Chapter 2

Double and triple integration

(Stewart (Ed. 7): Chapter 15, p998.)

Chapter Summary

Objective	**Tools**
Double integration over regular domains	Sketch the domain of integration, decide if the domain is type I or type II or both and with the aid of the graph find the limits of integration on the two integrals. If D is a type I domain defined by $g(x) \leq y \leq h(x)$ where $a \leq x \leq b$ then $\iint_D f(x,y)\,dxdy = \int_a^b dx \int_{g(x)}^{h(x)} f(x,y)\,dy$. If D is the type II domain defined by $g(y) \leq x \leq h(y)$ where $a \leq y \leq b$ then $\iint_D f(x,y)\,dxdy = \int_a^b dy \int_{g(y)}^{h(y)} f(x,y)\,dx$. Alternatively you may be asked to change the order of integration to enable you to carry out the integral. If the integrand is 1 then the double integral will give the area of the domain.
Double integration using Polar Coordinates	For domains or integrands that are related to circles change to polar coordinates by setting $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$. The limits on the integral will now range over the radius, r and the angle θ describing the domain. Beta functions can be used to help you evaluate the resulting integral. $\int_0^{\pi/2} \sin^m x \cos^n x \; dx = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \; K$ where $K = 1$ unless m and n are both even in which case $K = \pi/2$.

Objective	Tools
Change of variables for double integrals	To change from the variables x, y to $u(x,y), v(x,y)$ then $\iint_D f(x,y)\,dxdy = \iint_S f(x(u,v),y(u,v))\vert J\vert\,dudv$, where D is the domain in the xy-plane and S is the corresponding domain in the uv-plane, and $\vert J\vert$ is absolute value of $$J = 1\Big/\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$
Triple integration and Spherical Coordinates	Triple integrals have the form $\iiint_V f(x,y,z)\,dxdydz = \underbrace{\int_a^b}_{\text{Constants}} dx \underbrace{\int_{h_1(x)}^{h_2(x)}}_{\text{Curves}} dy \underbrace{\int_{g_1(x,y)}^{g_2(x,y)}}_{\text{Surfaces}} f(x,y,z)\,dz.$ where the domain is a volume. If the integrand is 1 the triple integral will give the volume of the domain. If the integrand is a density then the triple integral will give the mass of the volume. For domains or integrands that are related to spheres change to spherical coordinates by setting $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$ and $dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$. The limits on the integral will now range over the radius, ρ, the angle $\theta \in [0, 2\pi)$ between the positive x-axis and the angle $\phi \in [0,\pi)$ between the positive z-axis.

2.1 Area under a curve

Recall the way that definite integrals arise as "areas under curves". We can approximate the area under the curve $y = g(x)$ on the interval $[a, b]$ by the sum of areas of rectangles (called a *Riemann sum*) of widths δx_i and heights $g(x_i)$ as illustrated in Figure 2.1.

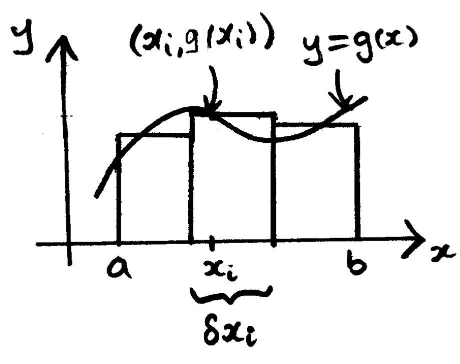

Figure 2.1: Approximating the area under a curve

If as the number of subintervals of $[a, b]$, N, increases the Riemann sums tend to a limit, this is the definite integral

$$\int_a^b g(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} g(x_i)\delta x_i.$$

2.2 Double integration on rectangular domains

(Stewart (Ed. 7): Section 15.1, p998.)

We wish to extend this idea to define the "volume under a surface" $z = f(x, y)$ on the set $D \subset \mathbb{R}^2$. We can approximate this volume by the sum of the volumes of cuboids. For simplicity, first consider a rectangular subset $R = [a, b] \times [c, d]$. This is divided into subrectangles of area $\delta A_{ij} = \delta x_i \delta y_j$ and the cuboid above this has height $f(x_i, y_j)$, as shown in Figure 2.2.

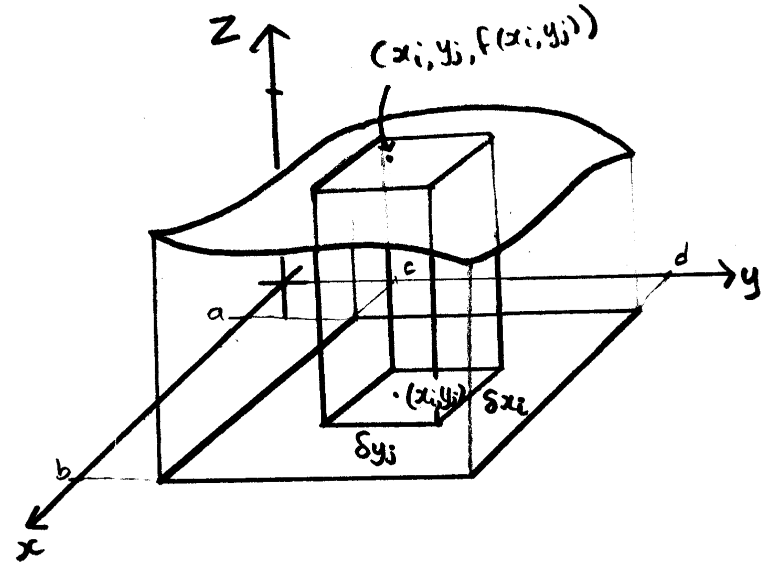

Figure 2.2: Approximating the volume under a surface

In this way, the whole volume is approximated by

$$\sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i, y_j)\delta A_{ij}.$$

If the limit as $M, N \to \infty$ exists we say that f is *integrable over* R and write it as

$$\iint_R f(x, y)\,dA \quad \text{or} \quad \iint_R f(x, y)\,dxdy.$$

This is called the *double integral of* f *over* R and $dA = dxdy$ is called the *area element*.

To evaluate the double integral we can think of the solid under the curve as made up of slices with y fixed (see Figure 2.3.) The area under the curve in such a cross section is

$$I(y) = \int_a^b f(x, y)\,dx,$$

where y is fixed in the integrand.

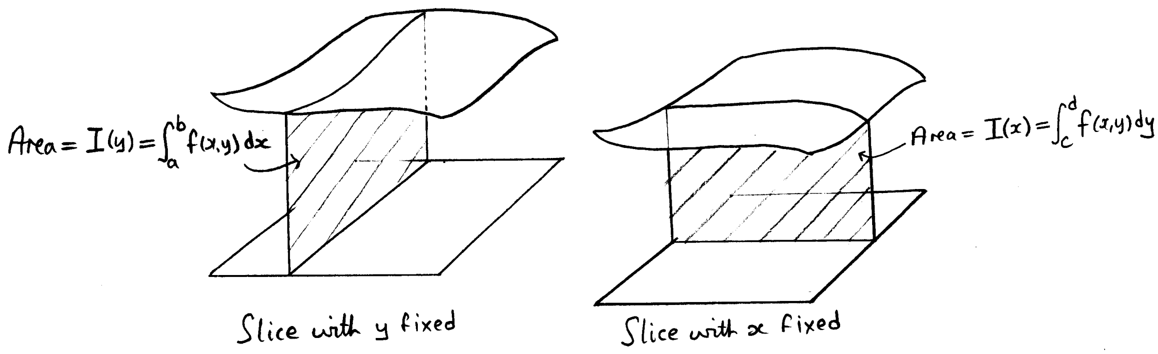

Figure 2.3: Cross sections through the solid under the surface

The sum of these areas

$$\int_c^d I(y)\,dy$$

gives the volume under the surface. This means that

$$\iint_R f(x,y)\,dxdy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

By summing the areas of cross sections of the solid with x fixed, we also have

$$\iint_R f(x,y)\,dxdy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx.$$

Notation 2.1 We usually write

$$\int_a^b dx \int_c^d f(x,y)\,dy \quad \text{for} \quad \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx.$$

Example 2.1 Evaluate

$$\iint_R x^2 + y^2\,dxdy$$

where R is $[1,3] \times [2,4]$.

Solution The integral may either be evaluated as

$$\begin{aligned}
\iint_R x^2 + y^2\,dxdy &= \int_1^3 dx \int_2^4 x^2 + y^2\,dy \\
&= \int_1^3 \left[x^2 y + \frac{1}{3}y^3 \right]_2^4 dx \\
&= \int_1^3 (4-2)x^2 + \frac{56}{3}\,dx \\
&= \left[\frac{2}{3}x^3 + \frac{56}{3}x \right]_1^3 \\
&= \frac{2(3^3 - 1^3) + 56(3-1)}{3} = \frac{164}{3},
\end{aligned}$$

or as

$$\iint_R x^2 + y^2 \, dxdy = \int_2^4 dy \int_1^3 x^2 + y^2 \, dx$$
$$= \int_2^4 \left[\frac{1}{3}x^3 + xy^2\right]_1^3 dy$$
$$= \int_2^4 \frac{(3^3 - 1^3)}{3} + (3-1)y^2 \, dy$$
$$= \left[\frac{26}{3}y + \frac{2}{3}y^3\right]_2^4$$
$$= \frac{26(4-2) + 2(4^3 - 2^3)}{3} = \frac{164}{3}.$$

Not surprisingly, each method gives the same answer. □

2.3 Double integration on regular domains

(Stewart (Ed. 7): Section 15.3, p1012.)

Consider now a more complicated domain T which is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,2)$ shown in Figure 2.4.

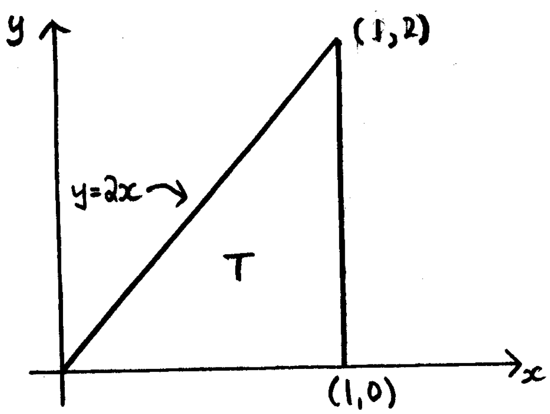

Figure 2.4: Triangular domain T

The domain T is bounded by the lines $y = 0$, $x = 1$ and $y = 2x$. As for a rectangular domain, to evaluate a double integral over T we could split T into a collection of vertical slices, integrate with respect to y and then integrate the result with respect to x. The difference here is that the limits in the first integral *depend on* x. A typical horizontal slice has end-points $y = 0$ and $y = 2x$, and there is a slice at each x from $x = 0$ to $x = 1$.

Hence we have

$$\iint_T f(x,y) \, dxdy = \int_0^1 dx \int_0^{2x} f(x,y) \, dy.$$

Alternatively, we could begin by looking at horizontal slices, with end-points $x = 1$, $x = \frac{1}{2}y$ and summing these from $y = 0$ to $y = 2$. This means that the integral is also

$$\iint_T f(x,y) \, dxdy = \int_0^2 dy \int_{\frac{1}{2}y}^1 f(x,y) \, dx.$$

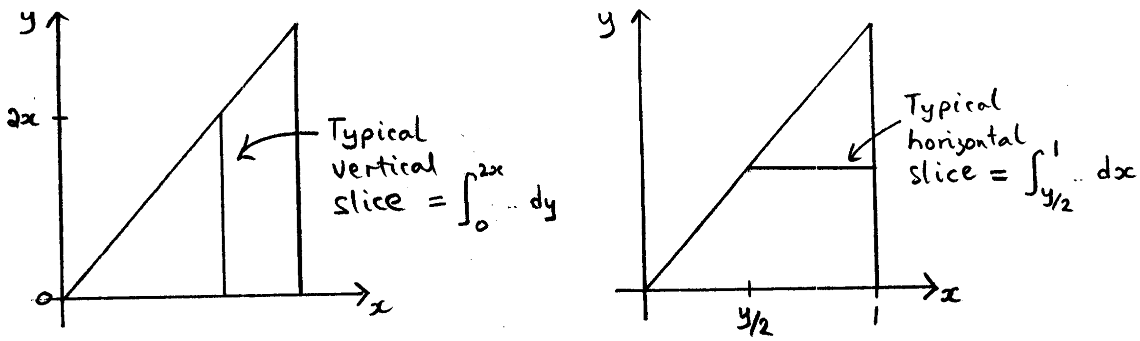

Figure 2.5: Vertical and horizontal slices through T

Definition 2.1 Let D be a domain in the x, y-plane. D is said to be

- *Type I* (or y-*simple*) if it is bounded by lines $x = a$, $x = b$ and curves $y = g(x)$, $y = h(x)$, the intersection of any vertical line $x = c$, where $c \in [a, b]$, is an interval or a single point,

- *Type I* (or x-*simple*) if it is bounded by curves $x = g(y)$, $x = h(y)$ and lines $y = a$, $y = b$, the intersection of any horizontal line $y = c$, where $c \in [a, b]$, is an interval or a single point,

. Typical examples are shown in Figure 2.6.

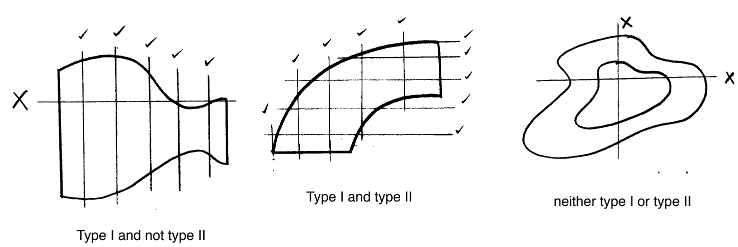

Figure 2.6: Type I and type II domains

D is said to be *regular* if it the union of finitely many disjoint type I and type II domains. Every type I and type II domain is regular.

Example 2.2 State whether each of the domains shown in Figure 2.7 are type I and/or type II or regular.

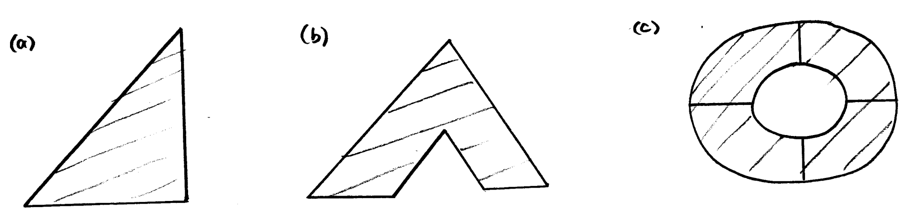

Figure 2.7: Example domains

Solution (a) Both horizontal and vertical lines intersect the triangle in an interval or a single point. Hence this domain is both type I and type II.

(b) Vertical lines intersect this domain in an interval. Hence it is type I. The intersection with some horizontal lines is a union of two intervals, hence it is not type II.

(c) Some horizontal and some vertical lines intersect the annulus in a union of intervals. Hence this domain is neither type I nor type II. The domain may be divided into four type I and type II domains as shown. Hence it is regular. □

Remark A double integral over a type I domain may be evaluated by integrating with respect to y over vertical slices and then integrating with respect to x. Thus a double integral is evaluated by carrying out two single integrals. For type II domains the order of integration is reversed.

If $C \cap D = \emptyset$ then

$$\iint_{C\cup D} f(x,y)\,dxdy = \iint_C f(x,y)\,dxdy + \iint_D f(x,y)\,dxdy.$$

Hence a double integral over a regular domain can be split into a sum of double integral over type I or type II domains.

Theorem *If D is the type I domain defined by $g(x) \le y \le h(x)$ where $a \le x \le b$ then*

$$\iint_D f(x,y)\,dxdy = \int_a^b dx \int_{g(x)}^{h(x)} f(x,y)\,dy.$$

If D is the type II domain defined by $g(y) \le x \le h(y)$ where $a \le y \le b$ then

$$\iint_D f(x,y)\,dxdy = \int_a^b dy \int_{g(y)}^{h(y)} f(x,y)\,dx.$$

The first integral performed (called the inner integral*) may have limit depending on the other variable but the second integral (the* outer integral*) has constant limits.*

Example 2.3 Evaluate

$$\iint_D xy^2\,dxdy,$$

where D is the region in the first quadrant bounded by the curve $y = 4x^2$, the x axis and the line $x = 1$.

Solution It is important to draw a sketch of the domain. This is given in Figure 2.8.

This domain is clearly both type I and type II but it is more readily thought of as type I; $0 \le y \le 4x^2$ where $0 \le x \le 1$. Hence

$$\begin{aligned}\iint_D xy^2\,dxdy &= \int_0^1 dx \int_0^{4x^2} xy^2\,dy \\ &= \int_0^1 \left[\frac{1}{3}xy^3\right]_0^{4x^2} dx = \int_0^1 \frac{1}{3}x((4x^2)^3 - 0^3)\,dx \\ &= \frac{64}{3}\int_0^1 x^7\,dx \\ &= \frac{64}{3}\frac{1}{8}\left[x^8\right]_0^1 = \frac{8}{3}.\end{aligned}$$

□

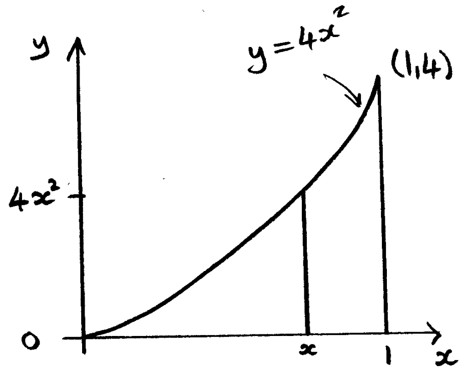

Figure 2.8: Type I domain

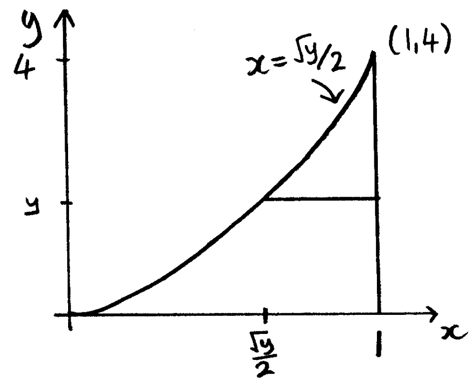

Figure 2.9: Type II domain

Remark In the domain D above $x \geq 0$ and so $y = 4x^2 \iff x^2 = y/4 \iff x = \sqrt{y}/2$. See Figure 2.9.

Hence the type II description of D is $\sqrt{y}/2 \leq x \leq 1$ where $0 \leq y \leq 4$. Consequently,

$$\iint_D xy^2 \, dxdy = \int_0^4 dy \int_{\sqrt{y}/2}^1 xy^2 \, dx.$$

The reader should verify that this evaluates to the same value, 8/3, as found above.

When a domain is both type I and type II it a matter of convenience which formulation is used.

Example 2.4 Evaluate

$$I = \iint_D 3x^2 + y^2 \, dxdy,$$

where D is the triangle with vertices $(0,0)$, $(1,1)$ and $(2,1)$.

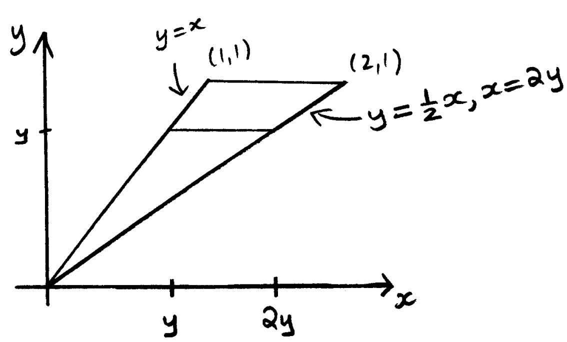

Figure 2.10: Type II domain

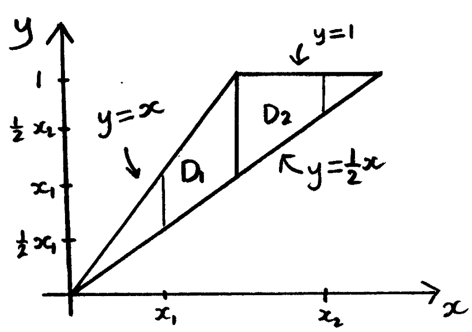

Figure 2.11: Two type I domains

Solution The region is sketched in Figure 2.10 and is both type I and type II. The type II formulation is easier

$$\begin{aligned}
I &= \int_0^1 dy \int_y^{2y} 3x^2 + y^2 \, dx \\
&= \int_0^1 \big[x^3 + xy^2\big]_y^{2y} \, dy = \int_0^1 (2y)^3 - y^3 + (2y-y)y^2 \, dy \\
&= 8\int_0^1 y^3 \, dy = 2\big[y^4\big]_0^1 \\
&= 2.
\end{aligned}$$

□

Remark The type I formulation is more awkward. The function describing the lower curve is $y = \frac{1}{2}x$ and the function describing the upper curve is

$$y = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 \le x \le 2. \end{cases}$$

To handle this it is best to split the domain into two pieces (see Figure 2.11) and evaluate the double integral as

$$I = \iint_{D_1} 3x^2 + y^2 \, dxdy + \iint_{D_2} 3x^2 + y^2 \, dxdy = \int_0^1 dx \int_{\frac{1}{2}x}^{x} 3x^2 + y^2 \, dy + \int_1^2 dx \int_{\frac{1}{2}x}^{1} 3x^2 + y^2 \, dy.$$

This is why the type II formulation is preferred.

Example 2.5 Evaluate

$$I = \int_0^1 dx \int_{\sqrt{x}}^1 \frac{e^{y^2}}{\sqrt{x}} \, dy.$$

Remark This double integral is expressed in type I form but it cannot be evaluated as it stands. The first step would be to find an antiderivative for e^{y^2} but this cannot be done (in terms of known function such as exp, log etc.).

The key to this example is to change the order of integration and convert the integral to type II form. To make this conversion it is vital to draw a sketch of the domain.

Solution A sketch of the domain with type I and type II descriptions is given in Figure 2.12. Using this sketch we can convert the double integral into type II form

$$\begin{aligned}
I &= \int_0^1 dy \int_0^{y^2} \frac{e^{y^2}}{\sqrt{x}} \, dx \\
&= \int_0^1 \Big[2\sqrt{x}e^{y^2}\Big]_0^{y^2} \, dy = 2\int_0^1 ye^{y^2} \, dy.
\end{aligned}$$

Now we can use the substitution

$$u = y^2, \quad du = 2y\,dy, \quad \boxed{\begin{array}{c|cc} y & 0 & 1 \\ \hline u & 0 & 1 \end{array}},$$

to give

$$I = \int_0^1 e^u \, du = \big[e^u\big]_0^1 = e - 1.$$

□

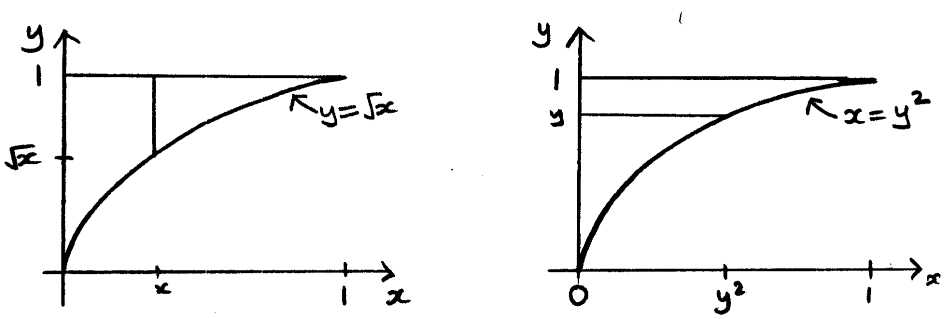

Figure 2.12: From type I to type II

Example 2.6 Find the volume of the tetrahedron T, bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

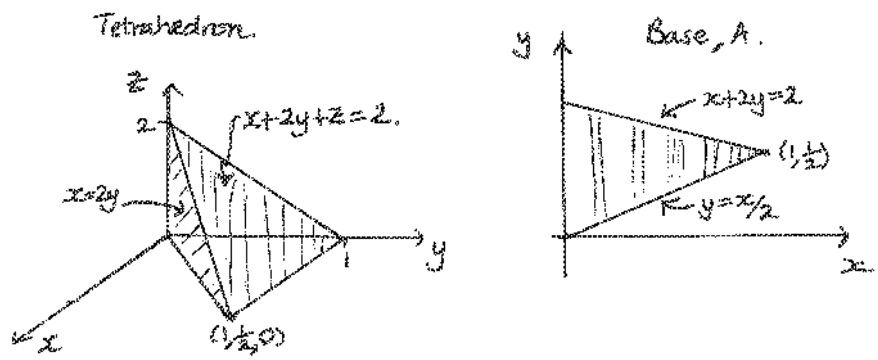

Figure 2.13: 3-D solid tetrahedron and the base, A.

Solution A sketch of the solid in 3-D and a sketch of the base, the planar region A over which we integrate are given in Figure 2.13. Using this sketch we can write down a double integral which describes the volume of the tetrahedron.

The plane $x + 2y + z = 2$ intersects the xy-plane ($z = 0$) in the line $x + 2y = 2$. So the tetrahedron lies above the triangular region A in the xy-plane. A is bounded by $x = 2y$, $x + 2y = 2$ and $x = 0$.

The plane $x + 2y + z = 2$ can be written as $z = 2 - x - 2y$. So the volume of the tetrahedron is the volume that lies under the graph of the function $z = 2 - x - 2y$ and above A, where $A = \{(x, y)|0 \le x \le 1, x/2 \le y \le 1 - x/2\}$.

$$
\begin{aligned}
T &= \iint_A (2 - x - 2y)\,dydx = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\,dydx \\
&= \int_0^1 \left[2y - xy - y^2\right]_{y=x/2}^{y=1-x/2} dx = \int_0^1 x^2 - 2x + 1\,dx \\
&= \left[\frac{x^3}{3} - x^2 + x\right]_0^1 = \frac{1}{3}.
\end{aligned}
$$

□

Remark The area of a surface $A \subset \mathbb{R}^2$ is given by the double integral

$$\iint_A 1\, dxdy.$$

2.4 Double integration in polar coordinates

(Stewart (Ed. 7): Section 15.4, p1021.)

The position of a point (x, y) on the cartesian plane can be specified by *polar coordinates* r, θ where

$$x = r\cos\theta, \quad y = r\sin\theta.$$

$\theta \in [0, 2\pi)$ is the anti-clockwise angle between the positive x-axis and the line joining (x, y) to $(0, 0)$ and $r \geq 0$ is the length of this line. See Figure 2.14.

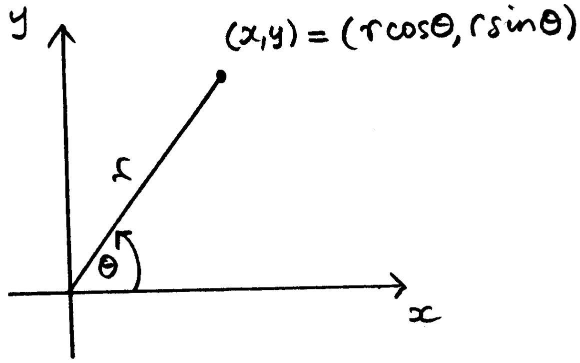

Figure 2.14: Polar coordinates

Remarks

1. This change of variables is *invertible* since every point on the plane can be uniquely described by polar coordinates.
2. Note that $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$ so that expression involving $x^2 + y^2$ can be written in terms of r alone.

In cartesian coordinates, the area of an elementary rectangle used in the Riemann sum is $\delta A = \delta x \delta y$ and for this reason the area element dA is $dxdy$. In polar coordinates, the area element is illustrated in Figure 2.15 and has area $\delta A \approx r\delta r\delta\theta$.

For this reason in polar coordinates, $dA = rdrd\theta$, i.e.,

$$\iint_D f(x, y)\, dxdy = \iint_D f(r\cos\theta, r\sin\theta) r\, drd\theta.$$

When either the domain is circular or the integrand is written in terms of $x^2 + y^2 (= r^2)$, the double integral should be rewritten in polar coordinates.

Example 2.7 Use polar coordinates to evaluate

$$I = \iint_D x + y\, dxdy,$$

where D is part of the annulus between circles of radius 1 and 2, centre $(0, 0)$ lying in upper half plane.

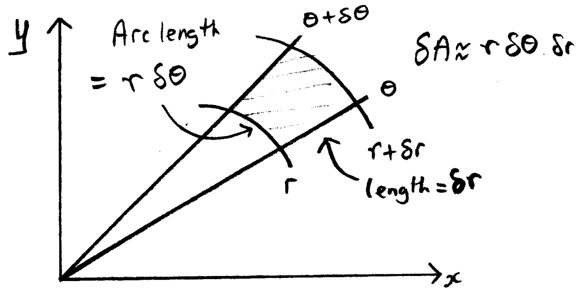

Figure 2.15: The area element in Polar coordinates

Solution In polar coordinates the domain is $1 \le r \le 2$, $0 \le \theta \le \pi$ (see Figure 2.16.)

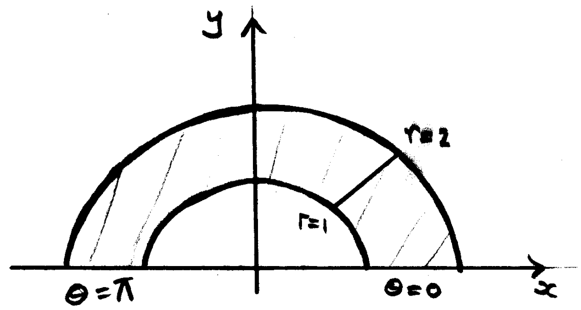

Figure 2.16: Annular domain

We have

$$
\begin{aligned}
I &= \int_0^\pi d\theta \int_1^2 (r\cos\theta + r\sin\theta) r\,dr = \int_0^\pi \cos\theta + \sin\theta\,d\theta \int_1^2 r^2\,dr \\
&= \big[\sin\theta - \cos\theta\big]_0^\pi \frac{1}{3}\big[r^3\big]_1^2 \\
&= \big(0 - 0 - (-1-1)\big)\,\frac{1}{3}(2^3 - 1^3) = \frac{14}{3}.
\end{aligned}
$$

□

Example 2.8 Evaluate

$$I = \iint_D y\,dxdy,$$

where D is the part of the disk of radius $a\,(>0)$ and centre $(a,0)$ lying in the first quadrant.

Solution The border of the disk has equation $(x-a)^2 + y^2 = a^2$, i.e., $x^2 + y^2 = 2ax$. In polar coordinates, this is

$$r^2 = 2ar\cos\theta, \quad \text{i.e., } r = 2a\cos\theta.$$

See Figure 2.17.

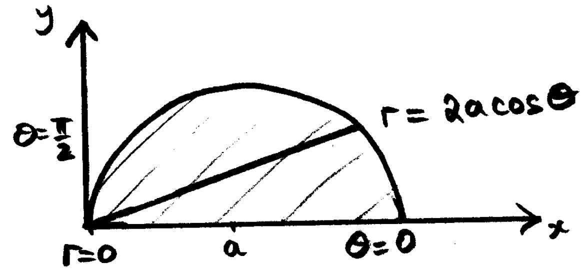

Figure 2.17: Semicircular domain

The domain is $0 \le r \le 2a\cos\theta$ where $0 \le \theta \le \pi/2$ and so

$$\begin{aligned} I &= \int_0^{\pi/2} d\theta \int_0^{2a\cos\theta} (r\sin\theta) r\, dr = \int_0^{\pi/2} d\theta \int_0^{2a\cos\theta} r^2 \sin\theta\, dr \\ &= \frac{1}{3}\int_0^{\pi/2} \sin\theta \big[r^3\big]_0^{2a\cos\theta}\, d\theta = \frac{8a^3}{3}\int_0^{\pi/2} \sin\theta\cos^3\theta\, d\theta. \end{aligned}$$

Using the change of variable

$$u = \cos\theta, \quad du = -\sin\theta, \quad \begin{array}{|c|cc|} \hline \theta & 0 & \pi/2 \\ \hline u & 1 & 0 \\ \hline \end{array},$$

we get

$$I = -\frac{8a^3}{3}\int_1^0 u^3\, du = \frac{8a^3}{3}\frac{1}{4}\big[u^4\big]_0^1 = \frac{2a^3}{3}.$$

□

2.5 Beta and Gamma functions

When dealing with polar coordinates we often need to integrate functions that involve powers of cosine and sine. While we can happily do this using trigonometric identities and integration by substitution, it can involve quite long calculations if the integrand involves very large powers of sine and cosine. To simplify these calculations we can draw on some properties of two functions: *Beta functions and Gamma functions.* We will not discuss these functions in depth here, this is dealt with in the second semester course 2D. Instead we show how these functions can be used and use them to simplify integration.

Definition 2.2

- *Beta function* is defined by

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx, \qquad p > 0 \text{ and } q > 0.$$

A particularly useful form that we shall use is,

$$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}(y)\cos^{2q-1}(y)\, dy\ .$$

This is found by substituting $x = \sin^2 y$ in the definition of the Beta function.

- *Gamma function* is defined by

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x}\, dx, \quad k > 0.$$

2.5.1 Properties of Beta and Gamma functions

1. $\Gamma(1) = 1$, $\Gamma(2) = 1$, $\Gamma(3) = 2$ and in general $\Gamma(n) = (n-1)!$ for every positive integer n.

2. $\Gamma(k) = (k-1)\Gamma(k-1)$ for all real numbers $k > 1$ Repeatedly applying the formula for $\Gamma(k)$ gives a formula in terms of $\Gamma(k-p)$, where $0 < k-p \leq 1$, e.g. $\Gamma(9/4) = \frac{5}{4}\frac{1}{4}\Gamma(\frac{1}{4})$.

3. For $0 < k < 1$, $\Gamma(k)\Gamma(1-k) = \pi/\sin(k\pi)$.

4. $\Gamma(1/2) = \sqrt{\pi}$.

5. $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

From these properties we can derive the following result that we will use in this course.

$$\boxed{\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} K}$$

where $K = 1$ unless m and n are both even in which case $K = \pi/2$. The notation means that the factors continue until 1 or 2 is reached. In the special cases, $m = 0$ or $m = 1$ none of the numerator factors involving m appear.

For example,

$$\int_0^{\pi/2} \sin^3 x \cos^6 x \, dx = \frac{2.5.3.1}{9.7.5.3.1} = \frac{2}{63}.$$

Now consider $\int_0^{2\pi} \sin^3 x \cos^6 x \, dx$. We can use properties of the graphs of sine and cosine to help us to simplify the integral before applying Beta functions. A definite integral calculates the area under the curve, with areas below the x-axis making a negative contribution to the integral. From the translational symmetry of sine and cosine functions seen in Figure 2.18 we can see that:

$$\int_0^{\pi} \sin x \, dx = 2\int_0^{\pi/2} \sin x \, dx; \quad \int_0^{2\pi} \sin x \, dx = 0; \quad \int_0^{\pi} \cos x \, dx = 0; \quad \int_0^{2\pi} \cos x \, dx = 0.$$

$$\int_0^{\pi} \sin^2 x \, dx = 2\int_0^{\pi/2} \sin^2 x \, dx; \quad \int_0^{2\pi} \sin^2 x \, dx = 4\int_0^{\pi/2} \sin^2 x \, dx.$$

$$\int_0^{\pi} \cos^2 x \, dx = 2\int_0^{\pi/2} \cos^2 x \, dx; \quad \int_0^{2\pi} \cos^2 x \, dx = 4\int_0^{\pi/2} \cos^2 x \, dx.$$

Similar rules apply to integrals where the integrand has the form $\sin^m x \cos^n x$, where m and n are non-negative integers.

Example 2.9 Evaluate:

$$\text{(a) } I = \int_0^{\pi} \sin^3 x \cos^4 x \, dx, \quad \text{(b) } I = \int_0^{\pi} \sin^3 x \cos^5 x \, dx, \quad \text{(c) } I = \int_0^{2\pi} \sin^2 x \cos^4 x \, dx.$$

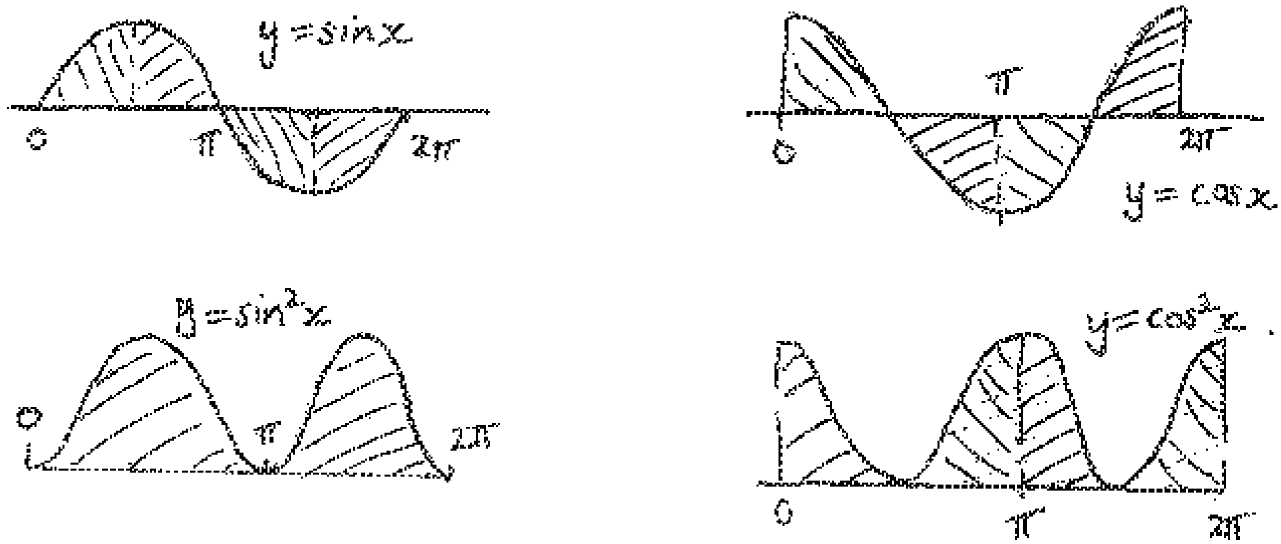

Figure 2.18: Graphs of $\sin x$, $\cos x$, $\sin^2 x$ and $\cos^2 x$.

Solution In each case we make a table of sign first to indicate if the integrand is above (+) or below (–) the x-axis in each quadrant. The symmetry of sine and cosine means that the absolute value of the area under the curve in each quadrant is the same, so the total integral is given by summing the number of plus signs minus the number of minus signs and multiplying the result by integral of the function over the first quadrant.

(a)

Quadrant	1	2
$\sin^3 x$	$+$	$+$
$\cos^4 x$	$+$	$+$
$\sin^3 x \cos^4 x$	$+$	$+$

Total=+2.

Hence,

$$I = 2\int_0^{\pi/2} \sin^3 x \cos^4 x \, dx = 2 \cdot \frac{3 \cdot 2 \cdot 1}{7 \cdot 5 \cdot 3 \cdot 1 \cdot} = \frac{4}{35}.$$

(b)

Quadrant	1	2
$\sin^3 x$	$+$	$+$
$\cos^5 x$	$+$	$-$
$\sin^3 x \cos^5 x$	$+$	$-$

Total=0.

Hence,

$$I = 0.$$

(c)

Quadrant	1	2	3	4
$\sin^2 x$	$+$	$+$	$+$	$+$
$\cos^4 x$	$+$	$+$	$+$	$+$
$\sin^2 x \cos^4 x$	$+$	$+$	$+$	$+$

Total=+4.

Hence,

$$I = 4\int_0^{\pi/2} \sin^2 x \cos^4 x \, dx = 4 \cdot \frac{1 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi}{8}.$$

□

We will put these tools to use in section 3.7 and in chapter 4.

2.6 Change of variables in double integration

(Stewart (Ed. 7): Section 15.10, p1064.)

The change to polar coordinates is a special case of the theorem stated below.

Definition 2.3 Consider a change of variables x, y to u, v. So $x = x(u, v)$ and $y = y(u, v)$. The *Jacobian* $\dfrac{\partial(u, v)}{\partial(x, y)}$ is the determinant

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

If the change of variables is invertible then the Jacobian is nonzero and

$$\frac{\partial(x, y)}{\partial(u, v)} = 1 \Bigg/ \frac{\partial(u, v)}{\partial(x, y)} .$$

Theorem *Let the change of variables x, y to u, v be invertible on the domain D. Then*

$$\iint_D f(x, y)\, dxdy = \iint_S f(x(u, v), y(u, v))|J|\, dudv,$$

where D is the domain in the xy-plane and S is the corresponding domain in the uv-plane, and $|J|$ is the absolute value of $\dfrac{\partial(x, y)}{\partial(u, v)}$. Often it is convenient to use

$$J = 1 \Bigg/ \frac{\partial(u, v)}{\partial(x, y)}$$

Remark The idea behind the change of variables is to transform curves in the xy-plane to lines (or simpler curves such as circles) in the uv-plane. This transformation is illustrated in Figure 2.19. Under the

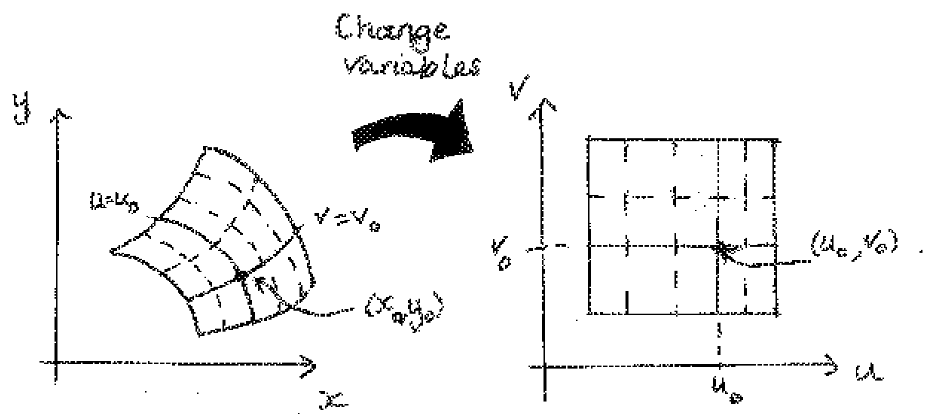

Figure 2.19: Domain in x, y and u, v coordinates

transformation $x = x(u, v)$, $y = y(u, v)$ the lines $u = u_0$ and $v = v_0$ in the uv-plane get mapped to the curves $x = x(u_0, v)$, $y = y(u_0, v)$ and $x = x(u, v_0)$, $y = y(u, v_0)$ in the xy-plane. Let $\mathbf{r} = (x, y)$. The tangent vector at (x_0, y_0) to the curve $(x(u, v_0), y(u, v_0))$ is $\mathbf{r}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right)$. Similarly the tangent vector at (x_0, y_0) to the curve $(x(u_0, v), y(u_0, v))$ is $\mathbf{r}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right)$.

We can consider what happens to a small element of the surface D under the change of variables by approximating the element by a parallelogram with sides $\Delta u\, \mathbf{r}_u$ and $\Delta v\, \mathbf{r}_v$ determined by the tangent vectors. This is illustrated in Figure 2.20.

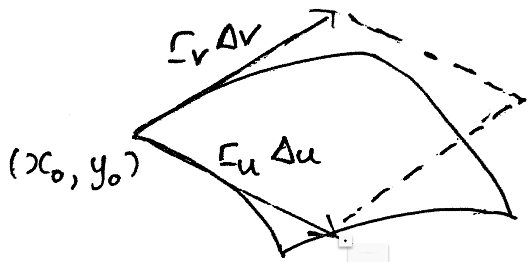

Figure 2.20: Element of D in x, y and u, v coordinates

The area of the small element of D is ΔA and is approximated by the area of the parallelogram which is given by the magnitude of the following cross product

$$\Delta u\, \mathbf{r}_u \times \Delta v\, \mathbf{r}_v = \Delta u\, \Delta v \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \Delta u\, \Delta v \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

$$= \Delta u\, \Delta v \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \Delta u\, \Delta v\, (x_u y_v - x_v y_u)\mathbf{k} = \Delta u\, \Delta v\, \frac{\partial(x, y)}{\partial(u, v)}\mathbf{k}.$$

Hence,

$$\Delta A = |\Delta u\, \mathbf{r}_u \times \Delta v\, \mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u \Delta v = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u\, \Delta v\ .$$

Summing the elements that make up the region D allows us to approximate the double integral of f over D,

$$\iint_D f(x, y)\, dxdy = \lim_{N,\, M \to \infty} \sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i, y_j) \Delta A = \lim_{N,\, M \to \infty} \sum_{i=1}^{N} \sum_{j=1}^{M} f(x(u_i, v_j), y(u_i, v_j)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u\, \Delta v$$

This is a Riemann sum for the integral

$$\iint_S f(x(u, v), y(u, v))|J|\, dudv.$$

Remark Note that for the change to polar coordinates,

$$J = \begin{vmatrix} (r\cos\theta)_r & (r\cos\theta)_\theta \\ (r\sin\theta)_r & (r\sin\theta)_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r,$$

giving the result

$$\iint_D f(x, y)\, dxdy = \iint_S f(r\cos\theta, r\sin\theta) r\, drd\theta,$$

derived in the previous section.

Example 2.10 By making a suitable change of variables, evaluate

$$\iint_D x + 3y \, dxdy,$$

where D is the region bounded by the lines

$$y = x - 1, \quad y = x + 1, \quad y = -x - 1, \quad y = -x + 3.$$

Remark The idea here is to choose variables u, v in which the domain is simply described, preferably with constant limits.

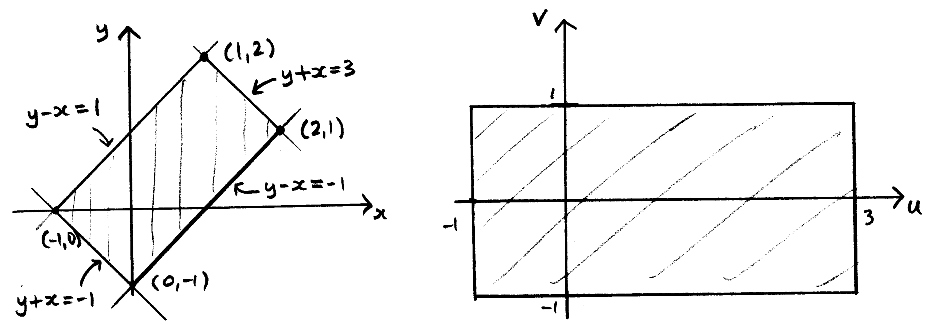

Figure 2.21: Domain in x, y and u, v coordinates

Solution If we define $u = x+y$ and $v = x-y$ then the domain D is described by $-1 \leq u \leq 3$, $-1 \leq v \leq 1$. See Figure 2.21.

We have

$$\frac{1}{J} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

Inverting the change of variable we get

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2},$$

and so the integrand is $x + 3y = \frac{1}{2}\big((u+v) + 3(u-v)\big) = 2u - v$.

Hence

$$\begin{aligned} I &= \iint_D (2u - v) \left| \frac{1}{-2} \right| dudv \\ &= \frac{1}{2} \int_{-1}^{3} du \int_{-1}^{1} 2u - v \, dv = \frac{1}{2} \int_{-1}^{3} \left[2uv - \frac{1}{2} v^2 \right]_{-1}^{1} du \\ &= \frac{1}{2} \int_{-1}^{3} 2\big(1 - (-1)\big) u \, du = \int_{-1}^{3} 2u \, du = \left[u^2 \right]_{-1}^{3} \\ &= \left(3^2 - (-1)^2\right) = 8. \end{aligned}$$

□

Example 2.11 Find the area bounded by the curves $y = e^x$, $y = 2e^x$, $y = e^{-x}$ and $y = 2e^{-x}$.

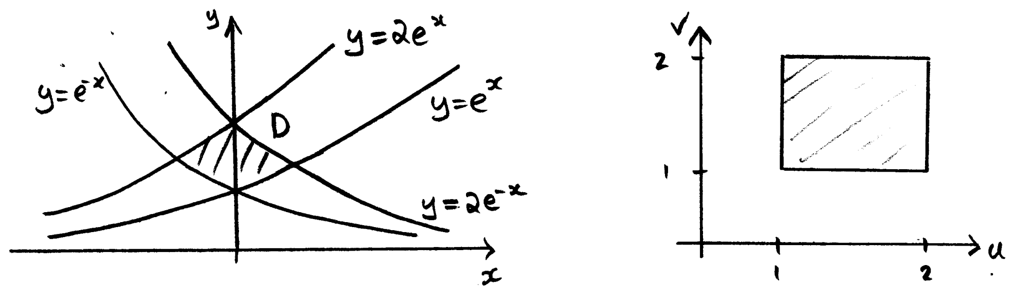

Figure 2.22: Domain in x, y and u, v coordinates

Solution Let this region be denoted by D then its area is

$$A = \iint_D dxdy,$$

(cf. $b - a =$ length of interval $[a, b] = \int_a^b dx$.) We use variables $u = ye^x$ and $v = ye^{-x}$ so that D is defined by $1 \leq u \leq 2$ and $1 \leq v \leq 2$. See Figure 2.22. Then

$$\frac{1}{J} = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} ye^x & e^x \\ -ye^{-x} & e^{-x} \end{vmatrix} = 2y = 2\sqrt{uv},$$

for $y > 0$. Hence

$$\begin{aligned} A &= \iint_D \left| \frac{1}{2\sqrt{uv}} \right| dudv \\ &= \frac{1}{2} \int_1^2 \frac{du}{\sqrt{u}} \int_1^2 \frac{dv}{\sqrt{v}} \\ &= \frac{1}{2} \left(\left[2\sqrt{u} \right]_1^2 \right)^2 \\ &= 2(\sqrt{2} - \sqrt{1})^2 = 2(2 - 2\sqrt{2} + 1) \\ &= 2(3 - 2\sqrt{2}). \end{aligned}$$

□

2.7 Triple integration

(Stewart (Ed. 7): Section 15.7, p1041.)

As we defined double integrals for functions of two variables we can define triple integrals for functions of three variables. Recall the definition of a double integral is given by

$$\iint_R f(x,y)\, dxdy = \lim_{N,\, M \to \infty} \sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i, y_j) \delta x_i \delta y_j.$$

For a triple integral instead of summing over an area $\delta A_{ij} = \delta x_i \delta y_j$, we sum over a volume $\delta V_{ijk} = \delta x_i \delta y_j \delta z_k$ which leads us to

$$\iiint_V f(x,y,z)\, dxdydz = \lim_{N,\, M,\, L \to \infty} \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{k=1}^{L} f(x_i, y_j, z_k) \delta x_i \delta y_j \delta z_k.$$

If the limit exists we say that f is *integrable over* V and we call this the *triple integral of* f *over* V and $dV = dxdydz$ is called the *volume element.*

Visualising a triple integral is not really possible, but they are useful as we see in the next section and later in Chapter 4.

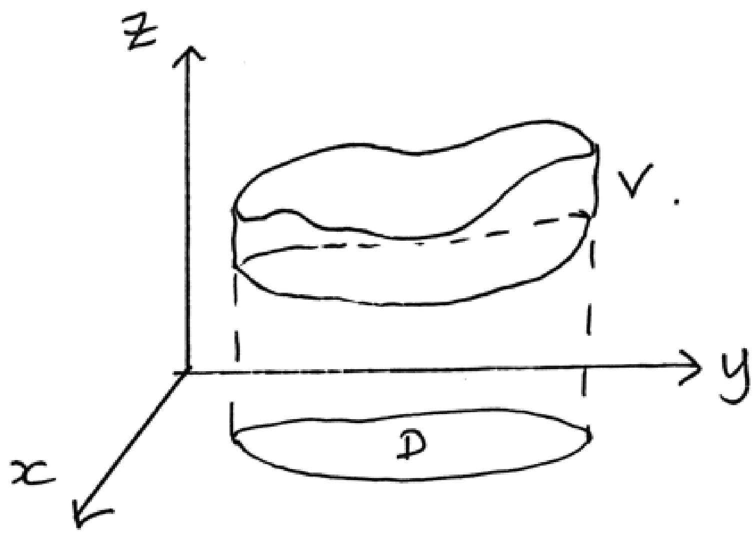

Figure 2.23: Illustration of V lying between two continuous functions of x and y, and its projection D onto the xy-plane.

If V lies between two continuous functions of x and y, that is

$$V = \{(x, y, z) : (x, y) \in D,\ u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is the projection of V onto the $xy-$plane and $u_1(x, y)$ is the upper boundary of V and $u_2(x, y)$ is the lower boundary of V as illustrated in Figure 2.23. If D is a type I region then the general form for a triple integral is given by

$$\iiint_V f(x, y, z)\, dxdydz = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dz \right) dxdy = \underbrace{\int_a^b}_{\text{Constants}} dx \underbrace{\int_{g_1(x)}^{g_2(x)}}_{\text{Curves}} dy \underbrace{\int_{u_1(x,y)}^{u_2(x,y)}}_{\text{Surfaces}} f(x, y, z)\, dz.$$

If D is a type II region then the general form for a triple integral is given by

$$\iiint_V f(x, y, z)\, dxdydz = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dz \right) dxdy = \underbrace{\int_c^d}_{\text{Constants}} dy \underbrace{\int_{h_1(y)}^{h_2(y)}}_{\text{Curves}} dx \underbrace{\int_{u_1(x,y)}^{u_2(x,y)}}_{\text{Surfaces}} f(x, y, z)\, dz.$$

We can generalise this further to deal with volumes V which lie between two continuous functions of y and z giving

$$\iiint_V f(x, y, z)\, dxdydz = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\ dx \right) dydz,$$

where D is the projection of V onto the yz-plane. Lastly, we can generalise to volumes V which lie between two continuous functions of x and z give

$$\iiint_V f(x,y,z)\,dxdydz = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\; dy \right) dxdz,$$

where D is the projection of V onto the xz-plane.

Remarks

1. (See Stewart (ed. 7) p1046). The volume of a solid $V \subset \mathbb{R}^3$ is given by the following triple integral
$$\iiint_V 1\,dxdydz.$$

2. (See Stewart (ed. 7) p1047). Extending this further we define the *mass* of the solid V, where the *density* $f(x,y,z)$ of the solid varies across V by
$$\iiint_V f(x,y,z)\,dxdydz.$$

Example 2.12 Evaluate

$$I = \iiint_V z\,dxdydz,$$

where V is the solid tetrahedron bounded by the four planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

Solution We draw two pictures, one of V and one of the projection of V onto the xy-plane, as shown in Figure 2.24.

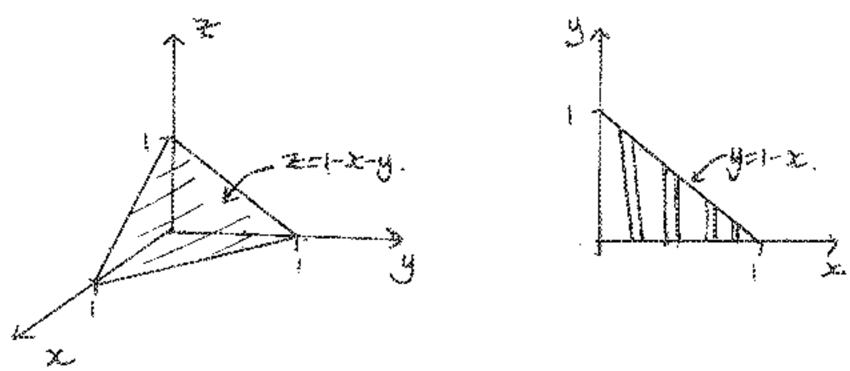

Figure 2.24: Solid tetrahedron and the projection in the xy-plane.

Thus the domain of integration is $0 \le x \le 1$, $0 \le y \le 1-x$ and $0 \le z \le 1-x-y$.

We have

$$
\begin{aligned}
I &= \iiint_V z\,dxdydz = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} z\,dzdydx \\
&= \int_0^1 dx\int_0^{1-x}\Big[\frac{z^2}{2}\Big]_0^{1-x-y} dy = \int_0^1 dx\int_0^{1-x}\frac{(1-x-y)^2}{2}\,dy \\
&= \frac{-1}{6}\int_{x=0}^{1}\big[(1-x-y)^3\big]_{y=0}^{1-x} dx = \frac{1}{6}\int_{x=0}^{1}(1-x)^3\,dx \\
&= \frac{1}{24}\big[-(1-x)^4\big]_{x=0}^{1} = \frac{1}{24}.
\end{aligned}
$$

□

Example 2.13 Set up (but do not evaluate) the integral for the volume of the solid that lies below the paraboloid $z = 9 - x^2 - y^2$ and above the plane $z = 5$.

Solution We draw two pictures, one of the volume V and one of the projection of V onto the xy-plane, as shown in Figure 2.25.

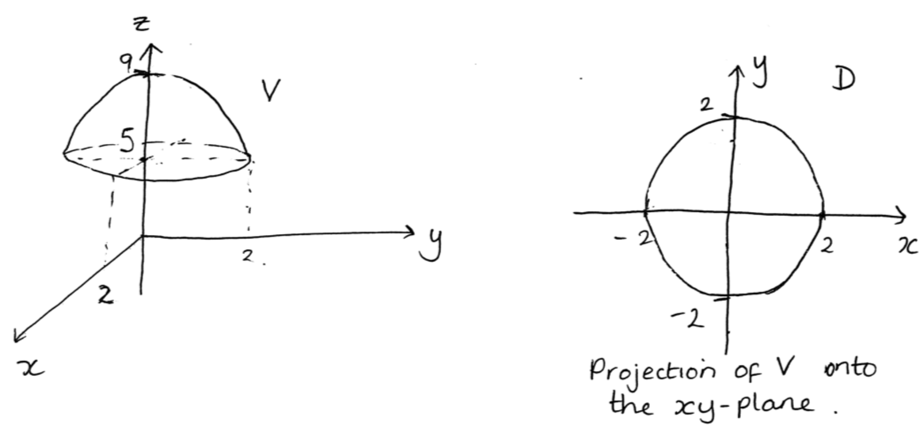

Figure 2.25: Solid paraboloid and the projection in the xy-plane.

Thus the domain of integration is $-2 \le x \le 2$, $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$ and $5 \le z \le 9 - x^2 - y^2$. We have

$$
\text{Volume} = \iiint_V 1\,dxdydz = \int_{x=-2}^{2} dx\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\int_{z=5}^{9-x^2-y^2} 1\,dz
$$

□

2.8 Triple integration in spherical coordinates

(Stewart (Ed. 7): Section 15.9, p1057.)

The position of a point (x, y, z) in cartesian coordinates can be specified by *spherical coordinates* ρ, θ, ϕ where

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

$\theta \in [0, 2\pi)$ is the anti-clockwise angle between the positive x-axis and the projection onto the xy-plane of the line joining (x, y, z) to $(0, 0, 0)$. The length of the line is $\rho \geq 0$ and $\phi \in [0, \pi)$ is the clockwise angle between the positive z-axis and the line joining (x, y, z) to $(0, 0, 0)$. See Figure 2.26.

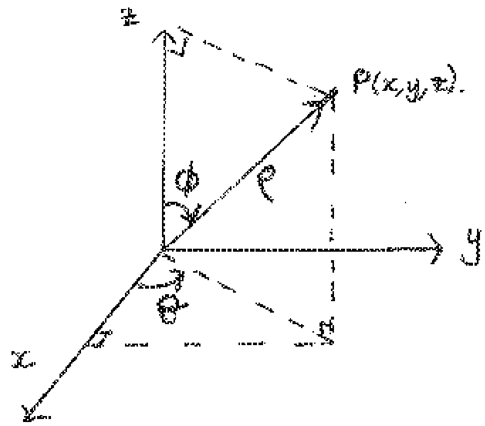

Figure 2.26: Spherical coordinates.

Remarks

1. Note that $x^2+y^2+z^2 = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + \rho^2 \cos^2\phi = \rho^2$ so that expressions involving $x^2 + y^2 + z^2$ can be written in terms of ρ alone.

In cartesian coordinates, the volume of an elementary cuboid used in the Riemann sum is $\delta V = \delta x\, \delta y\, \delta z$ and for this reason the volume element dV is $dx\, dy\, dz$. In spherical coordinates, the volume element is illustrated in Figure 2.27 and has volume $\delta V \approx \rho^2 \sin\phi\, \delta\theta\, \delta\phi\, \delta\rho$.

For this reason in spherical coordinates, $dV = \rho^2 \sin\phi\, d\theta\, d\phi d\rho$, i.e.,

$$\iiint_V f(x, y, z)\, dxdydz = \iiint_V f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi\; d\theta d\phi d\rho.$$

When either the domain is spherical or the integrand is written in terms of $x^2 + y^2 + z^2$ $(= \rho^2)$, the triple integral should be rewritten in spherical coordinates.

Example 2.14 Use spherical coordinates to evaluate

$$I = \iiint_B \exp((x^2 + y^2 + z^2)^{3/2})\, dxdydz,$$

where B is the unit ball, $B = \{(x, y, z)|x^2 + y^2 + z^2 \leq 1\}$.

Solution In spherical coordinates the domain is $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$.

We have

$$I = \int_0^\pi \int_0^{2\pi} \int_0^1 e^{\rho^3} \rho^2 \sin\phi\, d\rho d\theta d\phi = \int_0^\pi \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3}\, d\rho.$$

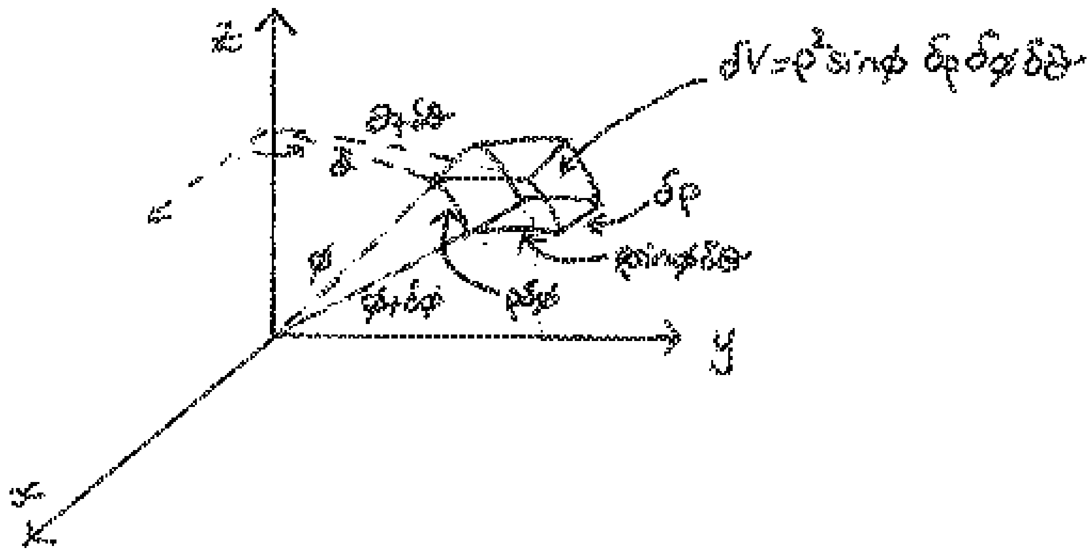

Figure 2.27: The volume element in spherical coordinates

Now we use the substitution

$$u = \rho^3, \quad du = 3\rho^2 d\rho, \quad \boxed{\begin{array}{c|cc} \rho & 0 & 1 \\ \hline u & 0 & 1 \end{array}},$$

we get

$$\begin{aligned} I &= \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^1 \frac{e^u}{3}\, du \\ &= \big[-\cos\phi\big]_0^{\pi} \big[\theta\big]_0^{2\pi} \big[\tfrac{1}{3}e^u\big]_0^1 \\ &= (-(-1)-(-1))\,(2\pi)\,\big(\tfrac{1}{3}(e-1)\big) = \frac{4}{3}\pi(e-1). \end{aligned}$$

□

Example 2.15 Find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$ as illustrated in Figure 2.28.

Solution By completing the square we can rewrite the equation of the sphere as $x^2+y^2+(z-1/2)^2 = 1/4$. Thus the sphere is centered at $(0,0,1/2)$ and has radius $1/2$.

Using spherical coordinates the equation for the sphere becomes

$$\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + \rho^2\cos^2\phi = \rho\cos\phi.$$

Hence, $\rho = \cos\phi$. So in describing the solid in spherical coordinates, we have $0 \le \rho \le \cos\phi$.

Using spherical coordinates the equation for the cone and sphere meet when $\rho\cos\phi = \rho\sin\phi$. Hence $\phi = \pi/4$. So we have $0 \le \phi \le \pi/4$. Lastly, θ varies from 0 to 2π. See Figure 2.29

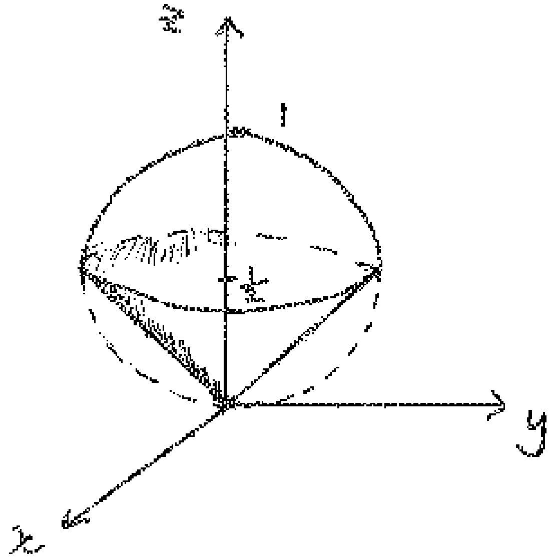

Figure 2.28: Solid

Thus, the volume of the solid is given by,

$$I = \iiint_V 1 \; dxdydz = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho d\phi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\cos\phi} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \cos^3\phi \, d\phi = \frac{2\pi}{3} \left[\frac{-\cos^4\phi}{4}\right]_0^{\pi/4} = \frac{\pi}{8}.$$

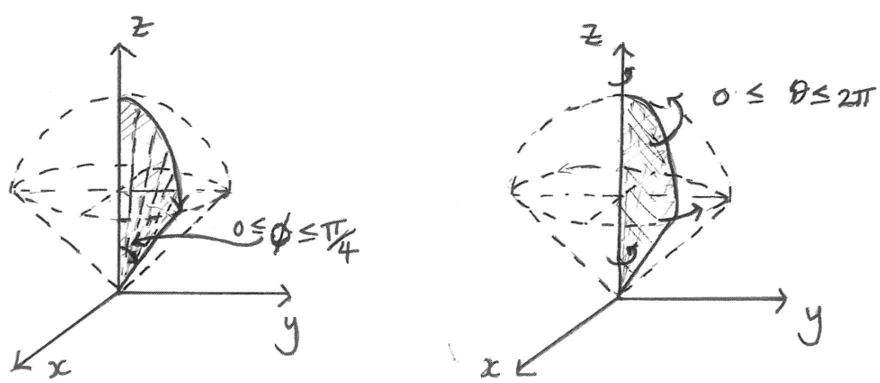

Figure 2.29: Finding the range of ϕ and θ.

□