2A Multivariable Calculus 2013

Ex Sheet **10**

Tutorial Exercises

T1 In $\mathbb{R}^3$ let S be the part of the plane $4x + 2y - z = 37$ enclosed within the infinite cylinder with rectangular section defined by $0 \leq x \leq 5$, $0 \leq y \leq 2$. Evaluate

$$\iint_S 2y \, dS.$$

T2 In $\mathbb{R}^3$ let S be the part of the plane $2x + y + 6z = 55$ that is enclosed within the infinite cylinder with triangular cross section determined by the planes $y = 0$, $x = 1$ and $y = 3x$. Using a surface integral find the area of the triangle in which the plane $2x + y + 6z = 55$ meets this cylinder.

T3 Evaluate

$$\iint_S z \, dS,$$

where S is the hemispherical surface given by $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

T4 Use Gauss's Divergence Theorem to evaluate

$$\iint_S x^4 + y^4 + z^4 \, dS,$$

where S is the entire surface of the sphere $x^2 + y^2 + z^2 = 1$. (You will have to write the integrand as $\mathbf{F} \cdot \mathbf{n}$ for a suitable $\mathbf{F}$ and for the unit normal $\mathbf{n}$.)

T5 A closed surface is made up of the cylinder $(x-1)^2 + y^2 = 1$ with $z \geq 0$ and $z \leq 3$. Use the Divergence Theorem to evaluate

$$\iint_S \mathbf{v} \cdot \mathbf{n} \, dS,$$

where $\mathbf{v} = (xy, y^2 + e^{xz^2}, \sin(xy))$ and $\mathbf{n}$ is the outward pointing unit normal.

T6 Use the Divergence Theorem to evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS,$$

where $\mathbf{F} = \frac{x}{z}\mathbf{i} - \frac{y}{x}\mathbf{j} + \frac{z}{y}\mathbf{k}$, $\mathbf{n}$ is the outward pointing unit normal and S is given by $S = \{(x, y, z) : 1 < x < 4, 2 < y < 3, 3 < z < 4\}$

Lecture 19

- **Key Points:**
  - calculating the surface integral of a scalar-valued field
  - Using a surface integral to calculate the area of a surface
- **Read:**
  - Stewart Section 16.7 (p1136)
- **Textbook Exercises:**
  - Exercises 16.7 (p1138) Qs 11–20

Lecture 20

- **Key Points:**
  - statement of Gauss' Divergence Theorem for surface integrals of closed surfaces
  - applying Gauss' Divergence Theorem to solve surface integrals
- **Read:**
  - Stewart Section 16.9 (p1152)
- **Textbook Exercises:**
  - Exercises 16.9 (p1157) Qs 1–14

Further Exercises

F1 A vineyard lies on a plane hillside. The base of the vineyard on a map of the area (i.e. the horizontal base section) is determined by by the rectangle $0 \leq x \leq 6$, $0 \leq y \leq 5$ and the plane of the vineyard (in the same coordinates) is $x + 3y + z = 21$. The distribution of the grape harvest (in mass per unit area) across the vineyard is given by the function xy at the point (x, y, z). Use surface integrals to find

a) the mass of the total crop of grapes from the vineyard,

b) the actual area on the hillside covered by the vineyard.

F2 Evaluate
$$\iint_S y \, dS,$$
where S is the plane surface given by the equations $x > 0$, $y > 0$, $z > 0$, and $x + y + z = 1$.

F3 Show that the surface area of the hemisphere given by $x^2 + y^2 + z^2 = a^2$ and $z \geq 0$, where $a > 0$, is $2\pi a^2$.

F4 A tent is in the form of the paraboloid $z = 6 - x^2 - y^2$ for $z > 0$. Find its surface area.

F5 Using the symmetry of the sine and cosine functions explain in one sentence why
$$\iiint_V x \, dxdydz = 0,$$
where V is the interior of the sphere $x^2 + y^2 + z^2 = a^2$. Use Gauss's Divergence Theorem to evaluate
$$\iint_S x^2z^2 + y^2z^2 + 3xz^2 \, dS,$$
where S is the entire surface of the same sphere.

F6 Show that for a well behaved closed surface S enclosing a three dimensional region R
$$\frac{1}{3} \iint_S \mathbf{r} \cdot \mathbf{n} \, dS$$
measures the volume of R. (As usual $\mathbf{r} = (x, y, z)$ and $\mathbf{n}$ denotes the outward drawn normal.)

F7 Use the Divergence Theorem to evaluate
$$\iint_S \mathbf{v} \cdot \mathbf{n} \, dS,$$
where $\mathbf{v} = 7x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$, $\mathbf{n}$ is the outward pointing unit normal and $S = \{x + y + z = 1, x = 0, y = 0, z = 0\}$

F8 Let the surface is given by two spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Use the Divergence Theorem to evaluate
$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS,$$
where $\mathbf{F} = (x, -y^2, xz)$ and $\mathbf{n}$ is the outward pointing unit normal.