Chapter 4

Line and surface integrals

Chapter Summary

Objective	**Tools**
Parametric equations of curves	The position vector of a point on the curve is $(x, y, z) = \mathbf{f}(t), \quad t \in \mathbb{R}$. This is called a *parametric* description of the curve and t is called a *parameter*. This may also be written in component form; if $\mathbf{f} = (f_1, f_2, f_3)$ then $$x = f_1(t),\ y = f_2(t),\ z = f_3(t), \quad t \in \mathbb{R}$$ A simple starting place for finding the parametric equation is try setting $x = t$ and see what the equation for y becomes after substituting $x = t$ into it. You also want to be on the look out for curves which are circular as these are best parameterised by polar coordinates.
Line integrals and work done	In $\mathbb{R}^2$ the work done by moving a particle along the curve C is: $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x, y)dx + Q(x, y)dy\ ,$$ where $P(x, y)$ is the force in the x direction and $Q(x, y)$ is the force in the y direction. To do the integrals on the right parameterise C using parameter t say and change variables on the integrals on the right to integrals involving t. If C is a closed curve you might be able to use Green's theorem instead.
Conservative vector fields and path independence	If there exists a scalar field ϕ such that the vector field $\mathbf{F} = \operatorname{grad}\phi$, we say that $\mathbf{F}$ is *conservative* and ϕ is called a *potential* for $\mathbf{F}$. When $\mathbf{F}$ is conservative then the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent and is equal to $\phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a))$, where $\mathbf{r}(t)$ is the parameterised curve and the parameter t satisfies $a \leq t \leq b$.

Objective	**Tools**
Green's Theorem	Deals with integrating over closed curves C. Green's Theorem states $$\int_C P(x,y)dx+Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \, .$$ where D is the region enclosed by the curve C. Applying Green's theorem means you are left with a double integral like those in Chapter 2, so you now need to use methods from that chapter to do the integration.
Surface integrals	The surface integrals describe the flux or flow across a surface S as follows, $$\iint_S f(x,y,z)dS =$$ $$\iint_D f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} dx\, dy \, ,$$ where D is the projection of S onto the xy-plane. To do the integral on the right we use methods from Chapter 2 to integrate a double integral.
Gauss' Divergence Theorem	Deals with integrating over closed surfaces. Let V be a closed bounded volume on $\mathbb{R}^3$ with boundary surface S, given with positive (*outward*) orientation. Let $\mathbf{F}$ be a vector field. Then $$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V \operatorname{div}\mathbf{F}\, dx\, dy\, dz \, ,$$ where $\mathbf{n}$ denotes the outward pointing *unit normal* at each point on the surface S. The resulting integral is a triple integral so we need to use methods from Chapter 2 to do the integration. Questions involving this theorem will either require you to first find $\mathbf{F}$ and $\mathbf{n}$, given you know what their dot product is, or $\mathbf{F}$ will be explicitly given. In both cases you need to use Chapter 3 to help you find div$\mathbf{F}$.

4.1 Line integrals of a vector field in two dimensions

(Stewart (Ed. 7): Section 16.2, p1087.)

Instead of integrating over an interval $[a, b]$ we can integrate over a curve C. Such integrals are called *line integrals*. They were invented in the early 19th century to solve problems involving forces, fluid flow and magnetism. Before studying these integrals we recall the notion of parametric equations which allow us to describe the curves we wish to integrate over.

4.1.1 Parametric equations of curves

(Stewart (Ed. 7): Section 13.1, p864.)

The simplest type of vector-valued function has the form $\mathbf{f}\colon I \to \mathbb{R}^2$, where $I \subset \mathbb{R}$. Such a function returns a 2D vector $\mathbf{f}(t)$ for each $t \in I$, which may be regarded as the position vector of some point on the plane.

For example, recall the Section Formula from Level 1. This states that the position vector of any point P on the line through points A and B is

$$\mathbf{p} = \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta},$$

for any scalars α, β. If we define $t = \beta/(\alpha + \beta)$, then this may be rewritten as

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}.$$

As t changes, we get different points on the line through A and B and in particular, $\mathbf{p}(0) = \mathbf{a}$ and $\mathbf{p}(1) = \mathbf{b}$.

In general, a curve, in 2D or 3D space, can be represented as the image of a vector-valued function on an interval I; the position vector of a point on the curve is

$$\mathbf{r} = \mathbf{f}(t), \quad t \in I.$$

This is called a *parametric* description of the curve and t is called a *parameter*. This may also be written in component form; if $\mathbf{r} = (x, y, z)$ and $\mathbf{f} = (f_1, f_2, f_3)$ then

$$x = f_1(t),\ y = f_2(t),\ z = f_3(t), \quad t \in I.$$

Standard types of parametric curve

Circle and ellipse The circle $(x-a)^2+(y-b)^2 = r^2$, having centre (a, b) and radius r, can be parameterised using polar coordinates $x - a = r\cos\theta$ and $y - b = r\sin\theta$. Recall that θ is the angle between the radius and the positive x-axis, measured in an anti-clockwise direction. Hence the circle has parametric form

$$x = a + r\cos\theta,\ y = b + r\sin\theta, \quad \theta \in [0, 2\pi).$$

If this circle were to be thought of as a curve on the xy-plane in 3D space then it would be

$$x = a + r\cos\theta,\ y = b + r\sin\theta,\ z = 0, \quad \theta \in [0, 2\pi).$$

In a similar way, the ellipse

$$\frac{(x-a)^2}{A^2} + \frac{(y-b)^2}{B^2} = 1,$$

has parametric form

$$x = a + A\cos\theta,\ y = b + B\sin\theta, \quad \theta \in [0, 2\pi).$$

Parabola The parabola $y^2 = 4ax$ can be parametrised as

$$x = at^2,\ y = 2at, \quad t \in (-\infty, \infty).$$

To show that the parametric curve is identical to the parabola we must prove that every point on the parametric curve lies on the parabola and vice versa. For any t, let $x = at^2$ and $y = 2at$ then $y^2 = 4a^2t^2 = 4a(at^2) = 4ax$ so that every point on the parametric curve lies on the parabola. Also, given any point (x, y) on the parabola, define $t = y/2a$ so that $y = 2at$ and then $x = y^2/4a = at^2$, so that (x, y) also lies on the parametric curve.

Line We have already seen that

$$\mathbf{r} = (1-t)\mathbf{a} + t\mathbf{b}, \quad t \in [0, 1],$$

is the parametric form of the line segment joining $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$. This may also be written in component form as

$$x = (1-t)a_1 + tb_1,\ y = (1-t)a_2 + tb_2,\ z = (1-t)a_3 + tb_3, \quad t \in [0, 1].$$

Also, if one is given a point $\mathbf{a}$ on the line and a direction vector $\mathbf{d}$ for the line then the parametric form is

$$\mathbf{r} = \mathbf{a} + t\mathbf{d}, \quad t \in \mathbb{R}.$$

4.1.2 Differentiation of vector-valued functions

Let us imagine that $\mathcal{C}$ is the path taken by a particle and t is time. The vector $\mathbf{r}(t)$ is the position vector of the particle at time t and $\mathbf{r}(t+h)$ is the position vector at a later time $t + h$. The *average velocity* of the particle in the time interval $[t, t+h]$ is then

$$\frac{\text{displacement vector}}{\text{length of time interval}} = \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

See Figure 4.1. In terms of the components of $\mathbf{r}$ this is $\left(\frac{x(t+h)-x(t)}{h}, \frac{y(t+h)-y(t)}{h}, \frac{z(t+h)-z(t)}{h}\right)$. If each of the

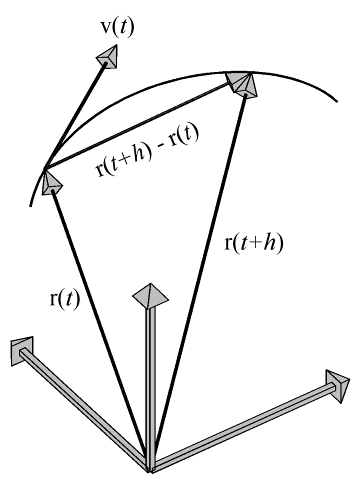

Figure 4.1: Velocity

scalar functions x, y and z are differentiable, then this vector has a limit

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (\dot{x}, \dot{y}, \dot{z}),$$

which is the *instantaneous velocity* of the particle $\mathbf{v} = \frac{d\mathbf{r}}{dt}$. This means that (if the motion is smooth) then

$$\frac{d\mathbf{r}}{dt} = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \frac{d}{dt}\mathbf{r}(t) = \dot{\mathbf{r}}(t).$$

This vector lies along the tangent to the curve at $\mathbf{r}$.

4.1.3 Work Done and line integrals of vector fields

In order to understand how line integrals arise we begin by recalling some basic ideas about *work done*. The *work done* W, by a variable force $f(x)$ in moving a particle from a point a to a point b along the x-axis is

$$W = \int_a^b f(x)dx = \sum f(x)\delta x = \text{Force} \times \text{distance=Work}.$$

We now generalise this idea to a particle moving a long a general curve C and this gives a line integral.

Suppose that the force is given by the vector $\mathbf{F}$ in the direction $\overrightarrow{PR}$ pointing as shown in Figure 4.2. If the force moves the object from P to Q, then the *displacement vector* is $\mathbf{D} = \overrightarrow{PQ}$. The *work done* done by this force is defined to be the product of the component of the force along $\mathbf{D}$ (i.e. $|\mathbf{F}|\cos\theta$), and the distance moved (i.e. $|\mathbf{D}|$), giving:

$$W = |\mathbf{D}||\mathbf{F}|\cos\theta = \mathbf{F}\cdot\mathbf{D}\ ,$$

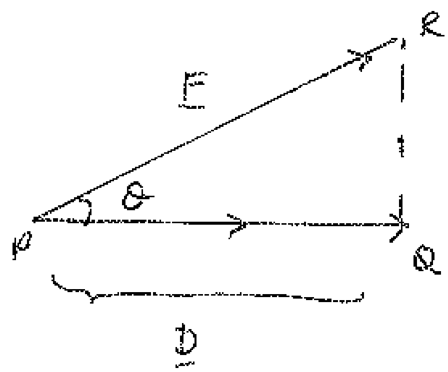

Figure 4.2: The force acting in the $\overrightarrow{PQ}$ direction is $|\mathbf{F}|\cos\theta$

So then if $\mathbf{r}(t) = (x(t), y(t))$ describes the parameterised curve C, it follows that $d\mathbf{r}$ is a small step along that curve and hence

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \text{Force} \times \text{distance}\ .$$

The notation is as follows, $d\mathbf{r} = (dx, dy)$ and $\mathbf{F} = (P(x,y), Q(x,y))$. For the purposes of this section we only consider two dimensions, but this can easily be extended to higher dimensions. So in two dimensions the work done by moving a particle along the curve C is:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P(x,y)dx + Q(x,y)dy\ ,$$

where $P(x,y)$ is the force in the x direction and $Q(x,y)$ is the force in the y direction.

It is usually helpful to parameterise the curve C using a parameter t, say. Starting with a plane curve C the parametric equations are given by

$$x = x(t), \quad y = y(t) \quad a \leq t \leq b$$

thus, $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. So we can change variables on the line integral by writing $d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt$. This gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}dt = \int_a^b P(x(t), y(t))\frac{dx}{dt}dt + \int_a^b Q(x(t), y(t))\frac{dy}{dt}dt\ .$$

Example 4.1 Find the work done by the force $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the curve which runs from $(1, 0)$ to $(0, 1)$ along the unit circle and then from $(0, 1)$ to $(0, 0)$ along the y-axis (see Figure 4.3).

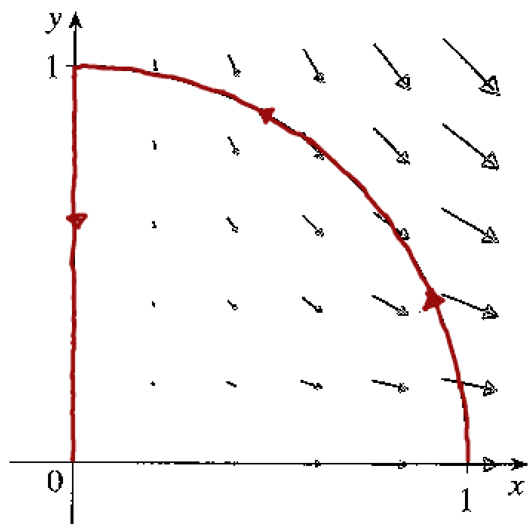

Figure 4.3: Shows the force field $\mathbf{F}$ and the curve C. The work done is negative because the field impedes the movement along the curve.

Solution : Split the curve C into two sections, the curve C_1 and the line that runs along the y-axis C_2. Then,

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}\ .$$

Curve C_1: Parameterise C_1 by $\mathbf{r}(t) = (x(t), y(t)) = (\cos t, \sin t)$, where $0 \leq t \leq \pi/2$ and $\mathbf{F} = (x^2, -xy)$ and $d\mathbf{r} = (dx, dy)$. Hence,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} x^2\,dx - xy\,dy = \int_0^{\pi/2} \cos^2 t\,\frac{dx}{dt}\,dt - \int_0^{\pi/2} \cos t \sin t\,\frac{dy}{dt}\,dt = -\int_0^{\pi/2} 2\cos^2 t \sin t\,dt = -2/3,$$

by applying Beta functions to solve the integral where $m = 2$, $n = 1$ and $K = 1$.

Curve C_2: Parameterise C_2 by $\mathbf{r}(t) = (x(t), y(t)) = (0, t)$, where $0 \leq t \leq 1$. Hence,

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_t^0 0\frac{dx}{dt}dt - \int_1^0 0t\,\frac{dy}{dt}\,dt = 0.$$

So the work done, $W = -2/3 + 0 = -2/3$. Notice the order of limits must reflect the direction along the curve. Work done is negative because the force field impedes the movement along the cure. $\square$

Example 4.2 Evaluate the line integral $\int_C (y^2)dx + (x)dy$, where C is the is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$

Solution :
Parameterise C by $\mathbf{r}(t) = (x(t), y(t)) = (4 - t^2, t)$, where $-3 \le t \le 2$, since $-3 \le y \le 2$. C is illustrated in Figure 4.4. $\mathbf{F} = (y^2, x)$ and $d\mathbf{r} = (dx, dy)$. Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y^2 dx + x dy = \int_{-3}^{2} t^2 \frac{dx}{dt} dt - \int_{-3}^{2} (4 - t^2) \frac{dy}{dt} dt = \int_{-3}^{2} -2t^3 + (4 - t^2) dt = 245/6.$$

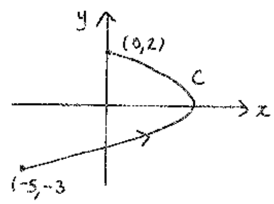

Figure 4.4: Curve C, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, 3)$ to $(0, 2)$.

$\square$

Remark When the curve C is something simple like a straight line then it is often easier to not parameterise the curve and instead use $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x, y)dx + Q(x, y)dy$ as it stands, as we shall see in the following example.

Example 4.3 Evaluate the line integral, $\int_C (x^2 + y^2)dx + (4x + y^2)dy$, where C is the straight line segment from $(6, 3)$ to $(6, 0)$.

Solution : We can do this question without parameterising C since C does not change in the x-direction. So $dx = 0$ and $x = 6$ with $0 \le y \le 3$ on the curve. Hence

$$I = \int_C (x^2 + y^2)0 + (4x + y^2)dy = \int_3^0 24 + y^2 dy = -81.$$

$\square$

4.2 Green's Theorem

(Stewart (Ed. 7): Section 16.4, p1108.)

If the plane curve C is a simple closed curve then we can use Green's Theorem to calculate the integral. Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane D bounded by C. (See Figure 4.5. We assume that D consists of all points inside C as well as all points on C). In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to moving round C in an **anticlockwise** direction. The region D is always on the left as we move round C. (Warning: if you move round C in the clockwise direction you get negative the integral you get when you go round in the anticlockwise direction).

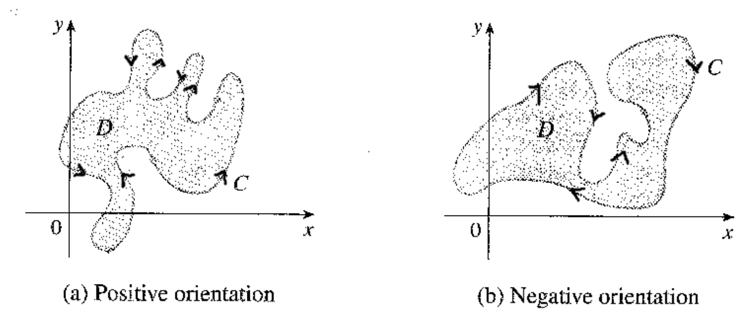

Figure 4.5: Closed curves C.

4.2.1 Green's Theorem in two dimensions

Theorem *Let C be a positively oriented simple closed curve in the plane and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open region that contains D, then*

$$\int_C P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy\ .$$

Proof For simplicity, we will only prove the theorem in the case where the regions are both type I **and** type II. The key element of the proof is to show that

$$\int_C P(x,y)dx = -\iint_D \frac{\partial P}{\partial y}dxdy \quad \text{and} \quad \int_C Q(x,y)dy = -\iint_D \frac{\partial Q}{\partial x}dxdy. \tag{4.1}$$

To prove the first expression in equation 4.1 we view D as an type I domain and let C_1 and C_2 be the lower and upper boundary curves of D. Then

$$\int_C P(x,y)dx = \int_{C_1} P(x,y)dx + \int_{C_2} P(x,y)dx = \int_{C_1} P(x,y)dx - \int_{-C_2} P(x,y)dx \tag{4.2}$$

The curves C_1 and C_2 can be expressed parametrically as, C_1 is $x = t$, $y = g_1(t)$, where $a \leq t \leq b$ and C_2 is $x = t$, $y = g_2(t)$, where $a \leq t \leq b$. In both cases a is the x-coordinate of C that is on the far left of the domain and B is the point on the far right of C. We can then use this to rewrite equation 4.2.

$$\begin{aligned}
\int_C P(x,y)dx &= \int_a^b P(t,g_1(t))\frac{dx}{dt}dt - \int_a^b P(t,g_2(t))\frac{dx}{dt}dt \\
&= \int_a^b P(t,g_1(t))dt - \int_a^b P(t,g_2(t))dt = -\int_a^b \left[P(t,y)\right]_{y=g_1(t)}^{y=g_2(t)} dt \\
&= -\int_a^b \left[\int_{g_1(t)}^{g_2(t)} \frac{\partial P}{\partial y}dy\right] dt = -\int_a^b \left[\int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}dy\right] dx = -\iint_D \frac{\partial P}{\partial y}dxdy
\end{aligned}$$

The proof of the second expression in equation 4.1 is obtained in a similar way, but treating D as type II. □

Example 4.4 Use Green's Theorem to evaluate $\int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4+1})dy$, where C is the circle $x^2 + y^2 = 9$.

Solution : $P(x,y) = 3y - e^{\sin x}$ and $Q(x,y) = 7x + \sqrt{y^4+1}$. Hence, $\frac{\partial Q}{\partial x} = 7$ and $\frac{\partial P}{\partial y} = 3$. Applying Green's Theorem where D is given by the interior of C, i.e. D is the disc such that $x^2 + y^2 \leq 9$.

$$\int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4+1})dy = \iint_D (7-3)dxdy = \int_0^{2\pi}\int_0^3 4r dr d\theta = \int_0^{2\pi} 18 d\theta = 36\pi$$

The D integral is solved by using polar coordinates to describe D. □

Example 4.5 Evaluate $\int_C (3x - 5y)dx + (x - 6y)dy$, where C is the ellipse $\frac{x^2}{4} + y^2 = 1$ in the anticlockwise direction. Evaluate the integral by (i) Green's Theorem, (ii) directly.

Solution : i) **Green's Theorem:** $P(x,y) = 3x - 5y$ and $Q(x,y) = x - 6y$. Hence, $\frac{\partial Q}{\partial x} = 1$ and $\frac{\partial P}{\partial y} = -5$. Applying Green's Theorem where D is given by the interior of C, i.e. D is the ellipse such that $x^2/4 + y^2 \leq 1$.

$$\int_C (3x-5y)dx + (x+6y)dy = \iint_D (1-(-5))dxdy = 6\iint_D 1 dxdy = 6 \times (\text{Area of the ellipse}) = 6 \times 2\pi.$$

See chapter 2 for calculating the area of an ellipse by change of variables for a double integral.

(i) **Directly:** Parameterise C by $x(t) = 2\cos t$, $y(t) = \sin t$, where $0 \leq t \leq 2\pi$.

$$\begin{aligned}
I = &\ \textstyle\int_0^{2\pi} (6\cos t - 5\sin t)\frac{dx}{dt}dt + (2\cos t - 6\sin t)\frac{dy}{dt}dt \\
= &\ \textstyle\int_0^{2\pi} 18\cos t\sin t + 10\sin^2 t + 2\cos^2 t\, dt \\
= &\ 0 + 40\textstyle\int_0^{\pi/2}\sin^2 t\, dt + 8\int_0^{\pi/2}\cos^2 t\, dt \\
= &\ 0 + 40\frac{\pi}{2}(1/2) + 8\frac{\pi}{2}(1/2) = 12\pi.
\end{aligned}$$

The integrals are calculated using symmetry properties of $\cos t$ and $\sin t$ and beta functions. Using the table of signs below we see that $\int_0^{2\pi}\sin^2 t = 4\int_0^{\pi/2}\sin t\, dt$ etc.

Quadrant	1	2	3	4	Total
$\cos t$	+	−	−	+	
$\sin t$	+	+	−	−	
$\cos t \sin t$	+	−	+	−	0
$\sin^2 t$	+	+	+	+	4
$\cos^2 t$	+	+	+	+	4

□

4.3 Line integrals of vector fields in $\mathbb{R}^3$

In section 4.1 we considered line integrals of the form $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C was a curve in $\mathbb{R}^2$ this formula works equally well in three dimensions. Take $\mathbf{F} = (P(x,y,z), Q(x,y,z), R(x,y,z))$ and $d\mathbf{r} = (dx, dy, dz)$ and now consider C as a curve in $\mathbb{R}^3$. This integral now represents the work done to move a particle along a curve in $\mathbb{R}^3$.

4.3.1 Independence of path

(Stewart (Ed. 7): Section 16.3, p1099.)

If we consider two curves C_1 and C_2 (which are called *paths*) with the same initial point A and the same end point B. We know that in general $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ since the work done to move a particle along one route to B may not be the same as another route because the forces impeding or assisting the movement may be different in the two regions of space where the two routes lie. However when $\mathbf{F} = \nabla\phi$ for some continuous scalar-valued function ϕ then we have $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ and we say that the line integral is **path independent**.

When we can find a scalar-valued function ϕ such that $\mathbf{F} = \nabla\phi$ we say that $\mathbf{F}$ is a **conservative vector field** and we denote ϕ as the **potential function**. The fact that $\mathbf{F}$ is conservative ensures the independence of path and gives an integral that is related to the Fundamental Theorem of Calculus which states $\int_a^b F'(x)dx = F(b) - F(a)$. In fact, we have the following theorem,

Theorem *Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let ϕ by a differentiable scalar function of 2 or 3 variables whose gradient vector $\nabla\phi$ is continuous on C. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla\phi \cdot d\mathbf{r} = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a))$$

Proof

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \nabla\phi(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}dt \\ &= \int_a^b \left(\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}\right) dt = \int_a^b \frac{d}{dt}\phi(\mathbf{r}(t))\, dt \qquad \text{(by the chain rule)} \\ &= \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)). \qquad \square.\end{aligned}$$

4.3.2 Conservative vector fields

(Stewart (Ed. 7): Section 16.3, p1099.)

The grad of every smooth scalar field is a vector field. It is natural to ask whether *all* vector fields are the grad of some scalar field. In general, the answer is "no", but we can characterise those vector fields $\mathbf{F}$ for which this is the case. If there exists a scalar field ϕ such that the vector field $\mathbf{F} = \operatorname{grad}\phi = \nabla\phi$, we say that $\mathbf{F}$ is *conservative* and ϕ is called a *potential* for $\mathbf{F}$.

These names reflect an application of this notion in physics; a force (vector field) that does not expend energy is said to be conservative and can be written as the gradient of a potential energy (scalar field). Gravitational force is a conservative force, whereas friction is not.

We have already seen (See Example 3.10) that $\operatorname{curl}\operatorname{grad}\phi = \mathbf{0}$ for all smooth scalar fields ϕ. This means that if $\mathbf{F} = \operatorname{grad}\phi$ for some ϕ then $\operatorname{curl}\mathbf{F} = \mathbf{0}$. This is a *necessary* condition for $\mathbf{F}$ to be conservative (i.e. if $\mathbf{F}$ is to be conservative then we must have $\operatorname{curl}\mathbf{F} = \mathbf{0}$). For a vector field that is defined everywhere then it is also *sufficient* (i.e. if $\mathbf{F}$ is defined everywhere and $\operatorname{curl}\mathbf{F} = \mathbf{0}$ then $\mathbf{F}$ is conservative).

Example 4.6 Vector fields $\mathbf{V}$ and $\mathbf{W}$ are defined by

$$\mathbf{V} = (2x - 3y + z, -3x - y + 4z, 4y + z)$$

$$\mathbf{W} = (2x - 4y - 5z, -4x + 2y, -5x + 6z)\ .$$

One of these is conservative while the other is not. Determine which is conservative and denote it by **F**. Find a potential function ϕ for **F** and evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r} \ ,$$

where C is the curve from A(1,0,0) to B(0,0,1) in which the plane $x + z = 1$ cuts the hemisphere given by $x^2 + y^2 + z^2 = 1$, $y \geq 0$.

Solution : We have

$$\begin{aligned} \operatorname{curl} \mathbf{V} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - 3y + z & -3x - y + 4z & 4y + z \end{vmatrix} \\ &= \big(0, 1, 0\big) \neq \mathbf{0}. \end{aligned}$$

Since $\operatorname{curl} \mathbf{V} \neq \mathbf{0}$, $\mathbf{V}$ is **NOT** conservative.

We have

$$\begin{aligned} \operatorname{curl} \mathbf{W} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - 4y - 5z & -4x + 2y & -5x + 6z \end{vmatrix} \\ &= \big(0, 0, 0\big) = \mathbf{0}. \end{aligned}$$

Since $\operatorname{curl} \mathbf{W} = \mathbf{0}$, $\mathbf{W}$ **is** conservative.

Suppose that $\operatorname{grad} \phi = \mathbf{W}$. Then

$$\frac{\partial \phi}{\partial x} = 2x - 4y - 5z, \tag{1}$$
$$\frac{\partial \phi}{\partial y} = -4x + 2y, \tag{2}$$
$$\frac{\partial \phi}{\partial z} = -5x + 6z. \tag{3}$$

Integrating (1) with respect to x, holding the other variables constant, we get

$$\phi = \int_{y,z \text{ fixed}} 2x - 4y - 5z \, dx = x^2 - 4yx - 5zx + A(y, z),$$

where A is an arbitrary function. Substituting this expression into (2) gives,

$$-4x + \frac{\partial A}{\partial y} = -4x + 2y, \quad \text{i.e. } \frac{\partial A}{\partial y} = 2y,$$

and therefore

$$A(y, z) = \int_{z \text{ fixed}} (2y) \, dy = y^2 + B(z),$$

where B is an arbitrary function, giving

$$\phi = x^2 - 4yx - 5zx + y^2 + B(z).$$

Finally, substituting this into (3) gives

$$-5x + \frac{dB}{dz} = -5x + 6z, \quad \text{i.e. } \frac{dB}{dz} = 6z,$$

so that $B = 3z^2 + C$, where C is a constant. Hence, by taking $C = 0$ we obtain a potential

$$\phi = x^2 - 4yx - 5zx + y^2 + 3z^2.$$

Notice that the potential function is not unique; we may always add an arbitrary constant to a potential and it remains a potential.

So the line integral is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \operatorname{grad} \phi \cdot d\mathbf{r} = \phi(0,0,1) - \phi(1,0,0) = 3 - 1 = 2.$$

□

4.4 Surface integrals

(Stewart (Ed. 7): Section 16.7, p1134.)

Instead of integrating over a domain that lies in the xy-plane as we did when doing double integration in Chapter 2 we can integrate over a domain that is a surface in $\mathbb{R}^3$, this gives a surface integral.

Consider a crop growing on a hillside S, suppose that the crop yield per unit surface area varies across the surface of the hillside and that it has the value $f(x, y, z)$ at the point (x, y, z). We may then ask what is the total yield of the crop over the whole surface of the hillside, a surface integral will give the answer to this question. Let a small element of surface ΔS contain the point (x, y, z). Then assuming that f is well behaved the contribution to the total crop from this small element of surface is $f(x, y, z)\Delta S$. Summing over all elements of surface and taking the limit as $\Delta S \to 0$ we obtain the *surface integral of f over the surface S*.

$$\iint_S f(x, y, z) dS \ .$$

Evaluating a surface integral We need to relate ΔS to the area of an element at the base $\Delta x \Delta y$ as shown in Figure 4.6. For a curved surface this relationship changes with x and y. In the special case where the surface S can be expressed as $z = z(x, y)$, or $\mathbf{r} = (x, y, z(x, y))$ the plane tangent to the surface at a point approximates a small piece of surface very well. The vector tangent to the surface in the x-direction is $\mathbf{r}_x = (1, 0, \frac{\partial z}{\partial x})$. The vector tangent to the surface in the y-direction is $\mathbf{r}_y = (0, 1, \frac{\partial z}{\partial y})$. The area of the tangent plane is then the area of the plane with sides of length Δx and Δy and direction given by the two tangent vectors. The area is the area of a parallelogram, with sides $\Delta x \mathbf{r}_x$ and $\Delta y \mathbf{r}_y$. The area of a parallelogram is given by the magnitude of the cross product which is given by

$$\Delta S \approx |\mathbf{r}_x \Delta x \times \mathbf{r}_y \Delta y| = |\mathbf{r}_x \times \mathbf{r}_y| \Delta x \Delta y = \left|(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1)\right| \Delta x \Delta y = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \Delta x \Delta y.$$

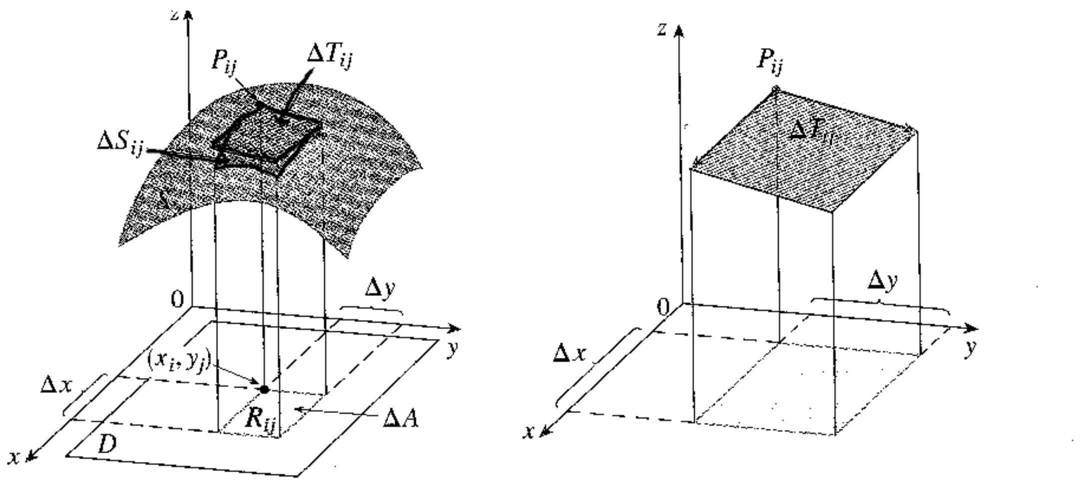

Figure 4.6: Shows the surface S and the tangent plane.

4.4.1 Rule for evaluating surface integrals

Using the above explanation we can replace dS by $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy$ in the surface integral

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\;dy\;,$$

where D is the projection of S onto the xy-plane.

Example 4.7 Evaluate

$$\iint_S z^2\;dS$$

where S is the hemisphere given by $x^2+y^2+z^2=1$ with $z\geq 0$.

Solution : We first find $\frac{\partial z}{\partial x}$ etc. These terms arise because $dS=\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}\,dxdy$. Since this change of variables relates to the surface S we find these derivatives by differentiating both sides of the surface $x^2+y^2+z^2=1$ with respect to x, giving $2x+2z\frac{\partial z}{\partial x}=0$. Hence, $\frac{\partial z}{\partial x}=-x/z$. Similarly, $\frac{\partial z}{\partial y}=-y/z$. Hence,

$$\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}=\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}=1/z.$$

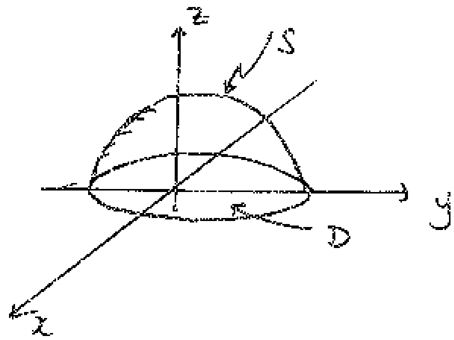

Figure 4.7: Shows the hemisphere S and the projection D onto the xy-plane.

Then the integral becomes the following, where D is the projection of the surface, S, onto the xy-plane. i.e. $D = \{(x, y) : x^2 + y^2 \leq 1\}$. (See Figure 4.7)

$$\begin{aligned}
\iint_S z^2 dS &= \iint_D z^2 \frac{1}{z}\, dxdy \\
&= \iint_D \sqrt{1 - x^2 - y^2}\, dxdy \\
&= \int_0^{2\pi} d\theta \int_0^1 \sqrt{1 - r^2}\, r\, dr \\
&= -\int_0^{2\pi} d\theta \int_1^0 \frac{1}{2}\sqrt{u}\, du \\
&= \int_0^{2\pi} \frac{1}{3}\, d\theta \\
&= 2\pi/3.
\end{aligned}$$

□

Remark A surface integral can also be used to calculate the area of a surface S.

$$\iint_S 1\ dS = \text{Area of surface S}$$

An intuition for this can be obtained be thinking about the crop analogy again. If the crop density is 1kg/square metre ($f = 1$), and the total crop is 65kg ($\iint_S 1\ dS = 65$), then the area of the crop is 65 square metres (Area of S=65).

Example 4.8 Find the area of the ellipse cut on the plane $2x + 3y + 6z = 60$ by the circular cylinder $x^2 + y^2 = 2x$.

Solution : The surface S lies in the plane $2x+3y+6z = 60$ so we use this to calculate $dS = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\, dxdy$. Differentiating the equation for the plane with respect to x gives,

$$2 + 6\frac{\partial z}{\partial x} = 0 \quad \text{thus,}\ \frac{\partial z}{\partial x} = -1/3.$$

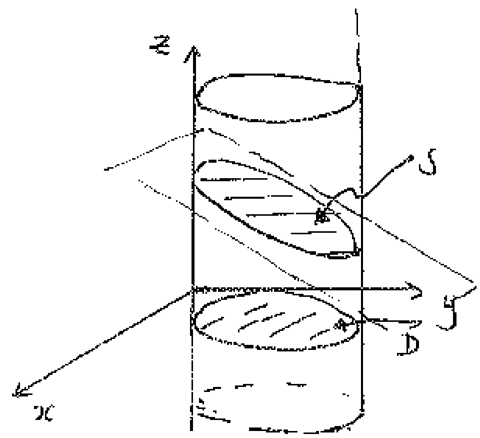

Figure 4.8: A sketch of the surface S and the projection onto the xy-plane.

Differentiating the equation for the plane with respect to y gives,

$$3 + 6\frac{\partial z}{\partial y} = 0 \quad \text{thus, } \frac{\partial z}{\partial y} = -1/2.$$

Hence,

$$\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} = \sqrt{1 + \frac{1}{9} + \frac{1}{4}} = 7/6.$$

Then the area of S is found be calculating the surface integral over S for the function $f(x, y, z) = 1$. The the projection of the surface, S, onto the $x-y$-plane is given by $D = \{(x, y) : x^2 - 2x + y^2 = (x-1)^2 + y^2 \leq 1\}$ (See Figure 4.8). Hence the area of S is given by

$$\begin{aligned}
\iint_S 1\, dS &= \iint_D 1\frac{7}{6}\, dxdy \\
&= \frac{7}{6}\iint_D 1\, dxdy \\
&= \frac{7}{6} \times \text{ Area of D} = \frac{7}{6}\pi.
\end{aligned}$$

Note, since D is a circle or radius 1 centred at $(1, 0)$ the area of D is the area of a unit circle which is π. □

4.5 Surface integral of a vector field

4.5.1 Normal direction to a surface

(Stewart (Ed. 7): Section 16.7, p1140.)

In order to define *surface integrals of vector fields*, we need to consider *orientable surfaces* (2 -sided). The möbius strip is an example of a nonorientable surface (1-sided). (You can construct a möbius strip by taking a long strip of paper and give a half twist and the tape the two short ends together.) We use the *normal* to the surface to give the surface orientation. The *normal* to the surface at a given point is the direction perpendicular to the tangent plane at that point. There are two possible normals, one points in the opposite direction to the other. So there are two possible orientations for any orientable surface (see Figure 4.9).

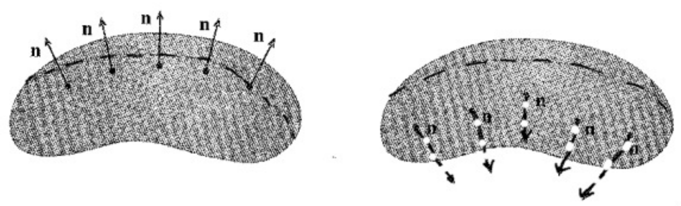

Figure 4.9: The two orientations of an orientable surface.

Remarks

1. For a surface in the form $f(x,y,z)=0$ the *normal vector* is given by

$$\mathbf{n} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

2. For a surface in the form $z = z(x,y)$ the *normal vector* is given by

$$\mathbf{n} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$$

This one follows from the fact that $\mathbf{r}_x \times \mathbf{r}_y$ is normal to the vectors $\mathbf{r}_x$ and $\mathbf{r}_y$ which lie in the tangent plane (see section 4.3).

Examples

1. For the plane $2x+7y+3z=50$ we have $f(x,y,z) = 2x+7y+3z-50 = 0$, so the normal is,

$$\mathbf{n} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2,7,3)$$

as expected.

2. For the sphere $x^2+y^2+z^2-a^2=0$, the normal is, $(2x,2y,2z)$ or (x,y,z) or $(x/a, y/a, z/a)$ i.e. along the radius vector from the centre of the sphere.

4.5.2 Surface integral of a vector field

Imagine a fluid flowing through a surface S. (Think of S as an imaginary fishing net, so it doesn't impede the flow). $\mathbf{F}$ is the force field and it is related to the velocity and density of the fluid flowing through the surface. A measure of the total flux (flow) across the surface is given by

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS,$$

where $\mathbf{n}$ is the *unit normal*. $\mathbf{F}\cdot\mathbf{n}\, dS$ tells us the mass of fluid flowing across a region dS in the direction of $\mathbf{n}$.

Remark Some books use the alternative notation

$$\iint \mathbf{F} \cdot \mathbf{dS}$$

for $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$. Notice in the alternative notation that $\mathbf{dS}$ is a vector.

4.5.3 Gauss' Divergence Theorem

(Stewart (Ed. 7): Section 16.9, p1152.)

Gauss' Divergence Theorem will help us calculate $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$ There are some similarities between Green's Theorem on $\mathbb{R}^2$ and Gauss' Divergence Theorem in $\mathbb{R}^3$ in the following respects:

Green's Theorem: Line integral round a boundary curve C of a *closed* region in $\mathbb{R}^2$=Double integral over the *enclosed* 2-dimensional region.

Gauss' Theorem: Surface integral over a boundary surface S of a *closed* region in $\mathbb{R}^3$=Triple integral over the *enclosed* 3-dimensional region.

Remark Note that for Green's Theorem the curve **must** be a *closed curve* and for Gauss' Theorem the surface **must** be a *closed surface*. Gauss's Divergence Theorem is named after Gauss (1777-1855) who discovered it during his work on electrostatics. In Eastern Europe the Divergence Theorem is known as Ostrogradsky's Theorem after the Russian mathematician who also discovered and published this result in 1826.

Gauss' Divergence Theorem

Theorem *Let V be a closed bounded volume on $\mathbb{R}^3$ with boundary surface S, given with positive (*outward*) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region containing V. Then*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \operatorname{div} \mathbf{F}\, dx\, dy\, dz\ ,$$

where $\mathbf{n}$ denotes the outward pointing unit normal *at each point on the surface S.*

Proof As with Green's theorem we only consider simple domains where V is a simple xy-solid and yz-solid and zx-solid a more general solid is to difficult to present here. Let $\mathbf{F} = (F_1(x,y,z), F_2(x,y,x), F_3(x,y,z))$ then we can express the main statement of the theorem as

$$\iint_S (F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}) \cdot \mathbf{n}\, dS = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dV$$

so suffices to prove

$$\iint_S F_1(x,y,z)\mathbf{i} \cdot \mathbf{n}\, dS = \iiint_V \left(\frac{\partial F_1}{\partial x} \right) dV \tag{4.3}$$

and similarly for the F_2 and F_3 terms. Since the proofs of all three equalities are similar, we will prove only the third.

Suppose that V has upper surface $z = g_2(x, y)$, lower surface $z = g_1(x, y)$, and projection D onto the xy-plane. Let S_1 denote the lower surface, S_2 the upper surface, and S_3 denote the lateral surface. If the upper surface and lower surface meet then there is no lateral surface. The proof will allow for both cases.

$$\iiint_V \frac{\partial F_3}{\partial z}\, dV = \iint_D \left[\int_{g_1(x,y)}^{g_2(x,y)} \frac{\partial F_3}{\partial z}\, dz\right] dA = \iint_D [F_3(x, y, z)]_{z=g_1(x,y)}^{z=g_2(x,y)}\, dA$$

hence,

$$\iiint_V \frac{\partial F_3}{\partial z}\, dV = \iint_D F_3(x, y, g_2(x, y)) - F_3(x, y, g_1(x, y))\, dA.$$

We now want to evaluate the surface integral in equation 4.3 by integrating over each surface of S separately. If there is a lateral surface S_3, then at each point of the surface $\mathbf{k} \cdot \mathbf{n} = 0$. Thus,

$$\iint_{S_3} F_3(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = 0.$$

Therefore regardless of whether S has a lateral surface, we can write

$$\iint_S F_3(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = \iint_{S_1} F_3(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS + \iint_{S_2} F_3(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS$$

on the upper surface S_2, the outer normal is upward and given by $\mathbf{n} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right)$, and on the lower surface S_1, the outer normal is a downward normal given by $\mathbf{n} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$, hence

$$\iint_{S_2} F_3(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = \iint_D F_3(x, y, g_2(x, y))\, dA$$

and

$$\iint_{S_2} F_3(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = -\iint_D F_3(x, y, g_1(x, y))\, dA$$

Hence,

$$\iint_S F_3(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = \iint_D F_3(x, y, g_2(x, y)) - F_3(x, y, g_1(x, y))\, dA$$

and equation 4.3 follows. □

Example 4.9 Use Gauss' Divergence Theorem to evaluate

$$I = \iint_S x^4 y + y^2 z^2 + xz^2\, dS,$$

where S is the entire surface of the sphere $x^2 + y^2 + z^2 = 1$.

Solution : In order to apply Gauss' Divergence Theorem we first need to determine $\mathbf{F}$ and the unit normal $\mathbf{n}$ to the surface S. The normal is $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 2y, 2z)$, where $f(z, y, z) = x^2+y^2+z^2-1 = 0$ describes the surface S. We require the unit normal, so $\mathbf{n} = (2x, 2y, 2z)/|(2x, 2y, 2z)| = (2x, 2y, 2z)/2 = (x, y, z)$. To find $\mathbf{F} = (F_1, F_2, F_3)$ we note that

$$\begin{aligned}\mathbf{F} \cdot \mathbf{n} &= x^4 y + y^2 z^2 + xz^2 \\ &= F_1 x + F_2 y + F_3 z\end{aligned}$$

Hence, comparing terms we have $F_1 = x^3y$, $F_2 = yz^2$ and $F_3 = xz$. Applying the Divergence Theorem noting that V is the volume enclosed by the sphere S gives

$$\begin{aligned}
I = \iint_S \mathbf{F}\cdot\mathbf{n}dS &= \iiint_V \operatorname{div}\mathbf{F}dxdydz \\
&= \iiint_V (3x^2y + z^2 + x)\, dxdydz \\
&= 0 + \iiint_V z^2\, dxdydz + 0 \\
&= \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^1 \rho^2\cos^2\phi\,\rho^2\sin\phi\, d\rho \\
&= 2\pi \int_0^{\pi} \cos^2\phi\sin\phi\, d\phi \int_0^1 \rho^4 d\rho \\
&= 2\pi \times 2 \times \frac{1\cdot 1}{3\cdot 1} \times 1 = \frac{4\pi}{15}.
\end{aligned}$$

Remarks

1. As V is a sphere it is natural to use spherical polar coordinates to solve the integral. Thus, $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$ and $dxdydz = \rho^2\sin\phi\, d\rho d\theta d\phi$.

2. $\iiint_V 3x^2y\ dxdydz = 0$ and $\iiint_V x\ dxdydz = 0$ from the symmetry of the cosine and sine functions. We look at the signs in each quadrant as θ changes. Think about a fixed ϕ. $\cos\theta$ and $\sin\theta$ terms in x^2y and x then have the following signs

Quadrant	1	2	3	4	Total
$\cos\theta$	$+$	$-$	$-$	$+$	
$\sin\theta$	$+$	$+$	$-$	$-$	
x^2y	$+$	$+$	$-$	$-$	0
x	$+$	$+$	$-$	$-$	0

The positive and negative contribution from the integral cancel out in these two cases so the integrals are zero.

$\square$

Example 4.10 Find $I = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS$ where $\mathbf{F} = (2x, 2y, 1)$ and where S is the entire surface consisting of S_2=the part of the paraboloid $z = 1 - x^2 - y^2$ with $z = 0$ together with S_1=disc $\{(x, y) : x^2 + y^2 \le 1\}$. Here $\mathbf{n}$ is the outward pointing unit normal.

Solution : Applying the Divergence Theorem noting that V is the volume enclosed by S_1 and S_2 (see

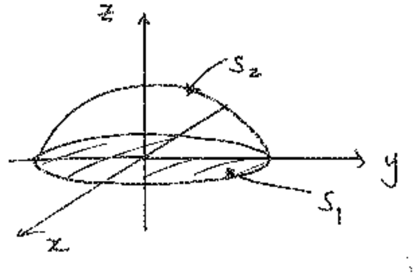

Figure 4.10: Illustration of surfaces S_1 and S_2.

Figure 4.10) and div $\mathbf{F} = 2 + 2 + 0$ gives

$$\begin{aligned}
I = \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \iiint_V \operatorname{div} \mathbf{F} dxdydz \\
&= \iiint_V 4dxdydz \\
&= 4 \iint_{\{(x,y):x^2+y^2 \leq 1\}} dxdy \int_0^{1-x^2-y^2} 1dz \\
&= 4 \iint_{\{(x,y):x^2+y^2 \leq 1\}} 1 - x^2 - y^2 dxdy \\
&= 4 \int_0^{2\pi} d\theta \int_0^1 (1-r^2) r \, dr \\
&= 4 \times 2\pi(1/2 - 1/4) = 2\pi.
\end{aligned}$$

$\square$