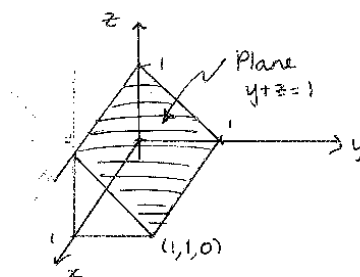


Tutorial Exercises

T1 Sketch the wedge shaped region W (in the first octant) enclosed by the five planes $x = 0$, $y = 0$, $z = 0$, $x = 1$ and $y + z = 1$. Then evaluate

$$\iiint_W xy \, dx \, dy \, dz.$$

Solution



Hence the integral is

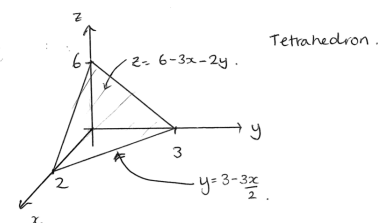
$$\begin{aligned} I &= \int_0^1 dx \int_0^1 dy \int_0^{1-y} xy \, dz = \int_0^1 x \, dx \int_0^1 y [z]_0^{1-y} dy \\ &= \int_0^1 x \, dx \int_0^1 y - y^2 \, dy = \int_0^1 x \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 dx \\ &= \int_0^1 \frac{1}{6} x \, dx = \frac{1}{6} \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{12}. \end{aligned}$$

T2 Sketch the solids whose volume is given by the following integrals

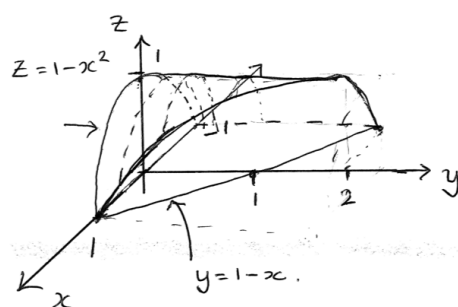
(a) $\int_0^2 dx \int_0^{3-3x/2} dy \int_0^{6-3x-2y} 1 \, dz$ (b) $\int_{-1}^1 dx \int_0^{1-x} dy \int_0^{1-x^2} 1 \, dz$

Solution

(a)



(b)



Tunnel shape
with a diagonal
slice removed.

T3 A solid shell of variable density is in the form of the region lying between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$. The density ρ of the shell at the point (x, y, z) is given by $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Find the mass of the shell.

Solution

Mass is given by the triple integral of the density ($\sqrt{x^2 + y^2 + z^2}$) over the volume V . Hence,

$$\begin{aligned} \text{Mass} &= \int \int \int_V \sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz = \int_0^{2\pi} d\theta \int_0^\pi d\phi \int_1^3 \rho^3 \sin \phi \, d\rho \\ &= 2\pi \int_0^\pi \sin \phi \, d\phi \int_1^3 \rho^3 \, d\rho = 4\pi \left[\frac{\rho^4}{4} \right]_1^3 = 80\pi. \end{aligned}$$

T4 Evaluate

$$\iiint z^2 \, dx \, dy \, dz$$

throughout

- the part of the sphere $x^2 + y^2 + z^2 = a^2$ ($a > 0$) in the first octant,
- the complete interior of the sphere $x^2 + y^2 + z^2 = a^2$ ($a > 0$).

Solution

(a) Hence the integral is

$$\begin{aligned} I &= \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \int_0^a \rho^4 \cos^2 \phi \sin \phi \, d\rho \\ &= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \cos^2 \phi \sin \phi \, d\phi \int_0^a \rho^4 \, d\rho = \frac{\pi}{2} \cdot \frac{1.1}{3.1} \left[\frac{\rho^5}{5} \right]_0^a = \frac{\pi a^5}{30}. \end{aligned}$$

(b) Hence the integral is

$$\begin{aligned}
 I &= \int_0^{2\pi} d\theta \int_0^\pi d\phi \int_0^a \rho^4 \cos^2 \phi \sin \phi d\rho \\
 &= \int_0^{2\pi} d\theta \int_0^\pi \cos^2 \phi \sin \phi d\phi \int_0^a \rho^4 d\rho \\
 &= 4\pi \int_0^{\pi/2} \cos^2 \phi \sin \phi d\phi \cdot \frac{a^5}{5} = \frac{4\pi a^5}{5} \cdot \frac{1.1}{3.1} = \frac{4\pi a^5}{15}.
 \end{aligned}$$

T5 Find $\text{grad } f$ at the point P for

(a) $f = x^2 + y^2 - 3yz$, $P(1, 2, 1)$, (b) $f = e^x \log(yz)$, $P(0, 2, 3)$,

(c) $f = \cos(yz) \log(xz)$, $P(1, 0, 3)$.

Solution

(a) $\text{grad } f = (2x, 2y - 3z, -3y) = (2, 1, -6)$ at $P(1, 2, 1)$.

(b) $\text{grad } f = (\log(yz), e^x/y, e^x/z) = (\log 6, \frac{1}{2}, \frac{1}{3})$ at $P(0, 2, 3)$.

(c) $\text{grad } f = (\cos(yz)/x, -z \sin(yz) \log(xz), -y \sin(yz) \log(xz) + \cos(yz)/z) = (1, 0, \frac{1}{3})$ at $P(1, 0, 3)$.

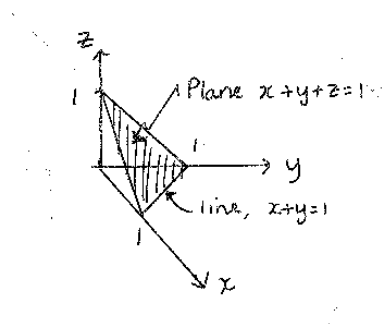
Further Exercises

F1 Evaluate

$$\iiint_T y \, dx dy dz$$

throughout the tetrahedron T given by $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$.

Solution



Hence the integral is

$$\begin{aligned} I &= \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} y dz = \int_0^1 dx \int_0^{1-x} y [z]_0^{1-x-y} dy \\ &= \int_0^1 dx \int_0^{1-x} y((1-x)-y) dy = \int_0^1 \left[\frac{1}{2} y^2 (1-x) - \frac{1}{3} y^3 \right]_0^{1-x} dx \\ &= \int_0^1 \frac{1}{6} (1-x)^3 dx = -\frac{1}{6} \left[\frac{(1-x)^4}{4} \right]_0^1 = \frac{1}{24}. \end{aligned}$$

F2 Use triple integration to express the volume of the solid that is bounded by the given surfaces and evaluate the volume:

- a) $z = x^2 + y^2 - 3$, $z = -x^2 - y^2 + 5$,
 b) $y = x^2$, $z = -y + 4$, $z = 0$.

Solution

(a) The projection of the volume onto the x, y -plane is given by a circle centre o radius 2. This is because the two surfaces meet at the widest point of the volume. They meet when $x^2 + y^2 - 3 = -x^2 - y^2 + 5$, rearranging this gives $x^2 + y^2 = 4$. We will call this region D . As we wish to find a volume the integrand is 1.

$$\text{Volume} = \iint_D dx dy \int_{x^2+y^2-3}^{-x^2-y^2+5} 1 dz = \iint_D -2x^2 - 2y^2 + 8 dx dy$$

Since D is a disc we carryout the remaining double integral using polar coordinates. D can be described by $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$ giving:

$$\text{volume} = \int_0^{2\pi} d\theta \int_0^2 (-2r^2 + 8)r dr = 2\pi \left[-\frac{2r^4}{4} + 4r^2 \right]_0^2 = 16\pi.$$

(b) The projection of the volume onto the x, y -plane is given by $x^2 \leq y \leq 4$ and $-2 \leq x \leq 2$. This is because the widest point where $z = -y + 4$ meets the plane $y = x^2$ is at $z = 0$. The top and bottom of the volume are given by the surfaces $z = -y + 4$ and $z = 0$ respectively. Taking the integrand to be 1 because we wish to find the volume gives:

$$\begin{aligned} \text{Volume} &= \int_{-2}^2 dx \int_{x^2}^4 dy \int_0^{-y+4} 1 dz = \int_{-2}^2 dx \int_{x^2}^4 -y + 4 dy \\ &= \int_{-2}^2 \left[\frac{-y^2}{2} + 4y \right]_{x^2}^4 dx = \int_{-2}^2 8 + \frac{x^4}{2} - 4x^2 dx = \frac{256}{15} \end{aligned}$$

F3 Find the mass of the solid of constant density ρ that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$ and $z = 1$.

Solution

The mass is found by integrating the density over the volume.

$$\begin{aligned}\text{mass} &= \iiint \rho \, dV = \int_{-1}^1 dy \int_{y^2}^1 dx \int_0^x \rho \, dz \\ &= \rho \int_{-1}^1 dy \int_{y^2}^1 x \, dx = \rho \int_{-1}^1 \left[\frac{x^2}{2} \right]_{y^2}^1 dy \\ &= \frac{\rho}{2} \int_{-1}^1 (1 - y^4) dy = \frac{4\rho}{5}.\end{aligned}$$

F4 Evaluate

$$\iiint_R \sqrt{x^2 + y^2 + z^2} e^{-(x^2 + y^2 + z^2)} \, dx dy dz$$

where R is the interior of the sphere $x^2 + y^2 + z^2 = 1$.

Solution

$$\begin{aligned}I &= \int_0^{2\pi} d\theta \int_0^\pi d\phi \int_0^1 \rho^3 e^{-\rho^2} \sin \phi \, d\rho \\ &= 2\pi \int_0^\pi \sin \phi \, d\phi \int_0^1 \rho^3 e^{-\rho^2} \, d\rho \\ &= 4\pi \int_0^1 u e^{-u} \frac{1}{2} du, \quad (\text{using } u = \rho^2 \text{ and } \frac{1}{2} du = \rho \, d\rho) \\ &= 2\pi \left[-u e^{-u} + \int e^{-u} \cdot 1 \, du \right]_0^1, \quad (\text{using integration by parts.}) \\ &= 2\pi \left[-u e^{-u} - e^{-u} \right]_0^1 = 2\pi \left[1 - \frac{2}{e} \right].\end{aligned}$$

F5 Let V be the interior of the sphere $x^2 + y^2 + z^2 = 1$. Without doing any integration, explain why

$$\iiint_V x^2 \, dx dy dz = \iiint_V y^2 \, dx dy dz = \iiint_V z^2 \, dx dy dz,$$

and why

$$\iiint_V z \, dx dy dz = 0 \quad \text{and} \quad \iiint_V z^3 \, dx dy dz = 0.$$

Solution

$\int \int \int z^2 \, dx dy dz$ gives the mass of a sphere $x^2 + y^2 + z^2 \leq 1$, which is symmetrical about $(0,0,0)$ and has density z^2 at (x,y,z) . Turning the x, y, z axes so that x becomes the y , the y the z , and z the x , then in the new coordinates the density will be x^2 at (x,y,z) , but the mass of the sphere will be unchanged, because all we have done is change coordinates. The mass is now expressed as $\int \int \int x^2 \, dx dy dz$ in the new coordinates and so the two integrals are equal. Similarly for y^2 .

For $\int \int \int z \, dx dy dz$ the integrand is as often positive as it is negative, and in a symmetrical way. So

the answer is zero. For the same reason, the same is true when the integrand is any odd power of z .

F6 Evaluate

$$\iiint_R \frac{z}{\sqrt{x^2 + y^2 + z^2}} dx dy dz$$

where R is the interior of the sphere $x^2 + y^2 + z^2 = 2z$.

Solution

Since, $x^2 + y^2 + z^2 = 2z$, in polar coordinates this is $r = 2 \cos \phi$.

$$\begin{aligned} I &= \int_0^{2\pi} d\theta \int_0^{\pi/2} d\phi \int_0^{2\cos\phi} \rho^2 \cos \phi \sin \phi d\rho \\ &= 2\pi \int_0^{\pi/2} \cos \phi \sin \phi d\phi \int_0^{2\cos\phi} \rho^2 d\rho = 2\pi \int_0^{\pi/2} \cos \phi \sin \phi \left[\frac{\rho^3}{3} \right]_0^{2\cos\phi} d\phi \\ &= 2\pi \int_0^{\pi/2} \cos \phi \sin \phi \frac{8\cos^3 \phi}{3} d\phi = \frac{16\pi}{3} \int_0^{\pi/2} \cos^4 \phi \sin \phi d\phi = \frac{16\pi}{3} \frac{3.1}{5.3.1} = \frac{16\pi}{15}. \end{aligned}$$

F7 Evaluate

$$\iiint_R \frac{1}{(x^2 + y^2 + z^2)^2} dx dy dz$$

where R is the region in the first octant *outside* the sphere $x^2 + y^2 + z^2 = 1$.

Solution

$$\begin{aligned} I &= \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \int_1^\infty \frac{1}{\rho^2} \sin \phi d\rho \\ &= \frac{\pi}{2} \int_0^{\pi/2} \sin \phi d\phi \int_1^\infty \frac{1}{\rho^2} d\rho = \frac{\pi}{2} \cdot \frac{1}{1} \left[\frac{-1}{\rho} \right]_1^\infty = \frac{\pi}{2}. \end{aligned}$$

F8 Find $\text{grad } f$ for

(a) $f = x \sin(y)$, (b) $f = x \log(x + 3z)$, (c) $f = \sqrt{zy} \cot(x + y)$.

Solution

- (a) $\text{grad } f = (\sin(y), x \cos(y), 0)$.
 (b) $\text{grad } f = (\log(x + 3z) + \frac{x}{x+3z}, 0, \frac{3x}{x+3z})$.
 (c) $\text{grad } f = (-\sqrt{zy} \sec^2(x + y), -\sqrt{zy} \sec^2(x + y) + \sqrt{z/(4y)} \cot(x + y), \sqrt{y/(4z)} \cot(x + y))$.

F9 Find the volume of the region lying inside the cylinder $x^2 + 4y^2 = 4$ above the xy -plane, and below the plane $2 + x$.

¹ Only attempt these if you have been able to do all the other problems successfully.

Solution

$$\begin{aligned}\text{Volume} &= \int \int \int_V 1 \, dx dy dz = \int_{-2}^2 dx \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} dy \int_0^{2+x} 1 \, dz \\ &= \int_{-2}^2 [2y + xy]_{y=-\sqrt{1-x^2/4}}^{y=\sqrt{1-x^2/4}} dx = 4\pi.\end{aligned}$$

The final integral can be done using the substitution $x = 2 \cos u$.

F10 Find $\iiint_R z \, dV$, over the region R satisfying $x^2 + y^2 \leq z \leq \sqrt{2 - x^2 - y^2}$.

Solution

The domain of integration consists of a paraboloid with a spherical top. So we use spherical polar coordinates to solve the problem. To do the integration we split up the domain into two sections, A and B. Firstly the sphere meets the paraboloid when $x^2 + y^2 = z = \sqrt{2 - x^2 - y^2} = \sqrt{2 - z}$, so $z^2 + z - 2 = 0$, since $z \geq 0$ we have $z = 1$ is the only solution. Considering the $x = 0$ cross section of the paraboloid we determine $y = 1$ when $z = 1$, hence the sphere and paraboloid meet at the angle $\phi = \pi/4$. So region A is given by $0 \leq \rho \leq \sqrt{2}$, $0 \leq \phi \leq \pi/4$ and $0 \leq \theta \leq 2\pi$. For $\phi > \pi/4$ the radius is determined by the paraboloid, $z = x^2 + y^2$, in spherical polar coordinates this gives $\rho = \frac{\cos \phi}{\sin^2 \phi}$. So the second section of the domain, B, is given by $0 \leq \rho \leq \frac{\cos \phi}{\sin^2 \phi}$, $\pi/4 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$.

Hence, $\iiint_V z \, dx dy dz = \iiint_A z \, dx dy dz + \iiint_B z \, dx dy dz$

$$\begin{aligned}\iiint_A z \, dx dy dz &= \int_0^{2\pi} d\theta \int_0^{\pi/4} d\phi \int_0^{\sqrt{2}} \rho^3 \cos \phi \sin \phi \, d\rho \\ &= 2\pi \int_0^{\pi/4} \cos \phi \sin \phi \, d\phi \left[\frac{\rho^4}{4} \right]_0^{\sqrt{2}} = \pi/2.\end{aligned}$$

$$\begin{aligned}\iiint_B z \, dx dy dz &= \int_0^{2\pi} d\theta \int_{\pi/4}^{\pi/2} d\phi \int_0^{\frac{\cos \phi}{\sin^2 \phi}} \rho^3 \cos \phi \sin \phi \, d\rho \\ &= 2\pi \int_{\pi/4}^{\pi/2} \cot^5 \phi \operatorname{cosec}^2 \phi \, d\phi = \pi/(12).\end{aligned}$$

(The last integral is calculated using the substitution $u = \cot \phi$.) Hence the integral is $\pi/2 + \pi/12 = 7\pi/12$.