Proof of Archimedes' Principle using the Divergence Theorem

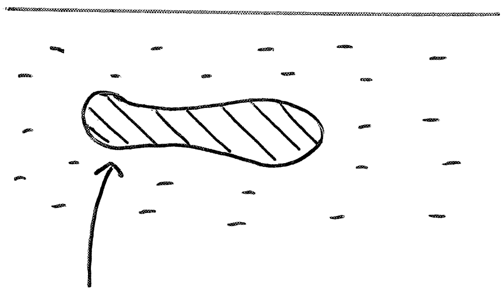
Around 250 BC Archimedes predicted that

> The buoyancy force on an object submerged in a fluid equals the weight of the fluid displaced by the object.

Over 2000 years later it was possible to prove this using the Divergence Theorem.

Proof

Let the object be completely submerged in the fluid:



Let S be the boundary of the object and V be the region occupied by the object.

It can be shown that the buoyancy force on the object due to the pressure p of the fluid is

$$\underline{b} = -\iint_S \underset{\text{pressure}}{p}\ \underset{\text{unit outer normal}}{\underline{n}}\ dS \qquad \leftarrow \text{surface integral}$$

It can also be shown (see the appendix) that

$$p(x,y,z) = -\rho g z$$

pressure

density of the fluid

acceleration due to gravity

Therefore

$$\underline{b} = -\iint_S p\,\underline{n}\,dS$$

$$= \iint_S \rho g z\,\underline{n}\,dS \qquad \left(\begin{aligned}\underline{n} &= n_1\underline{i} + n_2\underline{j} + n_3\underline{k} \\ &= (\underline{n}\cdot\underline{i})\underline{i} + (\underline{n}\cdot\underline{j})\underline{j} + (\underline{n}\cdot\underline{k})\underline{k}\end{aligned}\right)$$

$$= \underline{i}\iint_S (\rho g z\,\underline{i})\cdot\underline{n}\,dS$$

$$+ \underline{j}\iint_S (\rho g z\,\underline{j})\cdot\underline{n}\,dS + \underline{k}\iint_S (\rho g z\,\underline{k})\cdot\underline{n}\,dS$$

Divergence Theorem

$$= \underline{i}\iiint_V \operatorname{div}(\rho g z\,\underline{i})\,dx\,dy\,dz + \underline{j}\iiint_V \operatorname{div}(\rho g z\,\underline{j})\,dx\,dy\,dz$$

$$+ \underline{k}\iiint_V \operatorname{div}(\rho g z\,\underline{k})\,dx\,dy\,dz$$

$$= 0 + 0 + \underline{k}\iiint_V \rho g\,dx\,dy\,dz$$

$$= \underline{k}\, g \underbrace{\iiint_V \rho \, dx\,dy\,dz}_{\substack{= m \\ = \text{mass of fluid displaced by } V}}$$

$$= \underline{k} \underbrace{mg}_{= \text{weight of fluid displaced by object}}$$

Therefore $\boxed{\underline{b} = mg\,\underline{k}}$ as required. $\square$

Appendix: Derivation of $p = -\rho g z$ from the Navier-Stokes equations.

Let: $\underline{v}(\underline{x},t)$ be the velocity of the fluid

$p(\underline{x},t)$ be the pressure in the fluid

ρ = constant be the density of the fluid

μ = viscosity of the fluid

$\underline{f}$ = gravitational force per unit volume of the fluid.

The Navier-Stokes equations for a viscous incompressible fluid are

$$\boxed{\begin{gathered} \rho\left(\frac{\partial \underline{v}}{\partial t} + \begin{pmatrix} \underline{v}\cdot\nabla v_1 \\ \underline{v}\cdot\nabla v_2 \\ \underline{v}\cdot\nabla v_3 \end{pmatrix}\right) = -\nabla p + \mu \nabla^2 \underline{v} + \underline{f} \quad (*) \\ \\ \nabla\cdot\underline{v} = 0 \end{gathered}}$$

This is just Newton's Second Law $F = ma$ for the fluid.

Substituting $\underline{v} = \underline{0}$, $\underline{f} = -\rho g \underline{k}$ into $(*)$ gives

$$\nabla p = -\rho g \underline{k}.$$

Solving this PDE for p gives

$$\boxed{p = -\rho g z} \quad \text{as required.} \quad \square$$