2A Multivariable Calculus 2013

Ex Sheet 3

Tutorial Exercises

T1 By making the change of variables indicated, find the general solution of each of the following partial differential equations.

a) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 6xy$. Change to $u = \frac{y}{x}$ and $v = x$

b) $2x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = 2xy$. Change to $u = xy^2$, and $v = y$

Solution

(a) The chain rule gives

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{-y}{x^2}\frac{\partial f}{\partial u} + 1\cdot\frac{\partial f}{\partial v}\,.$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}\frac{1}{x} + 0\,.$$

Therefore the PDE is

$$x\left(\frac{-y}{x^2}\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) + y\left(\frac{1}{x}\frac{\partial f}{\partial u}\right) = 6xy$$

i.e. $f_v = 6y = 6x\frac{y}{x} = 6uv$. Integrating with respect to v gives $f = 3uv^2 + \phi(u)$. Hence the general solution is $f = 3xy + \phi(\frac{y}{x})$, where ϕ is an arbitrary function of one variable.

(b)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = y^2\frac{\partial f}{\partial u} + 0\,.$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 2xy\frac{\partial f}{\partial u} + 1\cdot\frac{\partial f}{\partial v}\,.$$

Therefore the PDE is

$$2xy^2\frac{\partial f}{\partial u} - 2xy^2\frac{\partial f}{\partial u} - y\frac{\partial f}{\partial v} = 2xy$$

i.e. $z_v = -2x$. Since $x = u/y^2 = u/v^2$, we have $f_v = -2u/v^2$ and $f = \frac{2u}{v} + \phi(u)$. Hence the general solution is $f = 2xy + \phi(xy^2)$, where ϕ is an arbitrary function.

T2 Evaluate

$$\text{(a)}\ \int_0^1 dx \int_0^2 3y^2 - 4x\,dy, \quad \text{(b)}\ \int_0^1 dx \int_0^1 2x + 10y\,dy.$$

Solution

(a) We have

$$\int_0^1 dx \int_0^2 3y^2 - 4x\,dy = \int_0^1 \left[y^3 - 4xy\right]_0^2 dx = 8\int_0^1 1 - x\,dx = 4\left[2x - x^2\right]_0^1 = 4.$$

(b) We have

$$\int_0^1 dx \int_0^1 2x + 10y\, dy = \int_0^1 \left[2xy + 5y^2\right]_0^1 dx = \int_0^1 2x + 5\, dx = \left[x^2 + 5x\right]_0^1 = 6.$$

T3 Evaluate

$$\text{(a)} \int_1^2 dx \int_1^x \frac{1}{x+y}\, dy, \quad \text{(b)} \int_0^{\pi/2} dy \int_y^4 x \sin y\, dx.$$

Solution

(a) The integral is

$$\int_1^2 \left[\log|x+y|\right]_1^x dx = \int_1^2 \log(2x) - \log(x+1)\, dx$$
$$= \left[x\log(2x) - (x+1)\log(x+1)\right]_1^2 = 5\log 2 - 3\log 3.$$

Recall, that to calculate the integral of $\log x$ with respect to x you can express the function as $1 \times \log x$ and then use integration by parts. Please see revision sheet 0 and your 1S/1Y notes.

(b) The integral is

$$\int_0^{\pi/2} \left[x^2 \sin y\right]_y^4 dy = \frac{1}{2}\int_0^{\pi/2} (16 - y^2)\sin y\, dy$$
$$= \frac{1}{2}\left[-18\cos y + y^2\cos y - 2y\sin y\right]_0^{\pi/2} = 9 - \frac{\pi}{2}.$$

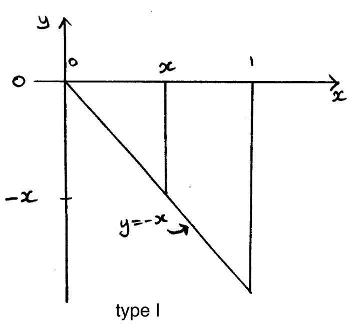
T4 Sketch the triangular domain T, bounded by the lines $y = -x$, $y = 0$ and $x = 1$ and illustrate that it is both type I and type II. Evaluate the double integral

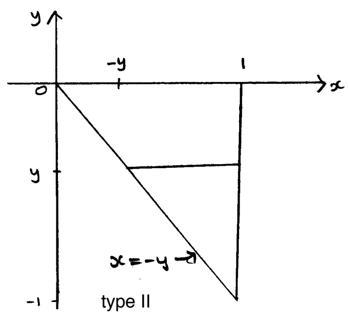
$$\iint_T x\, dxdy,$$

using (a) the type I formulation of T and (b) the type II formulation of T.[1]

[1] The answers you get to (a) and (b) should, of course, be the same.

Solution





(a) Using the type I formulation the integral is

$$\iint_T x\, dxdy = \int_0^1 \left(\int_{-x}^0 x\, dy\right) dx = \int_0^1 x^2\, dx = \frac{1}{3}.$$

(b) Using the type II formulation the integral is

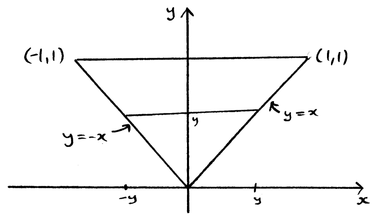
$$\iint_T x\,dxdy = \int_{-1}^{0}\left(\int_{-y}^{1} x\,dx\right)dy = \frac{1}{2}\int_{-1}^{0} 1-y^2\,dy = \frac{1}{3}.$$

T5 Evaluate

$$\iint_D e^{x+y}\,dxdy,$$

where D is the triangle with vertices $(0,0)$, $(1,1)$ and $(-1,1)$.

Solution



The type II formulation is simpler and we get

$$\iint_D e^{x+y}\,dxdy = \int_0^1\left(\int_{-y}^{y} e^{x+y}\,dx\right)dy = \int_0^1 \left[e^{x+y}\right]_{-y}^{y}dy = \int_0^1 e^{2y}-1\,dy$$
$$= \frac{1}{2}(e^2-3).$$

Further Exercises

F1 By making the change of variables indicated, find the general solution of each of the following partial differential equations.

a) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3y(y^2-x^2)$. Change to $u = x$, $v = \frac{y}{x}$.

b) $2x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{6x^4}{y^2}$. Change to $u = \frac{x}{y^2}$, and $v = x$

c) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{4x^3}{y}$. Change to $u = \frac{x}{y}$, $v = x$.

Solution

(a)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\left(\frac{-y}{x^2}\right).$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 + \frac{\partial f}{\partial v}\frac{1}{x}.$$

Therefore the PDE is

$$x\frac{\partial f}{\partial u} - \frac{y}{x}\frac{\partial f}{\partial v} + \frac{y}{x}\frac{\partial f}{\partial v} = 3y(y^2-x^2)$$

i.e. $f_u = 3\frac{y}{x}(y^2-x^2) = 3v((uv)^2-u^2) = 3u^2v^3 - 3u^2v$ and $f = u^3v^3 - u^3v + \phi(v)$. Hence the general solution is $f = y^3 - x^2y + \phi(\frac{y}{x})$, where ϕ is an arbitrary function.

(b)

$$\frac{\partial z}{\partial x} = \frac{1}{y^2}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \quad \frac{\partial z}{\partial y} = \frac{-2x}{y^3}\frac{\partial z}{\partial u}.$$

Therefore the PDE is $\frac{2x}{y^2}z_u + 2xz_v - \frac{2x}{y^2}z_u = \frac{6x^4}{y^2}$, i.e. $z_v = \frac{3x^3}{y^2} = 3uv^2$. Hence, $z = uv^3 + \phi(u)$, i.e. $z = \frac{x^4}{y^2} + \phi\left(\frac{x}{y^2}\right)$.

(c)

$$\frac{\partial z}{\partial x} = \frac{1}{y}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \quad \frac{\partial z}{\partial y} = \frac{-x}{y^2}\frac{\partial z}{\partial u}.$$

Therefore the PDE is $\frac{x}{y}z_u + xz_v - \frac{x}{y}z_u = \frac{4x^3}{y}$, i.e. $z_v = \frac{4x^2}{y} = 4uv$. Hence, $z = 2uv^2 + \phi(u)$, i.e. $z = 2\frac{x^3}{y} + \phi\left(\frac{x}{y}\right)$.

F2 Evaluate

$$\int\int x^2 + 2y \; dxdy$$

over the rectangle with vertices at $(0,0)$, $(2,0)$, $(2,3)$ and $(0,3)$.

Solution

We have

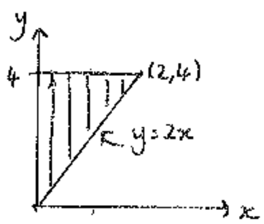
$$\int_0^2 dx \int_0^3 x^2 + 2y \, dy = \int_0^2 [x^2y + y^2]_0^3 \, dx = \int_0^2 3x^2 + 9 \, dx = [x^3 + 9x]_0^2 = 26.$$

F3 Evaluate

$$\int\int xy \; dxdy$$

over the triangle enclosed by the lines $y = 2x$, $y = 4$ and the y-axis.

Solution



We have

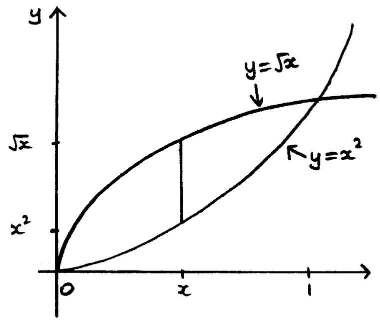
$$\int_0^2 dx \int_{2x}^4 xy \, dy = \int_0^2 x[\frac{1}{2}y^2]_{2x}^4 \, dx = \int_0^2 8x - 2x^3 \, dx = [4x^2 - \frac{2x^4}{4}]_0^2 = 8.$$

F4 Evaluate

$$\iint_D xy \, dxdy,$$

where D is the finite region bounded by the curves $y = x^2$ and $x = y^2$.

Solution



Using the type I formulation,

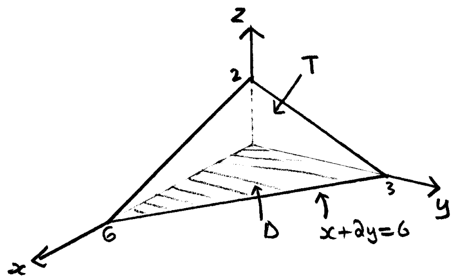
$$\iint_D xy\, dxdy = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} xy\, dy \right) dx = \frac{1}{2} \int_0^1 [xy^2]_{x^2}^{\sqrt{x}}\, dx = \frac{1}{2} \int_0^1 x^2 - x^5\, dx$$
$$= \frac{1}{12}.$$

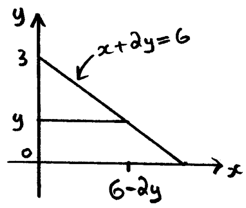
F5 Sketch the tetrahedron T formed by the plane $x + 2y + 3z = 6$ and the xy-, xz- and yz-planes. Show that the volume of T is

$$V = \frac{1}{3} \iint_D 6 - x - 2y\, dxdy,$$

where D is the finite region bounded by $x = 0$, $y = 0$ and $x + 2y = 6$. Hence evaluate V.

Solution





The volume of T is the volume under the the surface $z = \frac{1}{3}(6 - x - 2y)$ and so

$$V = \iint_D z\, dxdy = \frac{1}{3} \iint_D 6 - x - 2y\, dxdy,$$

where, as illustrated, D is the finite region bounded by $x = 0$, $y = 0$ and $x + 2y = 6$. Thus

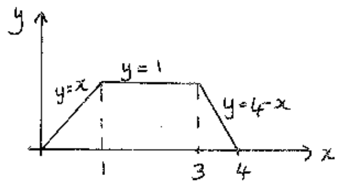
$$V = \frac{1}{3} \int_0^3 \left(\int_0^{6-2y} 6 - x - 2y\, dx \right) dy = \frac{1}{3} \int_0^3 [6x - \tfrac{1}{2}x^2 - 2xy]_0^{2(3-y)}\, dy$$
$$= \frac{1}{3} \int_0^3 12(3-y) - 2(3-y)^2 - 4y(3-y)\, dy = \int_0^3 6 - 4y + \tfrac{2}{3}y^2\, dy = 6.$$

F6 Evaluate

$$\int\int x \, dx \, dy$$

over the trapezium with vertices at $(0,0)$, $(4,0)$, $(3,1)$ and $(1,1)$.

Solution



To avoid splitting up the domain, we treat it as type II and integrate with respect to x first.

$$\int_0^1 dy \int_y^{4-y} x \, dx = \int_0^1 [\frac{x^2}{2}]_y^{4-y} \, dy$$
$$= \frac{1}{2}\int_0^1 (4-y)^2 - y^2 \, dy = \frac{1}{2}\int_0^1 16 - 8y \, dy = \frac{1}{2}\big[16y - 4y^2\big]_0^1 = 6.$$

With the other order of integration we would have bee to split the domain into 3 pieces (see diagram).

F7 Evaluate

$$\int\int e^{-(x+y)} \, dx \, dy$$

over the region given by the inequalities $y \geq 0$, $y \leq 1$ and $y \leq x$.

Solution

We have,

$$\int_0^1 dy \int_y^{\infty} e^{-x}e^{-y} \, dx = \int_0^1 e^{-y}\big[-e^{-x}\big]_y^{\infty} \, dy$$
$$= \int_0^1 e^{-y}e^{-y} \, dy = \int_0^1 e^{-2y} \, dy = [-\frac{1}{2}e^{-2y}]_0^1 = \frac{1}{2}(1-e^{-2}).$$

2 Harder challenge problems

[2] Only attempt these if you have been able to do all the other problems successfully.

F8 Find the volume of the given solid

a) Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant.

b) Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$.

Solution

a) We observe the solid bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant lies under the surface $z = \sqrt{4 - y^2}$ and above the triangle $x/2 \leq y \leq 2$ and $0 \leq x \leq 4$, hence

$$\int_0^4 \int_{x/2}^2 \sqrt{4 - y^2}\, dy\, dx = \int_0^2 \int_0^{2y} \sqrt{4 - y^2}\, dx\, dy = 16/3.$$

b) Using symmetry we find the volume in the first octant and multiply the answer by 8 to get the total volume of the solid. We observe that the solid bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$ lies under the surface $z = \sqrt{r^2 - y^2}$ and above the quarter circle $0 \leq y \leq r$ and $0 \leq x \leq \sqrt{r^2 - y^2}$, hence the total volume of the solid is

$$8 \int_0^r \int_0^{\sqrt{r^2 - y^2}} \sqrt{r^2 - y^2}\, dx\, dy = (16/3) r^3.$$

F9 Use geometry or symmetry, or both, to evaluate the double integral

$$\iint_D (2 + x^2 y^3 - y^2 \sin x)\, dA,$$

where $D = \{(x, y) \mid |x| + |y| \leq 1\}$.

Solution

$D = \{(x, y) \mid |x| + |y| \leq 1\}$ is a square with corners at $(0, 1)$, $(1, 0)$, $(-1, 0)$ and $(0, -1)$. We then look at the symmetry of the integrand and notice that $x^2 y^3$ is symmetrical about the y-axis and a rotation by π about the x-axis, so the integral of this term must be zero. Similarly the $y^2 \sin x$ is symmetrical about the x-axis and a rotation by π about the y-axis, so the integral of this term is zero also. This leaves $\iint_D 2\, dA$. The symmetry of the domain means this integral is

$$4 \int_0^1 \int_0^{1-x} 2\, dy\, dx = 4.$$