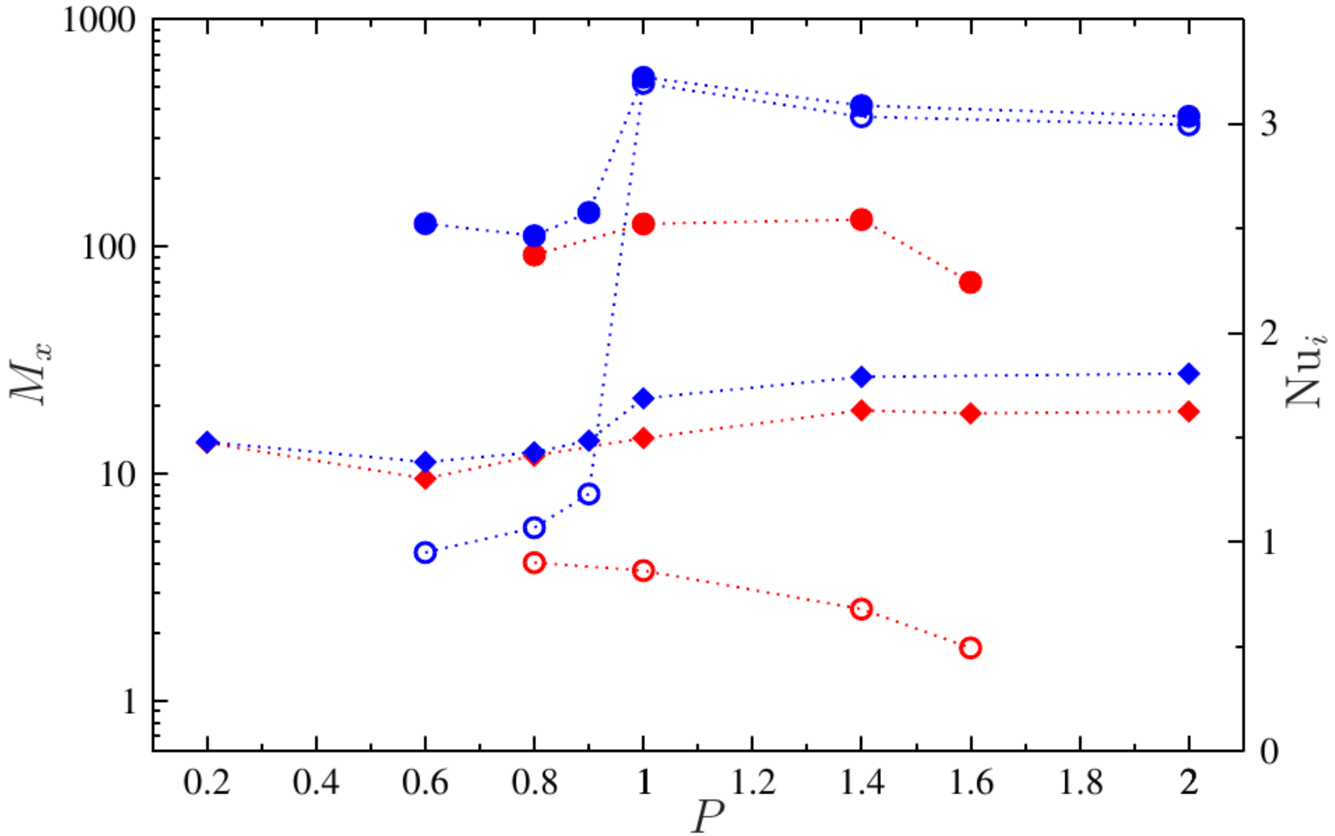
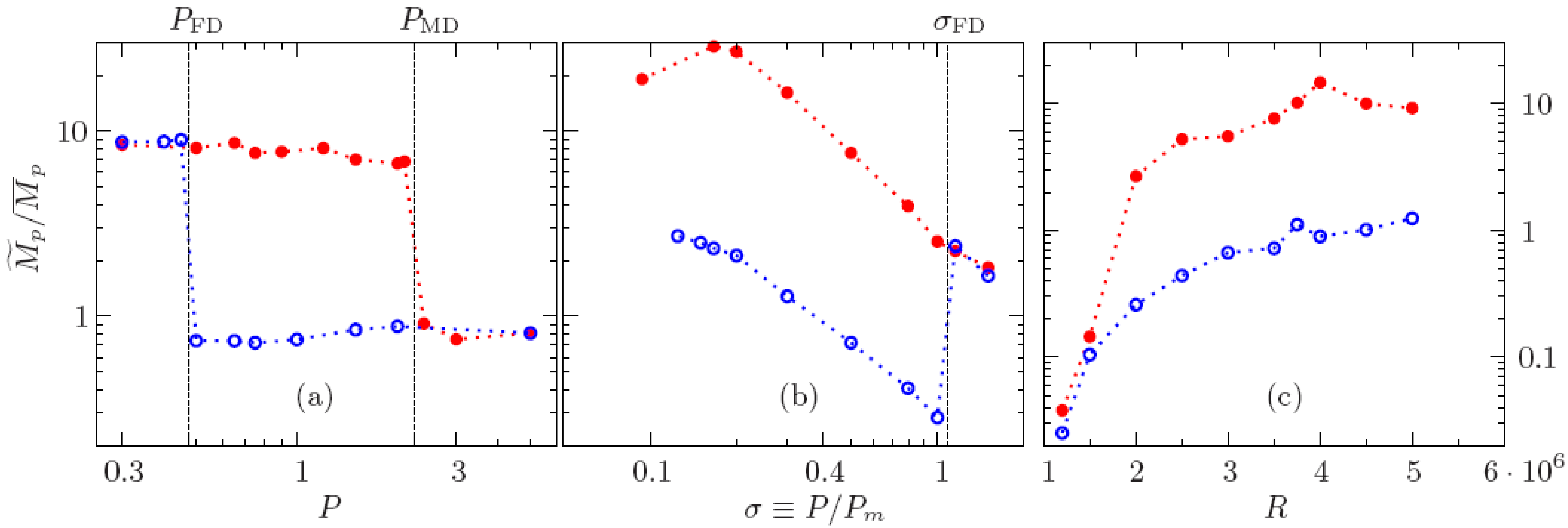


Figure 2: Co-existing nonlinear dynamo solutions at identical parameter values in the case $\eta = 0.65$ $\tau = 2000$, $R = 150000$ and $P/P_m = 0.2$. The fluctuating poloidal magnetic energy component $\widetilde{M}_p$ (solid circles) and the mean poloidal magnetic energy component $\overline{M}_p$ (empty circles) are scaled on the left ordinate, while the Nusselt number at $r = r_i$ (solid diamonds) is scaled on the right ordinate. Cases started from initial conditions

Bistability and hysteresis in the MD <==> FD transition (thick shells)

Bistability and hysteresis *in the ratio of fluctuating poloidal to mean poloidal magn energy*



(a) $R = 3.5 \cdot 10^6 \quad P/P_m = 0.5$

(b) $R = 3.5 \cdot 10^6, \; P = 0.75$

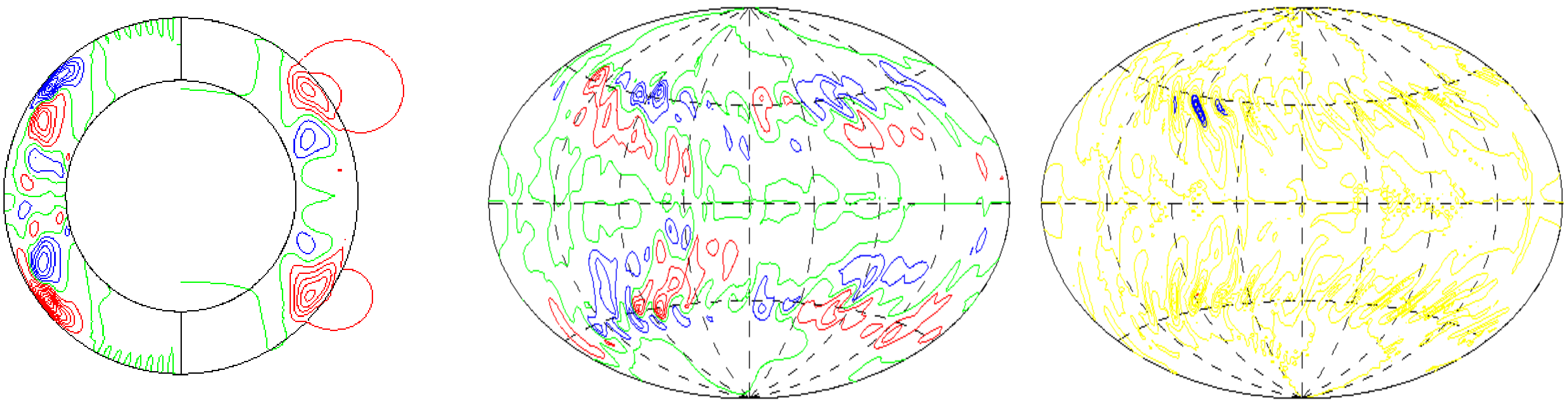
(c) $P = 0.75, \; P_m = 1.5$

in all cases: $\tau = 3 \cdot 10^4$

The coexistence is ***not an isolated phenomenon*** *but can be traced with variation of the parameters.*

$P_{MD} = 2.2$ $\quad P_{FD} = 0.5$
$\sigma_{MD} = 0.07$ $\quad \sigma_{FD} = 1$

Regular dipolar oscillations in thin shells

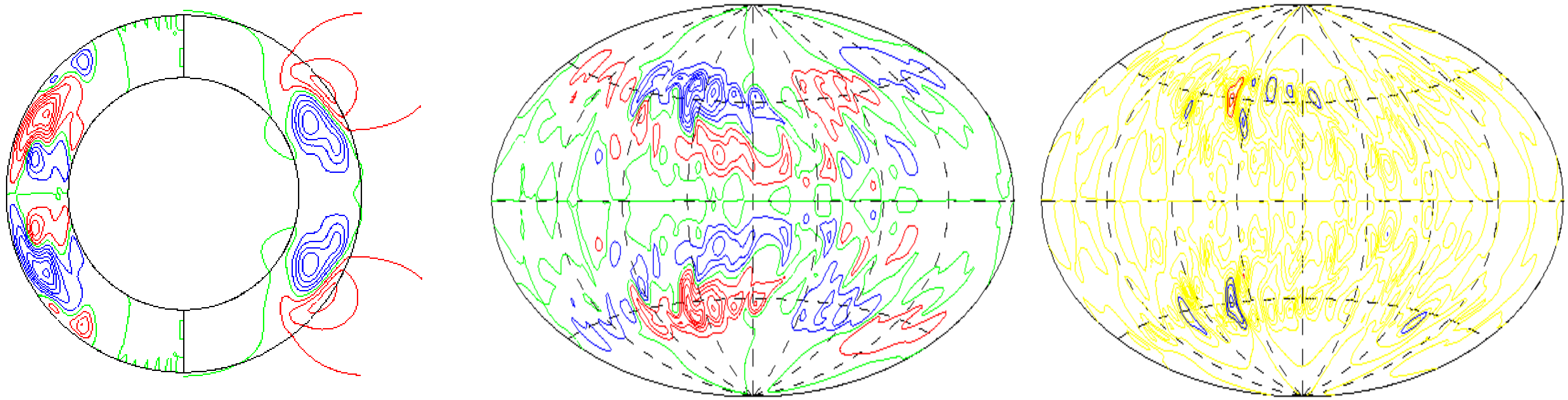


$\overline{B_\varphi}$ $\qquad r \sin\theta \partial_\theta \overline{h}$ $\qquad B_r$ at $r = r_o$ $\qquad \frac{\partial g}{\partial \theta}$ at $r = r_i + 0.98$

$\eta = 0.65$ *P=0.8, \tau= 1750, R=100000, Pm=4*

No-slip at inner boundary, stress free at outer boundary

Regular dipolar oscillations in thin shells



$\overline{B_\varphi}$ $\quad r \sin\theta \partial_\theta \overline{h}$ $\quad B_r$ at $r = r_o$ $\quad \dfrac{\partial g}{\partial \theta}$ at $r = r_i + 0.98$

$\eta = 0.65$ *P=1.2, \tau= 2000, R=100000, Pm=4.5*

No-slip at inner boundary, stress free at outer boundary

Conclusions

- ***Global dynamo models in rotating spherical shells and with the Boussinesq approximation capture the basic first order physics of the convective-dynamo problem***

  Thermally/chemically driven convection,

  Magnetic field generation,

  Lorentz force acts on fluid.

- ***Such models have been relatively successful in modelling the Geodynamo because compressibility is not so important.***

  Westward drift of magnetic structures,

  Reversals and excursions,

  Scaling properties.

We propose to use such models to gain insights into the basic physics of the regular solar magnetic cycle.