Chapter 0

Revision of differentiation, integration and vector algebra from level 1

0.1 Revision of differentiation

(Stewart (Ed. 7): Chapter 2, p103.)

0.1.1 Three important rules for differentiation

Product Rule This rule is for differentiating the product of functions. If $f(x)$ and $g(x)$ are differentiable functions then the *product rule* says the product $f(x)g(x)$ is also differentiable and

$$\frac{d}{dx}\left(f(x)g(x)\right) = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}.$$

Quotient Rule This rule is for differentiating the quotient of two functions. The *quotient rule* says that $f(x)/g(x)$ is differentiable and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}g(x) - f(x)\frac{dg}{dx}}{(g(x))^2}.$$

Chain Rule This rule is for differentiating the composite of two functions. The *chain rule* says that $f(g(x))$ is differentiable. If we let $u = g(x)$ then

$$\frac{d}{dx}\left(f(g(x))\right) = \frac{d}{dx}\left(f(u)\right) = \frac{df}{du}\frac{du}{dx}.$$

0.2 Revision of integration

(Stewart (Ed. 7): Chapter 7, p487.)

0.2.1 Some important rules for integration

Method of substitution This method allows us to integrate expressions that include the composition of two functions and has some analogies with the chain rule,

$$\int_a^b f(g(x))\frac{dg}{dx}\,dx = \int_A^B f(u)\,du,$$

where $u = g(x)$, $A = g(a)$ and $B = g(b)$.

Integration by parts This method allows us to integrate expressions that involve the product of two functions,

$$\int u(x)\frac{dv}{dx}\,dx = u(x)v(x) - \int v(x)\frac{du}{dx}\,dx.$$

Partial fractions and integrals of rational functions Let $P(x)$ and $Q(x)$ be polynomials in x with the degree of P less than the degree of Q, then we can use partial fractions to simplify the integrand in the integrals of the form,

$$\int \frac{P(x)}{Q(x)}\,dx.$$

The simplest example is when $P(x) = 1$ and $Q(x)$ is linear,

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C.$$

More generally,

- if $Q(x)$ can be factorised such that it contains a factor of the form

$$(x-a)^m$$

  then $\frac{P(x)}{Q(x)}$ can be expressed such that it contains the sum of fractions of the form

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots \frac{A_m}{(x-a)^m}.$$

- if $Q(x)$ contains an irreducible factor of the form x^2+bx+c then $P(x)/Q(x)$ contains a sum of following form

$$\frac{B_1x+C_1}{x^2+bx+c}.$$

0.3 Revision of vector algebra

(Stewart (Ed. 7): Chapter 12, p809.)

In this section we will recall some vector algebra from level 1. This will be used in two of the chapters in the course to explore differentiation of vectors and higher order integration. Some key aspects of this are recalling the way in which vectors can be 'multiplied'.

0.3.1 Scalar product in $\mathbb{R}^3$

(Stewart (Ed. 7): Section 10.2, p568.)

Recall the definition of a scalar product ($\mathbf{a}\cdot\mathbf{b}$), it is *a scalar* defined by:

1. $\mathbf{a}\cdot\mathbf{b}$=$|\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the vectors, or equivalently by,
2. $\mathbf{a}\cdot\mathbf{b}$=$a_1b_1 + a_2b_2 + a_3b_3$ in component form.

Some useful identities come from 1 and 2 above:

Remarks

1. $\mathbf{a}\cdot\mathbf{b} = 0 \iff \mathbf{a}$ and $\mathbf{b}$ are perpendicular.
2. The *length of* $\mathbf{a} = (a_1, a_2, a_3)$ is $|\mathbf{a}|$ and is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Example Let $\mathbf{a} = (7, 2, 1)$ and $\mathbf{b} = (5, -3, 4)$. Calculate $\mathbf{a} \cdot \mathbf{b}$ and $|\mathbf{a}|$ and $|\mathbf{b}|$.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= 7.5 + -3.2 = 1.4 = 33, \\ |\mathbf{a}| &= \sqrt{7^2 + 2^2 + 1^2} = \sqrt{54} \\ |\mathbf{b}| &= \sqrt{5^2 + (-3)^2 + 4^2} = \sqrt{50}. \end{aligned}$$

0.3.2 Vector product in $\mathbb{R}^3$

(Stewart (Ed. 7): Section 10.3, p578.)

Before we give the definition for a vector product we recall the formula for *determinants*.

Remarks

1. A 2×2 determinant is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \text{e.g.} \begin{vmatrix} 4 & 5 \\ 6 & 9 \end{vmatrix} = 36 - 30 = 6.$$

2. A 3×3 determinant given by

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

For example,

$$\begin{vmatrix} 4 & 2 & 3 \\ 1 & 5 & 4 \\ -2 & 7 & 6 \end{vmatrix} = 4(5 \cdot 6 - 4 \cdot 7) - 2(1 \cdot 6 - (-2) \cdot 4) + 3(1 \cdot 7 - (-2) \cdot 5) = 31.$$

We can now define the *vector product* $(\mathbf{a} \times \mathbf{b})$, which is *a vector* defined by,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

Remark An alternative formula for the vector product is

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \theta \mathbf{c},$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{c}$ is a unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

Example We can find $\mathbf{a} \times \mathbf{b}$ where $\mathbf{a} = (2, 5, 3)$ and $\mathbf{b} = (-1, 4, 8)$ as follows:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 5 & 3 \\ -1 & 4 & 8 \end{vmatrix} = (5.8 - 3.4)\mathbf{i} - (2.8 - 3.(-1))\mathbf{j} + (2.4 - 5.(-1))\mathbf{k} = (28, -19, 13).$$

Some useful observations about vector products,

Remarks

1. $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$

2. $\mathbf{a} \times \mathbf{b} = 0 \Longleftrightarrow \mathbf{b}$ is a multiple of $\mathbf{a}$.

3. In particular, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

4. $|\mathbf{a} \times \mathbf{b}|$ is the area of a paralellogram with sides given by the vectors $\mathbf{a}$ and $\mathbf{b}$.

0.4 Triple scalar product

One important combination of scale and vector products is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, the *triple scalar product* of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and is denoted by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$. (It is a **number** not a vector). Some properties of triple scalar products include,

Remarks

1. From the definitions of scalar and vector product we see that the easiest way to calculate the triple scalar product is as

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = [\mathbf{a}, \mathbf{b}, \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

   i.e. the 3×3 determinant of the components.

2. From the properties of interchanging rows of a determinant, we can see immediately that

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}].$$

3. The triple scalar $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is a *number* that measures the volume of the parallelepiped with sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (see Figure 1.)

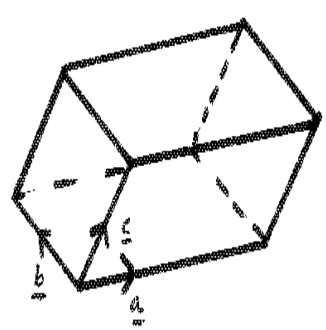

Figure 1: Parallelpiped with sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

4. In particular, $[\mathbf{a}, \mathbf{a}, \mathbf{c}] = 0$. This easily follows because the corresponding parallelepiped has volume 0.

Example Find $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ for $\mathbf{a} = (2, 1, 5)$, $\mathbf{b} = (0, 0, 3)$ and $\mathbf{c} = (7, 5, -6)$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 1 & 5 \\ 0 & 0 & 3 \\ 7 & 5 & -6 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & 3 \\ 2 & 1 & 5 \\ 7 & 5 & -6 \end{vmatrix} = -3\begin{vmatrix} 2 & 1 \\ 7 & 5 \end{vmatrix} = -9.$$

0.5 The triple vector product

The *triple vector product* is another key identity that you should try to learn:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Note that as the name suggests this is a vector. The proof of this formula is quite hard to construct from nothing and is not covered in this course, but it can easily be shown to be true by showing the left and right hand sides of the formula are equal for three general vectors **a**, **b** and **c**.

Example Show that if **a** and **b** are perpendicular, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{r})$ is a multiple of **b**.
Using the triple vector product definition we find,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{r}) = (\mathbf{a} \cdot \mathbf{r})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{r}.$$

But, $\mathbf{a} \cdot \mathbf{b} = 0$ since $\mathbf{a}, \mathbf{b}$ are perpendicular. So,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{r}) = (\mathbf{a} \cdot \mathbf{r})\mathbf{b},$$

i.e. the vector triple product is a multiple of **b**.