Chapter 2

Double and triple integration

Example 2.1 Evaluate

$$\iint_R x^2 + y^2 \, dxdy$$

where R is $[1,3] \times [2,4]$.

Solution The integral may either be evaluated as

$$\begin{aligned}
\iint_R x^2 + y^2 \, dxdy &= \int_1^3 dx \int_2^4 x^2 + y^2 \, dy \\
&= \int_1^3 \left[x^2 y + \frac{1}{3} y^3 \right]_2^4 dx \\
&= \int_1^3 (4-2)x^2 + \frac{56}{3} \, dx \\
&= \left[\frac{2}{3} x^3 + \frac{56}{3} x \right]_1^3 \\
&= \frac{2(3^3 - 1^3) + 56(3-1)}{3} = \frac{164}{3},
\end{aligned}$$

or as

$$\begin{aligned}
\iint_R x^2 + y^2 \, dxdy &= \int_2^4 dy \int_1^3 x^2 + y^2 \, dx \\
&= \int_2^4 \left[\frac{1}{3} x^3 + xy^2 \right]_1^3 dy \\
&= \int_2^4 \frac{(3^3 - 1^3)}{3} + (3-1)y^2 \, dy \\
&= \left[\frac{26}{3} y + \frac{2}{3} y^3 \right]_2^4 \\
&= \frac{26(4-2) + 2(4^3 - 2^3)}{3} = \frac{164}{3}.
\end{aligned}$$

Not surprisingly, each method gives the same answer. □

Example 2.2 State whether each of the domains shown in Figure 2.1 are type I and/or type II or regular.

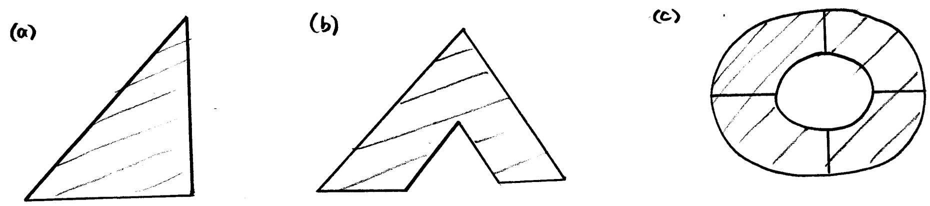


Figure 2.1: Example domains

Solution (a) Both horizontal and vertical lines intersect the triangle in an interval or a single point. Hence this domain is both type I and type II.

(b) Vertical lines intersect this domain in an interval. Hence it is type I. The intersection with some horizontal lines is a union of two intervals, hence it is not type II.

(c) Some horizontal and some vertical lines intersect the annulus in a union of intervals. Hence this domain is neither type I nor type II. The domain may be divided into four type I and type II domains as shown. Hence it is regular. □

Example 2.3 Evaluate

$$\iint_D xy^2\, dxdy,$$

where D is the region in the first quadrant bounded by the curve $y = 4x^2$, the x axis and the line $x = 1$.

Solution It is important to draw a sketch of the domain. This is given in Figure 2.2.

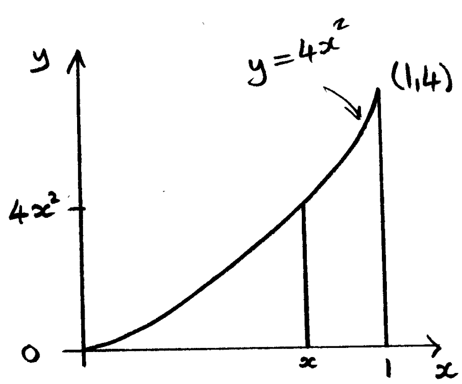


Figure 2.2: type I domain

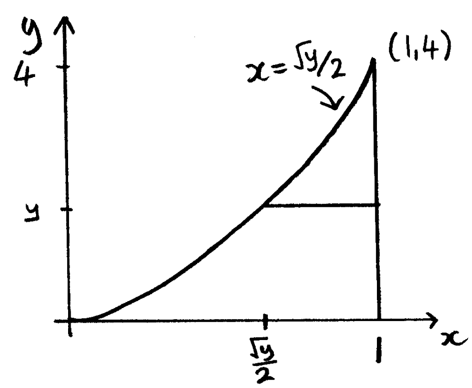


Figure 2.3: type II domain

This domain is clearly both type I and type II but it is more readily thought of as type I; $0 \leq y \leq 4x^2$

where $0 \leq x \leq 1$. Hence

$$\begin{aligned}\iint_D xy^2\,dxdy &= \int_0^1 dx \int_0^{4x^2} xy^2\,dy \\ &= \int_0^1 \left[\frac{1}{3}xy^3\right]_0^{4x^2} dx = \int_0^1 \frac{1}{3}x((4x^2)^3 - 0^3)\,dx \\ &= \frac{64}{3}\int_0^1 x^7\,dx \\ &= \frac{64}{3}\frac{1}{8}\left[x^8\right]_0^1 = \frac{8}{3}.\end{aligned}$$

□

Example 2.4 Evaluate

$$I = \iint_D 3x^2 + y^2\,dxdy,$$

where D is the triangle with vertices $(0,0)$, $(1,1)$ and $(2,1)$.

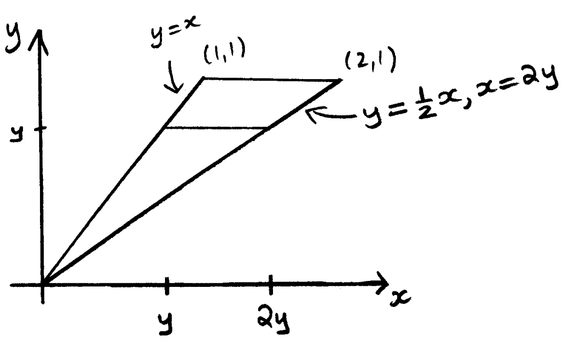


Figure 2.4: type II domain

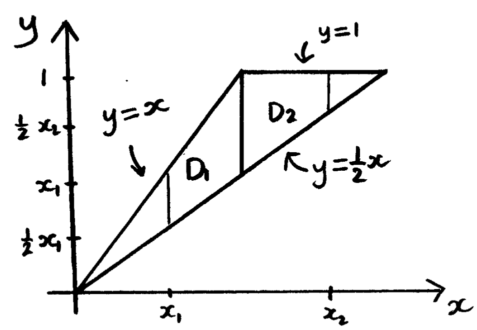


Figure 2.5: Two type I domains

Solution The region is sketched in Figure 2.4 and is both type I and type II. The type II formulation is easier

$$\begin{aligned}I &= \int_0^1 dy \int_y^{2y} 3x^2 + y^2\,dx \\ &= \int_0^1 \left[x^3 + xy^2\right]_y^{2y} dy = \int_0^1 (2y)^3 - y^3 + (2y-y)y^2\,dy \\ &= 8\int_0^1 y^3\,dy = 2\left[y^4\right]_0^1 \\ &= 2.\end{aligned}$$

□

Example 2.5 Evaluate

$$I = \int_0^1 dx \int_{\sqrt{x}}^1 \frac{e^{y^2}}{\sqrt{x}}\,dy.$$

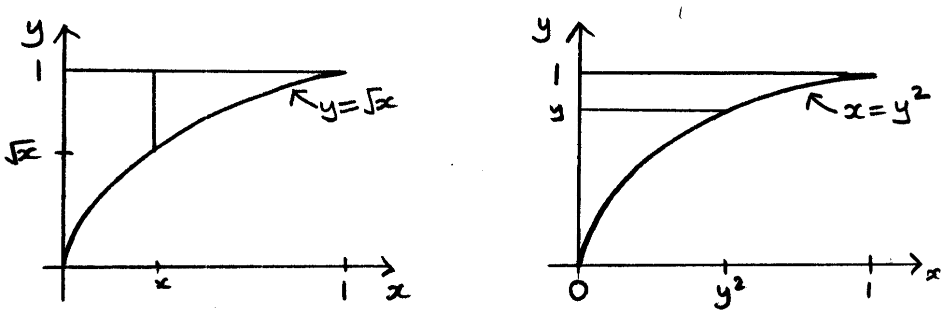


Figure 2.6: From type I to type II

Solution A sketch of the domain with type I and type II descriptions is given in Figure 2.6. Using this sketch we can convert the double integral into type II form

$$
\begin{aligned}
I &= \int_0^1 dy \int_0^{y^2} \frac{e^{y^2}}{\sqrt{x}}\,dx \\
&= \int_0^1 \left[2\sqrt{x}e^{y^2}\right]_0^{y^2} dy = 2\int_0^1 y e^{y^2}\,dy.
\end{aligned}
$$

Now we can use the substitution

$$
u = y^2, \quad du = 2y\,dy, \quad \begin{array}{|c|cc|} \hline y & 0 & 1 \\ \hline u & 0 & 1 \\ \hline \end{array},
$$

to give

$$
I = \int_0^1 e^u\,du = \left[e^u\right]_0^1 = e - 1.
$$

□

Example 2.6 Find the volume of the tetrahedron T, bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

Solution A sketch of the solid in 3-D and a sketch of the base, the planar region A over with we integrate are given in Figure 2.7. Using this sketch we can write down a double integral which describes the volume of the tetrahedron.

The plane $x + 2y + z = 2$ intersects the xy-plane ($Z = 0$) in the line $x = 2y = 2$. So the tetrahedron lies above the triangular region A in the xy-plane. A is bounded by $x + 2y$, $x + 2y = 2$ and $x = 0$.

The plane $x + 2y + z = 2$ can be written as $z = 2 - x - 2Y$. So the volume of the tetrahedron is the volume that lies under the graph of the function $z = 2 - x - y$ and above A, where $A = \{(x, y)|0 \leq x \leq 1, x/2 \leq y \leq 1 - x/2\}$.

$$
\begin{aligned}
T &= \iint_A (2 - x - 2y)\,dydx = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\,dydx \\
&= \int_0^1 \left[2y - xy - y^2\right]_{y=x/2}^{y=1-x/2} dx = \int_0^1 x^2 - 2x + 1\,dx \\
&= \left[\frac{x^3}{3} - x^2 + x\right]_0^1 = \frac{1}{3}.
\end{aligned}
$$

□

Example 2.7 Use polar coordinates to evaluate

$$I = \iint_D x + y\,dxdy,$$

where D is part of the annulus between circles of radius 1 and 2, centre $(0,0)$ lying in upper half plane.

Solution In polar coordinates the domain is $1 \le r \le 2$, $0 \le \theta \le \pi$ (see Figure 2.8.)

We have

$$\begin{aligned}
I &= \int_0^\pi d\theta \int_1^2 (r\cos\theta + r\sin\theta) r\,dr = \int_0^\pi \cos\theta + \sin\theta\,d\theta \int_1^2 r^2\,dr \\
&= \big[\sin\theta - \cos\theta\big]_0^\pi \frac{1}{3}\big[r^3\big]_1^2 \\
&= \big(0 - 0 - (-1-1)\big)\frac{1}{3}(2^3 - 1^3) = \frac{14}{3}.
\end{aligned}$$

□

Example 2.8 Evaluate

$$I = \iint_D y\,dxdy,$$

where D is the part of the disc of radius $a\,(>0)$ and centre $(a,0)$ lying in the first quadrant.

Solution The border of the disk has equation $(x-a)^2 + y^2 = a^2$, i.e., $x^2 + y^2 = 2ax$. In polar coordinates, this is

$$r^2 = 2ar\cos\theta, \quad \text{i.e., } r = 2a\cos\theta.$$

See Figure 2.9.

The domain is $0 \le r \le 2a\cos\theta$ where $0 \le \theta \le \pi/2$ and so

$$\begin{aligned}
I &= \int_0^{\pi/2} d\theta \int_0^{2a\cos\theta} (r\sin\theta) r\,dr = \int_0^{\pi/2} d\theta \int_0^{2a\cos\theta} r^2 \sin\theta\,dr \\
&= \frac{1}{3}\int_0^{\pi/2} \sin\theta \big[r^3\big]_0^{2a\cos\theta} d\theta = \frac{8a^3}{3}\int_0^{\pi/2} \sin\theta\cos^3\theta\,d\theta.
\end{aligned}$$

Using the change of variable

$$u = \cos\theta, \quad du = -\sin\theta, \quad \begin{array}{|c|cc|} \hline \theta & 0 & \pi/2 \\ \hline u & 1 & 0 \\ \hline \end{array},$$

we get

$$I = -\frac{8a^3}{3}\int_1^0 u^3\,du = \frac{8a^3}{3}\frac{1}{4}\big[u^4\big]_0^1 = \frac{2a^3}{3}.$$

□

Example 2.9 Evaluate:

$$\text{(a) } I = \int_0^\pi \sin^3 x \cos^4 x\;dx, \quad \text{(b) } I = \int_0^\pi \sin^3 x \cos^5 x\;dx, \quad \text{(c) } I = \int_0^{2\pi} \sin^2 x \cos^4 x\;dx.$$

Solution In each case we make a table of sign first to indicate if the integrand is above (+) or below (−) the x-axis in each quadrant. The symmetry of sine and cosine means that the absolute value of the area under the curve in each quadrant is the same, so the total integral is given by summing the number of plus signs minus the number of minus signs and multiplying the result by integral of the function over the first quadrant.

(a)

Quadrant	1	2
$\sin^3 x$	+	+
$\cos^4 x$	+	+
$\sin^3 x \cos^4 x$	+	+

Total=+2.

Hence,

$$I = 2\int_0^{\pi/2} \sin^3 x \cos^4 x \, dx = 2 \cdot \frac{3 \cdot 2 \cdot 1}{7 \cdot 5 \cdot 3 \cdot 1 \cdot} = \frac{4}{35}.$$

(b)

Quadrant	1	2
$\sin^3 x$	+	+
$\cos^5 x$	+	-
$\sin^3 x \cos^5 x$	+	-

Total=0.

Hence,

$$I = 0.$$

(c)

Quadrant	1	2	3	4
$\sin^2 x$	+	+	+	+
$\cos^4 x$	+	+	+	+
$\sin^2 x \cos^4 x$	+	+	+	+

Total=+4.

Hence,

$$I = 4\int_0^{\pi/2} \sin^2 x \cos^4 x \, dx = 4 \cdot \frac{1 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi}{8}.$$

□

Example 2.10 By making a suitable change of variables, evaluate

$$\iint_D x + 3y \, dxdy,$$

where D is the region bounded by the lines

$$y = x - 1, \quad y = x + 1, \quad y = -x - 1, \quad y = -x + 3.$$

Solution If we define $u = x + y$ and $v = x - y$ then the domain D is described by $-1 \leq u \leq 3$, $-1 \leq v \leq 1$. See Figure 2.10.

We have

$$\frac{1}{J} = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

Inverting the change of variable we get

$$x = \frac{u + v}{2}, \quad y = \frac{u - v}{2},$$

and so the integrand is $x + 3y = \frac{1}{2}\big((u + v) + 3(u - v)\big) = 2u - v$.

Hence

$$
\begin{aligned}
I &= \iint_D (2u - v) \left| \frac{1}{-2} \right| \, dudv \\
&= \frac{1}{2} \int_{-1}^{3} du \int_{-1}^{1} 2u - v \, dv = \frac{1}{2} \int_{-1}^{3} \left[2uv - \frac{1}{2} v^2 \right]_{-1}^{1} \, du \\
&= \frac{1}{2} \int_{-1}^{3} 2\big(1 - (-1)\big) u \, du = \int_{-1}^{3} 2u \, du = \left[u^2\right]_{-1}^{3} \\
&= \left(3^2 - (-1)^2\right) = 8.
\end{aligned}
$$

□

Example 2.11 Find the area bounded by the curves $y = e^x$, $y = 2e^x$, $y = e^{-x}$ and $y = 2e^{-x}$.

Solution Let this region be denoted by D then its area is

$$A = \iint_D dxdy,$$

(cf. $b - a =$ length of interval $[a, b] = \int_a^b dx$.) We use variables $u = ye^x$ and $v = ye^{-x}$ so that D is defined by $1 \le u \le 2$ and $1 \le v \le 2$. See Figure 2.11. Then

$$\frac{1}{J} = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} ye^x & e^x \\ -ye^{-x} & e^{-x} \end{vmatrix} = 2y = 2\sqrt{uv},$$

for $y > 0$. Hence

$$
\begin{aligned}
A &= \iint_D \left| \frac{1}{2\sqrt{uv}} \right| \, dudv \\
&= \frac{1}{2} \int_1^2 \frac{du}{\sqrt{u}} \int_1^2 \frac{dv}{\sqrt{v}} \\
&= \frac{1}{2} \left(\left[2\sqrt{u}\right]_1^2 \right)^2 \\
&= 2(\sqrt{2} - \sqrt{1})^2 = 2(2 - 2\sqrt{2} + 1) \\
&= 2(3 - 2\sqrt{2}).
\end{aligned}
$$

□

Example 2.12 Evaluate

$$I = \iiint_V z \, dxdydz,$$

where V is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Solution We draw two pictures, one of V and one of the projection of V onto the xy-plane, as shown in Figure 2.12.

Thus the domain of integration is $0 \le x \le 1$, $0 \le y \le 1 - x$ and $0 \le z \le 1 - x - y$.

We have

$$\begin{aligned}
I &= \iiint_V z\, dxdydz = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} z\, dzdydx \\
&= \int_0^1 dx \int_0^{1-x} \left[\frac{z^2}{2}\right]_0^{1-x-y} dy = \int_0^1 dx \int_0^{1-x} \frac{(1-x-y)^2}{2}\, dy \\
&= \frac{-1}{6} \int_{x=0}^{1} \left[(1-x-y)^3\right]_{y=0}^{1-x} dx = \frac{1}{6} \int_{x=0}^{1} (1-x)^3\, dx \\
&= \frac{1}{24} \left[-(1-x)^4\right]_{x=0}^{1} = \frac{1}{24}.
\end{aligned}$$

□

Example 2.13 Set up (but do not evaluate) the integral for the volume of the solid that lies below the paraboloid $z = 9 - x^2 - y^2$ and above the plane $z = 5$.

Solution We draw two pictures, one of the volume V and one of the projection of V onto the xy-plane, as shown in Figure 2.13.

Thus the domain of integration is $-2 \le x \le 2$, $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$ and $5 \le z \le 9 - x^2 - y^2$.

We have

$$\text{Volume} = \iiint_V 1\, dxdydz = \int_{x=-2}^{2} dx \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \int_{z=5}^{9-x^2-y^2} 1\, dz$$

□

Example 2.14 Use spherical coordinates to evaluate

$$I = \iiint_B \exp((x^2+y^2+z^2)^{3/2})\, dxdydz,$$

where B is the unit ball, $B = \{(x,y,z) | x^2+y^2+z^2 \le 1\}$.

Solution In spherical coordinates the domain is $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

We have

$$I = \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{\rho^3} \rho^2 \sin\phi\, d\rho d\theta d\phi = \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3}\, d\rho.$$

Now we use the substitution

$$u = \rho^3, \quad du = 3\rho^2 d\rho, \quad \begin{array}{|c|cc|} \hline \rho & 0 & 1 \\ \hline u & 0 & 1 \\ \hline \end{array},$$

we get

$$\begin{aligned}
I &= \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^1 \frac{e^u}{3}\, du \\
&= \left[-\cos\phi\right]_0^{\pi} \left[\theta\right]_0^{2\pi} \left[\frac{1}{3} e^u\right]_0^1 \\
&= (-(-1) - (-1))\,(2\pi)\,\left(\frac{1}{3}(e-1)\right) = \frac{4}{3}\pi(e-1).
\end{aligned}$$

□

Example 2.15 Find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$ as illustrated in Figure 2.14.)

Solution By completing the square we can rewrite the equation of the sphere as $x^2+y^2+(z-1/2)^2 = 1/4$. Thus the sphere is centered at $(0,0,1/2)$ and has radius $1/2$.

Using spherical polar coordinates the equation for the sphere becomes

$$\rho^2 \sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi = \rho\cos\phi.$$

Hence, $\rho = \cos\phi$. So in describing the solid in polar coordinates, we have $0 \le \rho \le \cos\phi$.

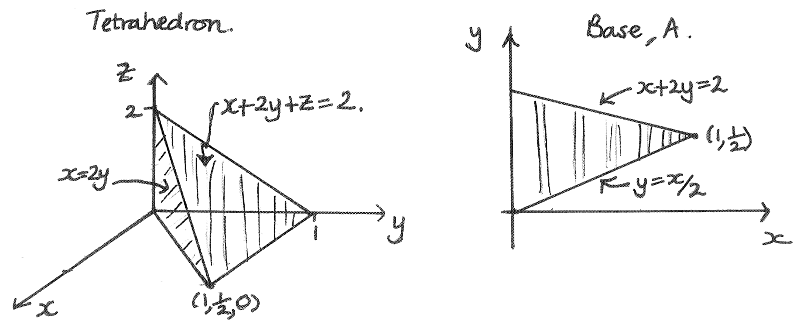
Using spherical polar coordinates the equation for the cone becomes $\rho\cos\phi = \rho\sin\phi$. Hence $\phi = \pi/4$. So we have $0 \le \phi \le \pi/4$. Lastly, θ varies from 0 to 2π. See Figure 2.15

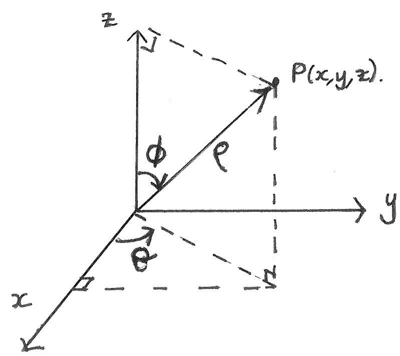
Thus, the volume of the solid is given by,

$$\begin{aligned} I &= \iiint_V 1 \, dxdydz = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho d\phi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\cos\phi} d\phi \\ &= \frac{2\pi}{3}\int_0^{\pi/4} \sin\phi\cos^3\phi \, d\phi = \frac{2\pi}{3}\Big[\frac{-\cos^4\phi}{4}\Big]_0^{\pi/4} = \frac{\pi}{8}. \end{aligned}$$

□

Fig 1.13. ex-domain5.pdf





spherical.pdf
. Fig 1.22

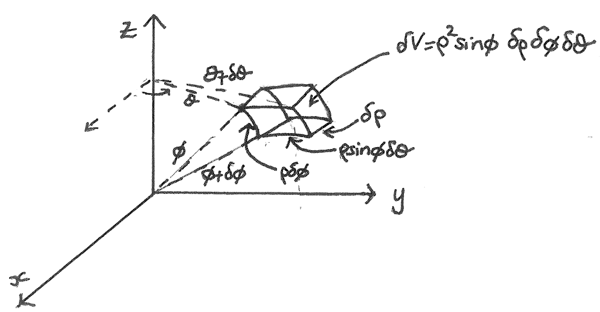


Fig 1.23.
sphericalarea.pdf

Figure 2.7: 3-D solid tetrahedron and the base, A.

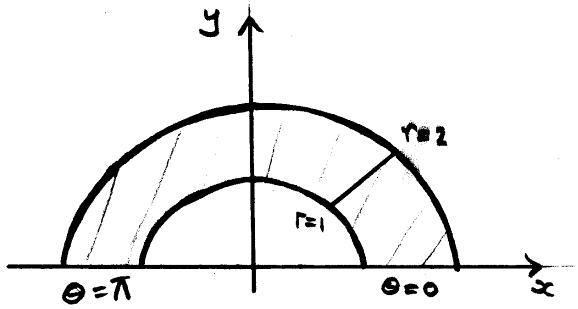


Figure 2.8: Annular domain

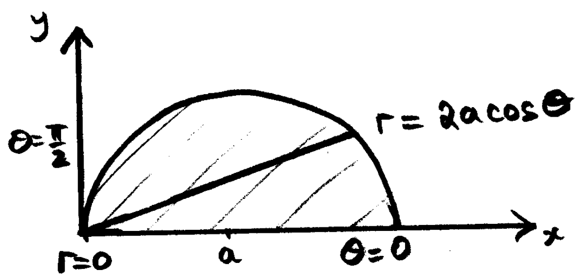


Figure 2.9: Semicircular domain

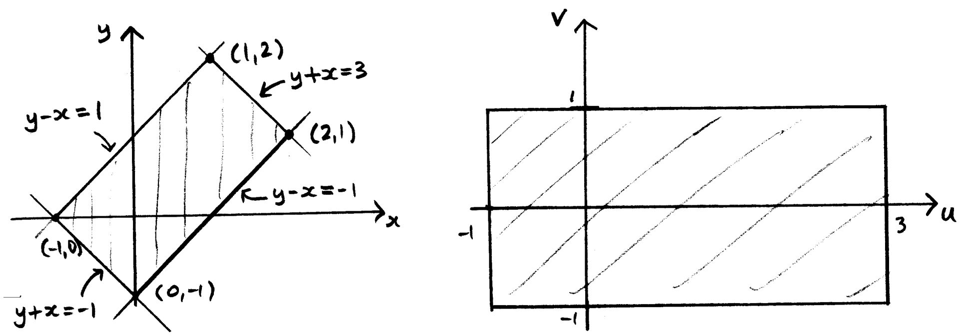


Figure 2.10: Domain in x, y and u, v coordinates

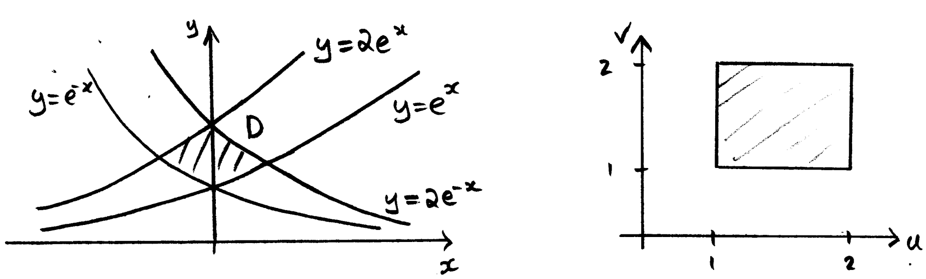


Figure 2.11: Domain in x, y and u, v coordinates

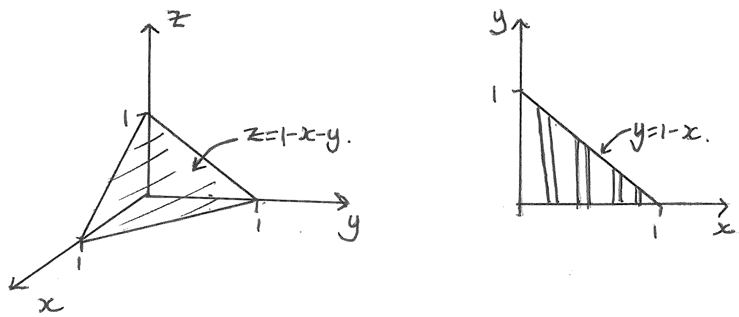
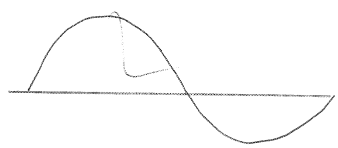


Fig 1.21.



SineCosine.pdf .

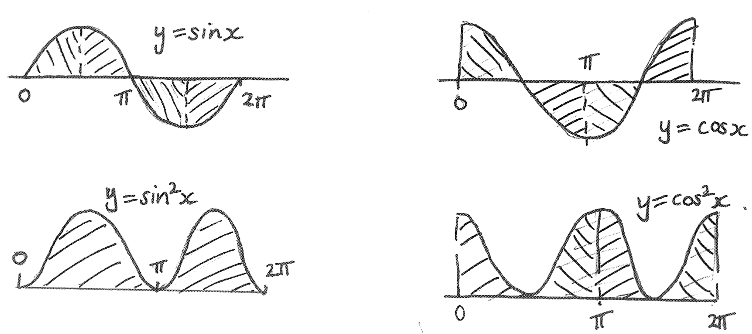


Figure 2.12: Solid tetrahedron and the projection in the xy-plane.

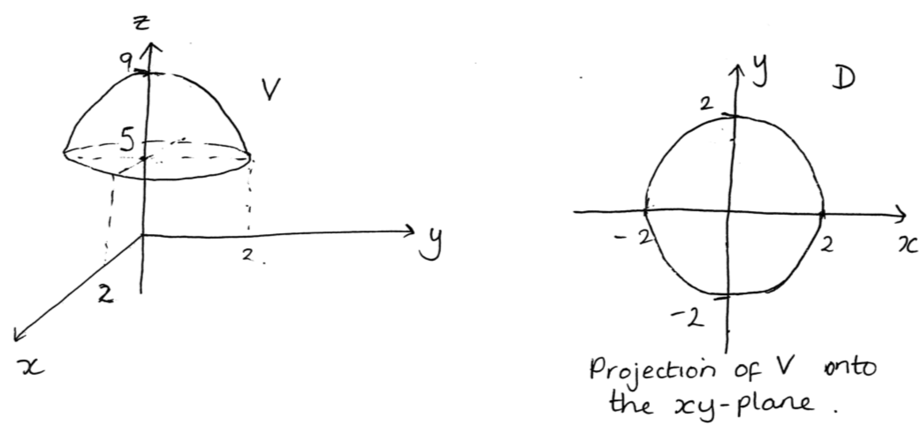


Figure 2.13: Solid paraboloid and the projection in the xy-plane.

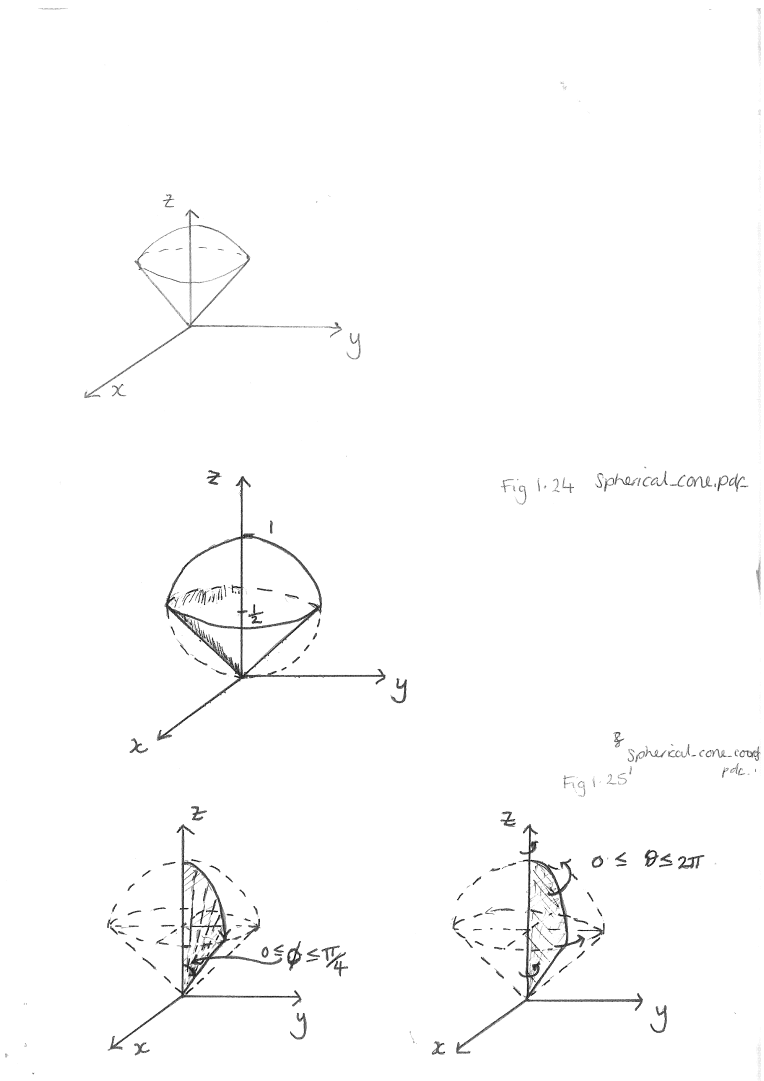


Figure 2.14: Solid

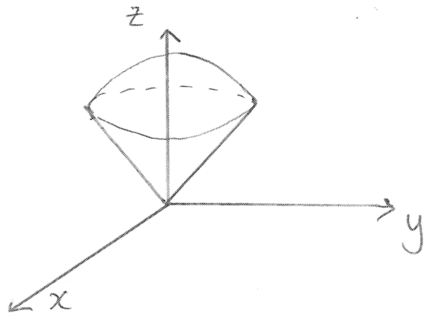
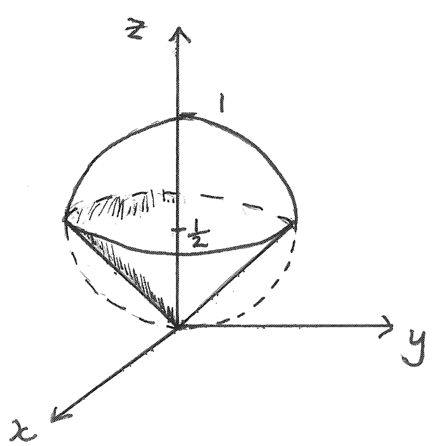


Fig 1.24 Spherical_cone.pdf



& Spherical_cone_coord pdf.
Fig 1.25'

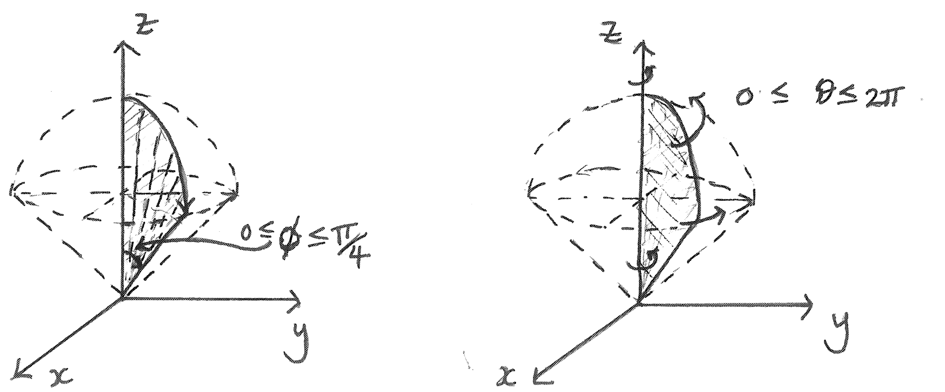


Figure 2.15: Finding the range of ϕ and θ.