Chapter 3

Differentiation of vectors

Chapter Summary

Objective	**Tools**
Know and use the definition of grad, div and curl and understand the meaning of vector and scalar fields	Nabla is the differential operator, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. For a scalar field f, $\operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$. For a vector field $\mathbf{F} = (F_1, F_2, F_3)$, $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$, $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$.
Find the directional derivative	$\frac{\partial f}{\partial \mathbf{u}} = \mathbf{u} \cdot \nabla f = u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} + u_3 \frac{\partial f}{\partial z}$. The formula for a directional derivative can only be used for *unit* vectors. To calculate the directional derivative in the direction of a non-unit vector $\mathbf{v}$, one must use the unit vector with the same direction as $\mathbf{v}$, that is $\mathbf{u} = \frac{\mathbf{v}}{\lvert\mathbf{v}\rvert}$.
Know the definition of the Laplacian and be able to calculate it.	Given a scalar field f, the Laplacian of f, written $\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.. This definition may be extended to the Laplacian of a vector field $\mathbf{F} = (F_1, F_2, F_3)$, to give $\nabla^2 \mathbf{F} = (\nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3)$.
Prove the nabla identities and use them to prove other results	Nabla identities: $\operatorname{grad}(f+g) = \operatorname{grad} f + \operatorname{grad} g$, $\operatorname{div}(\mathbf{F}+\mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$, $\operatorname{curl}(\mathbf{F}+\mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$, $\operatorname{grad}(fg) = f(\operatorname{grad} g) + (\operatorname{grad} f)g$, $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + (\operatorname{grad} f) \cdot \mathbf{F}$, $\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl} \mathbf{F} + \operatorname{grad} f \times \mathbf{F}$, $\operatorname{curl} \operatorname{grad} f = \mathbf{0}$, and $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$.

3.1 Vector-valued functions

In the previous chapters we have considered real functions of several (usually two) variables $f\colon D \to \mathbb{R}$, where D is a subset of $\mathbb{R}^n$, where n is the number of variables. These are *scalar-valued* functions in the sense that the result of applying such a function is a real number, which is a scalar quantity. We now wish to consider *vector-valued* functions $\mathbf{f}\colon D \to \mathbb{R}^m$. In principal, m can be any positive integer, but we will only consider the cases where $m = 2$ or 3, and the results of applying the function is either a 2D or 3D vector.

The simplest type of vector-valued function has the form $\mathbf{f}\colon I \to \mathbb{R}^2$, where $I \subset \mathbb{R}$. Such a function returns a 2D vector $\mathbf{f}(t)$ for each $t \in I$, which may be regarded as the position vector of some point on the plane.

3.2 Vector and scalar fields

(Stewart (Ed. 7): Section 16.1, p1081.)

A function of two or three variables mapping to a vector is called a *vector field*. In contrast, a function of two or three variables mapping to a scalar is called a *scalar field*. As we saw in Chapter 1 (using different terminology), one can represent the graph of a scalar field as a curve or surface. A vector field $\mathbf{F}(x, y)$ (or $\mathbf{F}(x, y, z)$) is often represented by drawing the vector $\mathbf{F}(\mathbf{r})$ at point $\mathbf{r}$ for representative points in the domain. A good example of a vector field is the velocity at a point in a fluid; at each point we draw an arrow (vector) representing the velocity (the speed and direction) of fluid flow (see Figure 3.1). The length of the arrow represents the fluid speed at each point.

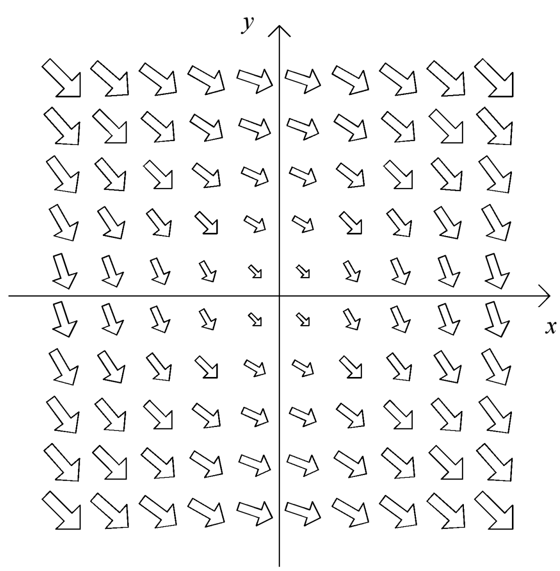

Figure 3.1: Vector field representing fluid velocity

3.3 Different types of derivative

(Stewart (Ed. 7): Section 14.6, p952.)

We have already discussed the derivatives and partial derivatives of scalar functions. Next we will consider discuss other different types of "derivatives" of scalar and vector functions; in some cases the result is a scalar and sometimes a vector.

Recall that if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors and α is a scalar, there are a number of different products that can be made;

Name of product	Formula	Type of result
Scalar multiplication	$\alpha\mathbf{u}$	Vector
Scalar or dot product	$\mathbf{u}\cdot\mathbf{v}$	Scalar
Vector or cross product	$\mathbf{u}\times\mathbf{v}$	Vector

Now consider the vector differential operator

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

This is read as *del* or *nabla* and is not to be confused with Δ, the capital Greek letter delta. One can form "products" of this vector with other vectors and scalars, but because it is an operator, it always has to be the first term if the product is to make sense. For example, if f is a scalar field, we can form the scalar "multiple" with ∇ as the first term

$$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

the result being a vector.

Below we will introduce the "derivatives" corresponding to the product of vectors given in the above table.

3.3.1 Gradient ("multiplication by a scalar")

This is just the example given above. We define the *gradient* of a scalar field f to be

$$\boxed{\operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)}.$$

We will use both of the notation $\operatorname{grad} f$ and ∇f interchangeably.

Remark Note that f must be a *scalar* field for $\operatorname{grad} f$ to be defined and $\operatorname{grad} f$ itself is a *vector* field.

Example 3.1 Find the gradient of the scalar field $f(x, y, z) = x^2y + x\cosh yz$. (Recall from 1S/1Y that $\cosh x = \frac{e^x+e^{-x}}{2}$ is the hyperbolic cosine and the hyperbolic sine is given by $\sinh x = \frac{e^x-e^{-x}}{2}$.)

Solution : We have

$$\frac{\partial f}{\partial x} = 2xy + \cosh yz, \quad \frac{\partial f}{\partial y} = x^2 + xz\sinh yz, \quad \frac{\partial f}{\partial z} = xy\sinh yz.$$

Therefore,

$$\operatorname{grad} f = (2xy + \cosh yz, x^2 + xz\sinh yz, xy\sinh yz).$$

□

Example 3.2 Let $\mathbf{r} = (x, y, z)$ so that $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. Show that

$$\nabla(r^n) = nr^{n-2}\mathbf{r},$$

for any integer n and deduce the values of $\operatorname{grad}(r)$, $\operatorname{grad}(r^2)$ and $\operatorname{grad}(1/r)$.

Solution : We have

$$\begin{aligned}\frac{\partial}{\partial x} r^n &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} \\ &= 2x\,\frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \\ &= nxr^{n-2}.\end{aligned}$$

Then, using the symmetry of r with respect to x, y and z, we get

$$\frac{\partial}{\partial y} r^n = nyr^{n-2}, \quad \frac{\partial}{\partial z} r^n = nzr^{n-2},$$

and thus

$$\nabla(r^n) = \left(\frac{\partial}{\partial x}(r^n), \frac{\partial}{\partial y}(r^n), \frac{\partial}{\partial z}(r^n)\right) = (nxr^{n-2}, nyr^{n-2}, nzr^{n-2}) = nr^{n-2}\mathbf{r}.$$

Hence

$$\begin{aligned}\text{grad}(r) &= \nabla(r) = 1r^{1-2}\mathbf{r} = \frac{\mathbf{r}}{r}, \\ \text{grad}(r^2) &= 2r^{2-2}\mathbf{r} = 2\mathbf{r},\end{aligned}$$

and

$$\text{grad}(1/r) = \nabla(r^{-1}) = (-1)r^{-1-2}\mathbf{r} = -\mathbf{r}/r^3.$$

□

Example 3.3 Determine grad$(\mathbf{c}\cdot\mathbf{r})$, when c is a constant (vector).

Solution : Let $\mathbf{c} = (c_1, c_2, c_3)$ so that

$$\begin{aligned}\text{grad}(\mathbf{c}\cdot\mathbf{r}) &= \text{grad}(c_1x + c_2y + c_2z) \\ &= \left(\frac{\partial(c_1x + c_2y + c_2z)}{\partial x}, \frac{\partial(c_1x + c_2y + c_2z)}{\partial y}, \frac{\partial(c_1x + c_2y + c_2z)}{\partial z}\right) \\ &= (c_1, c_2, c_3) = \mathbf{c}.\end{aligned}$$

□

Directional derivative This is the rate of change of a scalar field f in the direction of a *unit* vector $\mathbf{u} = (u_1, u_2, u_3)$. As with normal derivatives it is defined by the limit of a difference quotient, in this case the directional derivative of f at $\mathbf{p}$ in the direction $\mathbf{u}$ is defined to be

$$\lim_{h\to 0^+} \frac{f(\mathbf{p}+h\mathbf{u}) - f(\mathbf{p})}{h}, \qquad (*)$$

(if the limit exists) and is denoted

$$\frac{\partial f}{\partial \mathbf{u}}(\mathbf{p}).$$

This definition is rarely used directly. The key formula for the directional derivative of f in the direction $\mathbf{u}$ is

$$\boxed{\frac{\partial f}{\partial \mathbf{u}} = \mathbf{u}\cdot\nabla f = u_1\frac{\partial f}{\partial x} + u_2\frac{\partial f}{\partial y} + u_3\frac{\partial f}{\partial z}}\;.$$

To prove this, first notice that

$$\frac{d}{dt}f(\mathbf{p}+t\mathbf{u}) = \lim_{h\to 0^+}\frac{f(\mathbf{p}+(t+h)\mathbf{u})-f(\mathbf{p}+t\mathbf{u})}{h}$$

so that $(*)$ can be obtained as

$$\left.\frac{d}{dt}f(\mathbf{p}+t\mathbf{u})\right|_{t=0}.$$

Also, using the chain rule, we have

$$\frac{d}{dt}f(\mathbf{p}+t\mathbf{u}) = u_1\frac{\partial f}{\partial x}(\mathbf{p}+t\mathbf{u}) + u_2\frac{\partial f}{\partial y}(\mathbf{p}+t\mathbf{u}) + u_3\frac{\partial f}{\partial z}(\mathbf{p}+t\mathbf{u}) = \mathbf{u}\cdot\nabla f(\mathbf{p}+t\mathbf{u}).$$

Combining these results gives the required formula.

Remarks

1. The formula for a directional derivative can only be used for *unit* vectors. To calculate the directional derivative along a non-unit vector $\mathbf{v}$, one must use the unit vector having the same direction as $\mathbf{v}$, that is
$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

2. Partial derivatives are special cases of directional derivatives. For example, the partial x-derivative is the directional derivative in the direction $(1,0,0)$.

Example 3.4 Find the directional derivative of $f = x^2yz^3$ at the point $P(3,-2,-1)$ in the direction of the vector $(1,2,2)$.

Solution : The *unit* vector with the same direction as $(1,2,2)$ is

$$\mathbf{u} = \frac{(1,2,2)}{\sqrt{1^2+2^2+2^2}} = \left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right).$$

Hence the required directional derivative is

$$\begin{aligned}\mathbf{u}\cdot\nabla f &= \left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)\cdot(2xyz^3, x^2z^3, 3x^2yz^2)\\ &= \frac{1}{3}(2xyz^3+2x^2z^3+6x^2yz^2).\end{aligned}$$

At the point P, this gives

$$\frac{\partial f}{\partial \mathbf{u}}(3,-2,-1) = \frac{1}{3}(12-18-108) = -38.$$

□

If we fix a point $\mathbf{p}$ and are given a function f, then by considering all possible directional derivatives of f at the point $\mathbf{p}$ we can ask:

- in which direction does f change fastest?
- what is the maximal rate of change?

The following theorem answers these questions.

Theorem *Suppose f is a differentiable function for which $\nabla f(\mathbf{p}) \neq 0$ then the maximal value of $\frac{\partial f}{\partial \mathbf{u}}(\mathbf{p})$ is $|\nabla f(\mathbf{p})|$ and occurs when $\mathbf{u}$ is in the same direction as ∇f.*

Remark Proof: See Exercise Sheet 7, question F1.

Example 3.5 Consider $f = \ln(xy + z^3)$ at the point $P(1,1,1)$. In what direction does f have the maximal rate of change? What is this maximal rate of change?

Solution : The theorem above states that f increases the fastest in the direction of the gradient vector at the point $P(1,1,1)$.

$$\nabla f = \left(\frac{y}{xy + z^3}, \frac{x}{xy + z^3}, \frac{3z^2}{xy + z^3} \right),$$

hence $\nabla f(1,1,1) = (1/2, 1/2, 3/2)$ is the direction of the maximal rate of change. The maximal rate of change is

$$|\nabla f(1,1,1)| = \frac{\sqrt{11}}{2}.$$

□

3.3.2 Divergence of a vector field ("scalar product")

(Stewart (Ed. 7): Section 16.5, p1118.)

The *divergence* of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is the *scalar* obtained as the "scalar product" of ∇ and $\mathbf{F}$,

$$\boxed{\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}\,.$$

It is so called, because it measures the tendency of a vector field to diverge (positive divergence) or converge (negative divergence). In particular, a vector field is said to be *incompressible* (or *solenoidal*) if its divergence is zero.

Figure 3.2 shows the vector fields $\mathbf{F} = (x, y, 0)$, $\mathbf{G} = (x, -y, 0)$ and $\mathbf{H} = (-x, -y, 0)$ in the xy-plane. We have

$$\operatorname{div} \mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 2 > 0$$

and similarly, $\operatorname{div} \mathbf{G} = 0$ and $\operatorname{div} \mathbf{H} = -2 < 0$. Notice how the arrows on the plot of $\mathbf{F}$ diverge and on the plot of $\mathbf{H}$ converge.

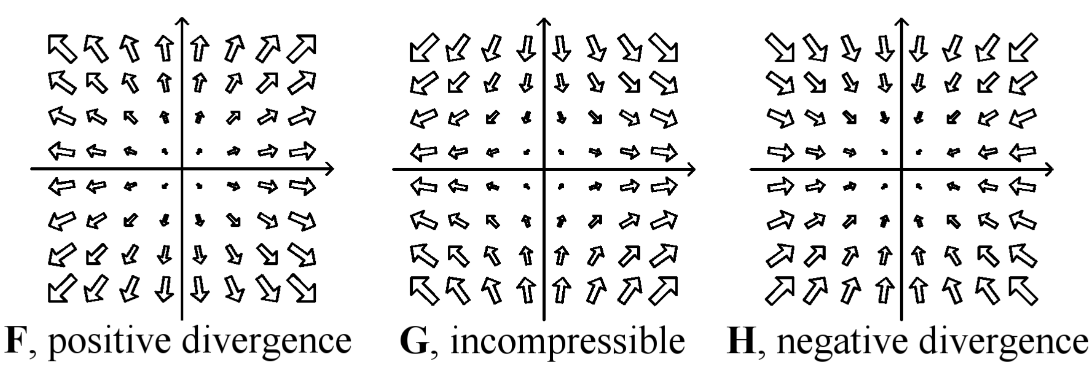

Figure 3.2: Positive and negative divergence

Example 3.6 Show that the divergence of $\mathbf{F} = (x - y^2, z, z^3)$ is positive at all points in $\mathbb{R}^3$.

Solution : We have

$$\operatorname{div}\mathbf{F} = \frac{\partial(x-y^2)}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial(z^3)}{\partial z} = 1 + 0 + 3z^2 = 1 + 3z^2.$$

Hence for every (x, y, z), $\operatorname{div}\mathbf{F} \geq 1 > 0$. □

A particular example of divergence is the *Laplacian* of a scalar field. Given a scalar field f, $\operatorname{grad} f = \nabla f$ is a vector field and the divergence of ∇f is the Laplacian of f, written $\nabla^2 f$. This means that

$$\boxed{\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}}\,.$$

This definition may be extended in a natural way to the Laplacian of a vector field $\mathbf{F} = (F_1, F_2, F_3)$,

$$\boxed{\nabla^2 \mathbf{F} = (\nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3)}\,.$$

Example 3.7 Find the values of n for which $\nabla^2(r^n) = 0$.

Solution : We have $r = \sqrt{x^2+y^2+z^2}$ and so from Example 3.2,

$$\frac{\partial(r^n)}{\partial x} = nxr^{n-2}.$$

Therefore,

$$\begin{aligned}\frac{\partial^2(r^n)}{\partial x^2} &= nr^{n-2} + nx\,(n-2)xr^{n-4}\\ &= nr^{n-4}\big(r^2 + (n-2)x^2\big),\end{aligned}$$

and because of the symmetry in r with respect to x, y and z, we also have

$$\frac{\partial^2(r^n)}{\partial y^2} = nr^{n-4}\big(r^2+(n-2)y^2\big), \quad \frac{\partial^2(r^n)}{\partial z^2} = nr^{n-4}\big(r^2+(n-2)z^2\big).$$

Taking the sum of these we get

$$\begin{aligned}\nabla^2(r^n) &= nr^{n-4}\big(3r^2 + (n-2)(x^2+y^2+z^2)\big)\\ &= n(n+1)r^{n-2}.\end{aligned}$$

Hence $\nabla^2(r^n) = 0$ if and only if $n = 0$ or $n = -1$. □

3.3.3 Curl of a vector field ("vector product")

(Stewart (Ed. 7): Section 16.5, p1115.)

The *curl* of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is the *vector* obtained as the "vector product" of ∇ and $\mathbf{F}$

$$\boxed{\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}}\,.$$

Like any other vector product, $\operatorname{curl}\mathbf{F}$ can be calculated using a 3×3 determinant,

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}\,.$$

The curl of a vector field measures its tendency to rotate. In particular, a vector field is said to be *irrotational* if its curl is the zero vector. Figure 3.3 shows the vector fields $\mathbf{F} = (-y, x, 0)$, $\mathbf{G} = (y, x, 0)$ and $\mathbf{H} = (y, -x, 0)$. We have

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\mathbf{k}$$

and similarly, $\operatorname{curl}\mathbf{G} = \mathbf{0}$ and $\operatorname{curl}\mathbf{H} = -2\mathbf{k} < 0$. The coefficient of $\mathbf{k}$ in $\operatorname{curl}\mathbf{F}$ being negative indicates anticlockwise rotation.

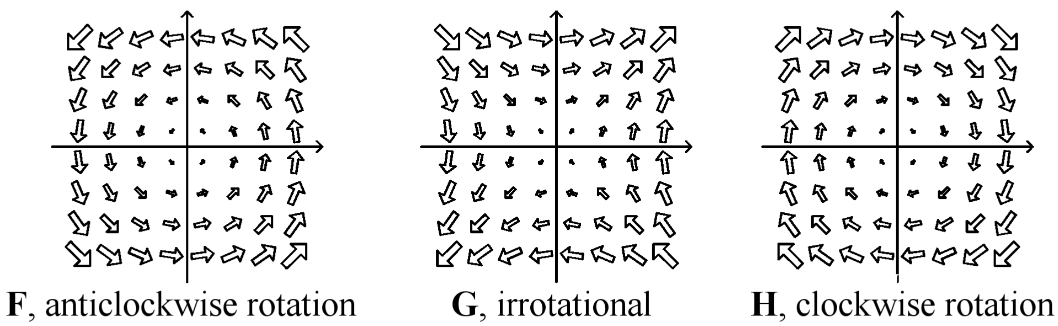

Figure 3.3: Clockwise and anticlockwise rotation

Example 3.8 Determine $\operatorname{curl}\mathbf{F}$ when $\mathbf{F} = (x^2y, xy^2 + z, xy)$.

Solution : We have

$$\begin{aligned} \operatorname{curl}\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & xy^2 + z & xy \end{vmatrix} \\ &= (x-1)\mathbf{i} + (0-y)\mathbf{j} + (y^2 - x^2)\mathbf{k} \\ &= (x-1, -y, y^2 - x^2). \end{aligned}$$

□

Example 3.9 If $\mathbf{c}$ is a constant vector, find $\operatorname{curl}(\mathbf{c} \times \mathbf{r})$.

Solution : We have $\mathbf{r} = (x, y, z)$ and let $\mathbf{c} = (c_1, c_2, c_3)$. First, we calculate

$$\mathbf{c} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & c_3 \\ x & y & z \end{vmatrix} = (c_2 z - c_3 y, c_3 x - c_1 z, c_1 y - c_2 x).$$

Then,

$$\begin{aligned} \operatorname{curl}(\mathbf{c} \times \mathbf{r}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ c_2 z - c_3 y & c_3 x - c_1 z & c_1 y - c_2 x \end{vmatrix} \\ &= \big(c_1 - (-c_1)\big)\mathbf{i} + \big(c_2 - (-c_2)\big)\mathbf{j} + \big(c_3 - (-c_3)\big)\mathbf{k} \\ &= 2\mathbf{c}. \end{aligned}$$

□

3.4 Nabla identities

(Stewart (Ed. 7): Section 16.5, p1118.)

There are analogues involving div, grad and curl of the elementary rules of differentiation such as linearity $(f + g)'(x) = f'(x) + g'(x)$ the product rule $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$.

Let f and g be smooth scalar fields and $\mathbf{F}$ and $\mathbf{G}$ smooth vector fields. Then all of the following are straightforward to prove (as illustrated in Example 3.10) just using definitions

$$\boxed{\begin{array}{rlrl} \operatorname{grad}(f+g) &= \operatorname{grad} f + \operatorname{grad} g & \operatorname{grad}(fg) &= f(\operatorname{grad} g) + (\operatorname{grad} f)g, \\ \operatorname{div}(\mathbf{F}+\mathbf{G}) &= \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G} & \operatorname{div}(f\mathbf{F}) &= f\operatorname{div}\mathbf{F} + (\operatorname{grad} f)\cdot\mathbf{F}, \\ \operatorname{curl}(\mathbf{F}+\mathbf{G}) &= \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G} & \operatorname{curl}(f\mathbf{F}) &= f\operatorname{curl}\mathbf{F} + \operatorname{grad} f \times \mathbf{F}, \\ \operatorname{curl}\operatorname{grad} f &= \mathbf{0}, & \operatorname{div}\operatorname{curl}\mathbf{F} &= 0. \end{array}}$$

In particular, note the special cases

$$\operatorname{grad}(cf) = c \operatorname{grad} f, \quad \operatorname{div}(c\mathbf{F}) = c \operatorname{div}\mathbf{F}, \quad \operatorname{curl}(c\mathbf{F}) = c \operatorname{curl}\mathbf{F},$$

when c is a (scalar) constant.

All of the identities are easier to remember if written using ∇. For example,

$$\begin{aligned} \operatorname{curl}(f\mathbf{F}) &= \nabla \times (f\mathbf{F}) \\ &= f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F} \\ &= f \operatorname{curl}\mathbf{F} + \operatorname{grad} f \times \mathbf{F}. \end{aligned}$$

Example 3.10 Prove the identities

(i) $\operatorname{curl}\operatorname{grad} f = 0$, (ii) $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + \operatorname{grad} f \times \mathbf{F}$, (iii) $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + (\operatorname{grad} f)\cdot\mathbf{F}$

Solution : We have (i)

$$\begin{aligned}
\operatorname{curl}\operatorname{grad} f &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} \\
&= \big((f_z)_y - (f_y)_z\big)\mathbf{i} + \big((f_x)_z - (f_z)_x\big)\mathbf{j} + \big((f_y)_x - (f_x)_y\big)\mathbf{k} \\
&= (0,0,0) = \mathbf{0},
\end{aligned}$$

and (ii),

$$\begin{aligned}
\operatorname{curl}(f\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix} \\
&= \big((fF_3)_y - (fF_2)_z\big)\mathbf{i} + \big((fF_1)_z - (fF_3)_x\big)\mathbf{j} + \big((fF_2)_x - (fF_1)_y\big)\mathbf{k} \\
&= f\left[\big((F_3)_y - (F_2)_z\big)\mathbf{i} + \big((F_1)_z - (F_3)_x\big)\mathbf{j} + \big((F_2)_x - (F_1)_y\big)\mathbf{k}\right] \\
&\qquad + (f_yF_3 - f_zF_2)\mathbf{i} + (f_zF_1 - f_xF_3)\mathbf{j} + (f_xF_2 - f_yF_1)\mathbf{k} \\
&= f\operatorname{curl}\mathbf{F} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= f\operatorname{curl}\mathbf{F} + \operatorname{grad} f \times \mathbf{F},
\end{aligned}$$

as required.

(iii) Left as an exercise. □

Example 3.11 Let $\mathbf{r} = (x, y, z)$ denote a position vector with length $r = \sqrt{x^2 + y^2 + z^2}$ and $\mathbf{c}$ is a constant (vector). Determine

$$\text{(i)}\ \operatorname{div}(r^n(\mathbf{c}\times\mathbf{r})), \quad \text{(ii)}\ \operatorname{curl}(r^n(\mathbf{c}\times\mathbf{r})).$$

Solution :

$$\mathbf{c}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & c_3 \\ x & y & z \end{vmatrix} = (c_2z - c_3y, c_3x - c_1z, c_1y - c_2x).$$

(i) Using the identity $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \operatorname{grad}(f)\cdot\mathbf{F}$, and setting $f = r^n$ and $\mathbf{F} = \mathbf{c}\times\mathbf{r}$ gives

$$\begin{aligned}
\operatorname{div}(r^n(\mathbf{c}\times\mathbf{r})) &= r^n\operatorname{div}(\mathbf{c}\times\mathbf{r}) + \operatorname{grad}(r^n)\cdot(\mathbf{c}\times\mathbf{r}) \\
&= 0 + nr^{n-2}\mathbf{r}\cdot(\mathbf{c}\times\mathbf{r}) \\
&= 0.
\end{aligned}$$

This uses the result from Example 3.2, and the fact that $\mathbf{c}\times\mathbf{r})$ is perpendicular to $\mathbf{r}$.

(ii), using the identity $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + \operatorname{grad}(f)\times\mathbf{F}$ gives

$$\begin{aligned}
\operatorname{curl}(r^n(\mathbf{c}\times\mathbf{r})) &= r^n\operatorname{curl}((\mathbf{c}\times\mathbf{r})) + \operatorname{grad}(r^n)\times(\mathbf{c}\times\mathbf{r}) \\
&= r^n 2\mathbf{c} + nr^{n-2}\mathbf{r}\times(\mathbf{c}\times\mathbf{r}) \quad \text{(by Ex 3.9 and 3.2)} \\
&= 2r^n\mathbf{c} + nr^{n-2}((\mathbf{r}\cdot\mathbf{r})\mathbf{c} - (\mathbf{r}\cdot\mathbf{c})\mathbf{r}) \quad \text{(by the vector triple product)} \\
&= (2+n)r^n\mathbf{c} - nr^{n-2}(\mathbf{r}\cdot\mathbf{c})\mathbf{r}
\end{aligned}$$

□