University of Glasgow

Mathematics 2A—Calculus (2013/14)

Chapter 1

Partial differentiation

(Stewart (Ed. 7): Chapter 14, p901.)

Chapter Summary

Objective	**Tools**
Sketch and identify surfaces in 2-D and 3-D.	**Level curves** are curves $f(x, y) = c$, where c is a constant. They describe where the surface $z = f(x, y)$ intersects the plane $z = c$ and can be used to build up a picture of the surface (think coutours of a moutain on a map). **Cross sections** are a generalisation of this, for example an $x = 0$ cross section is found by setting $x = 0$ in the equation for the surface $z = f(x, y)$.
Find partial derivatives and use these to show simple results, such as $xu_x+yu_y = 0$. Deduce related second order formula such as $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$.	Partial derivatives are given by $\dfrac{\partial f}{\partial x_i}$ which means differentiate f with respect to variable x_i and treat the other variables as constants. To find related second order results differentiate the first order equations again.
Apply the chain rule to functions of several variables	If $F(x, y) = f(u(x, y), v(x, y))$ then the chain rule states that $\dfrac{\partial F}{\partial x} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x}$, similarly for $\dfrac{\partial F}{\partial y}$. Note you need be happy with the idea that the function f might not be given explictly or it might be the product of a known function and an unkown one in which case you will need to use the product rule too.
Solve partial differential equations, eg solve $u_x+u_y = 0$ to find $u(x, y)$.	(i) Integrate with respect to one variable while treating the other variables as constants. Remember instead of a constant of integration we have an arbitary function dependent on the variables that were held constant. (ii) **Change variables by using the chain rule** for functions of several variables to rewrite the PDE in the new variables, then proceed as in (i).

1.1 Functions of one variable

We begin by recalling some basic ideas about real functions of one variable. For example, the volume V of a sphere only depends on its radius r and is given by the formula

$$V = \tfrac{4}{3}\pi r^3.$$

We write $V = f(r)$, where $f(r) = \frac{4}{3}\pi r^3$ to emphasise the fact that volume is a function f of the radius (only). Two related ideas should also be recalled.

Domain In general, the domain D is the set of points at which the formula is to be calculated. In the present example, since the radius should be real and cannot be negative, the domain consists of all non-negative real numbers, $[0, \infty)$ (it is debatable whether or not 0 should be included or excluded but this is not an important issue).

To be precise when we define a real function f, we should specify not only the formula but also its domain D by writing $f\colon D \to \mathbb{R}$. If we do not specify the domain, we assume that the domain is the *maximal domain*, that is the set of *all* points at which the formula makes sense. For the present example, f is defined by

$$f\colon [0, \infty) \to \mathbb{R}, \quad \text{where} \quad f(r) = \tfrac{4}{3}\pi r^3.$$

If we were to say simply that f was defined by

$$f(r) = \tfrac{4}{3}\pi r^3,$$

then it would be assumed that the domain of f is the maximal domain which is $\mathbb{R} = (-\infty, \infty)$ since the formula makes sense for all real numbers r.

Graph In general, this is the set of all ordered pairs $(a, f(a))$ where a is a point in the domain. This is usually shown as a curve in the cartesian plane. In the present example, the graph is the set of points $(r, \frac{4}{3}\pi r^3)$ for all $r \geq 0$ and is illustrated in Figure 1.1.

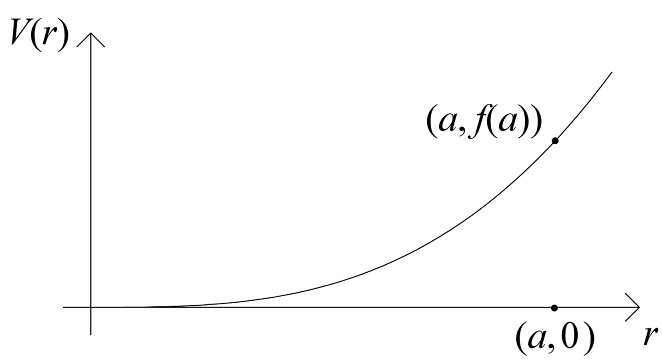

Figure 1.1: Graph of $f\colon D \to \mathbb{R}$

The volume V of a cylinder, on the other hand, depends on two dimensions, the radius r and the height h. In this case we might write $V = g(r, h)$, where $g(r, h) = \pi r^2 h$ defines a *function of two variables*. In the next section we will extend the notions of domain and graph to functions of several variables.

1.2 Functions of several variables

We will only discuss the case of two variables but the main ideas are valid for any number of variables.

Let D be a subset of $\mathbb{R}^2$, that is, a region in a plane. A typical element of D is a point (x, y). A function $f: D \to \mathbb{R}$ is a rule which determines a unique real number $z = f(x, y)$ for each $(x, y) \in D$. The graph of g is the set of points $(a, b, c) \in \mathbb{R}^3$ such that $(a, b) \in D$ and $c = f(a, b)$. This is typically represented as a *surface* in (three dimensional) space. Figure 1.2 illustrates this.

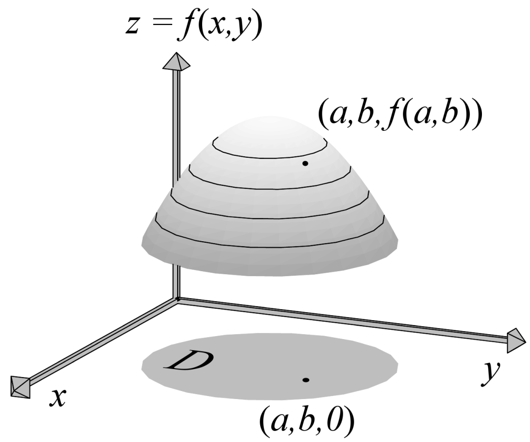

Figure 1.2: Graph of $f: D \to \mathbb{R}$

Similar definitions exist for functions of any number of variables but the graph of a function of more than two variables cannot be simply represented.

Remark As with real functions of one variable, we often don't give the domain of a function f of several variables explicitly; instead we assume that the domain of f is maximal.

Aids to visualisation of surfaces

In several parts of this course it will be important to be able to visualise a surface which is either the graph of a function of two variables $z = f(x, y)$ or, more generally, is a relation $F(x, y, z) = 0$. We will here give several examples illustrating some useful techniques.

Example surfaces: Spheres

A sphere of radius r, centre (a, b, c) consists of those points (x, y, z) which are a distance r from (a, b, c). Thus, by Pythagoras's theorem, this sphere is defined by

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

Furthermore, if we solve for z we get

$$z = c \pm \sqrt{r^2 - (x - a)^2 - (y - b)^2}.$$

Because of this, for any given a, b, c, the graph of a function $f(x, y) = c + \sqrt{r^2 - (x - a)^2 - (y - b)^2}$ is the "northern" hemisphere and $f(x, y) = c - \sqrt{r^2 - (x - a)^2 - (y - b)^2}$ the corresponding "southern" hemisphere.

Given an equation

$$x^2 + y^2 + z^2 + \alpha x + \beta y + \gamma z + \delta = 0,$$

one may always complete the square to write this in the form

$$(x + \tfrac{1}{2}\alpha)^2 + (y + \tfrac{1}{2}\beta)^2 + (z + \tfrac{1}{2}\gamma)^2 = \frac{1}{4}(\alpha^2 + \beta^2 + \gamma^2) - \delta$$

which defines a sphere if and only if $\frac{1}{4}(\alpha^2 + \beta^2 + \gamma^2) - \delta > 0$.

Example 1.1 Sketch the graph of $f(x, y) = -\sqrt{1 - 2x - x^2 - y^2}$.

Solution :

Let $z = f(x, y)$. Completing the square, we have

$$z^2 = 1 - 2x - x^2 - y^2 = 2 - (x + 1)^2 - y^2,$$

i.e. $(x + 1)^2 + y^2 + z^2 = 2$. This is the sphere with centre $(-1, 0, 0)$ and radius $\sqrt{2}$. The part given by $z = -\sqrt{1 - 2x - x^2 - y^2}$ (≤ 0) is the hemisphere *below* the x, y-plane. See Figure 1.3.

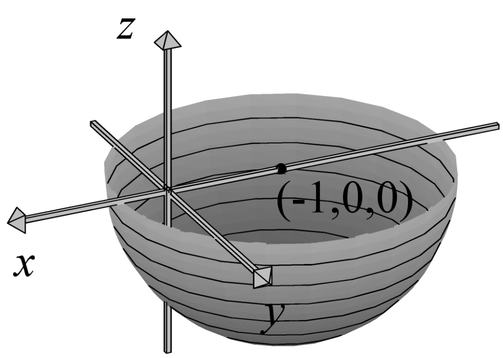

Figure 1.3: Graph of $f(x, y) = -\sqrt{1 - 2x - x^2 - y^2}$

□

Cross-sections and level curves

The plane $x =$ constant is parallel to the yz-plane and may, or may not, have a non-empty intersection with the surface $F(x, y, z) = 0$. This intersection is called a *cross-section* of the surface (or Stewart uses the term *trace of the surface*). Typically, this cross-section will be a curve on the plane can give useful clues to the overall nature of the surface. Similarly, we may take cross-section with the planes $y =$ constant and $z =$ constant.

In particular, for a surface $z = f(x, y)$, the cross-section with the plane $z = c$, where c is a constant, is the curve $f(x, y) = c$ and is called a *level curve* or *contour*. The second name is used because of the close connection with contour lines on a map (lines linking points with the same height above sea-level). In this analogy, $z = f(x, y)$ represents part of the surface of the earth and each level curve represents a particular contour line on a map.

For each choice of c the level curve is denoted L_c and is the set of points (x, y) in D for which $f(x, y)$ has the value c. For different choices of c, L_c may be a curve, a point or points, or the empty set. Note that each point in the domain of f lies on a particular level curve.

Example 1.2 By considering the level curves and the cross-sections $x = 0$ and $y = 0$, obtain a sketch of $z = \sqrt{x^2 + y^2}$.

Solution : The level curves are defined by

$$L_c = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = c\}.$$

For $c < 0$, $L_c = \emptyset$ (since $\sqrt{\cdots} \geq 0$), $L_0 = \{(0, 0)\}$ (since $x^2 + y^2 = 0 \implies x = y = 0$) and for $c > 0$, $L_c = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = c^2\}$, the circle of radius c, centre $(0, 0)$.

Fixing $x = 0$ we get $z = \sqrt{y^2} = |y|$ and fixing $y = 0$ we get $z = |x|$. These cross-sections are illustrated in Figure 1.4.

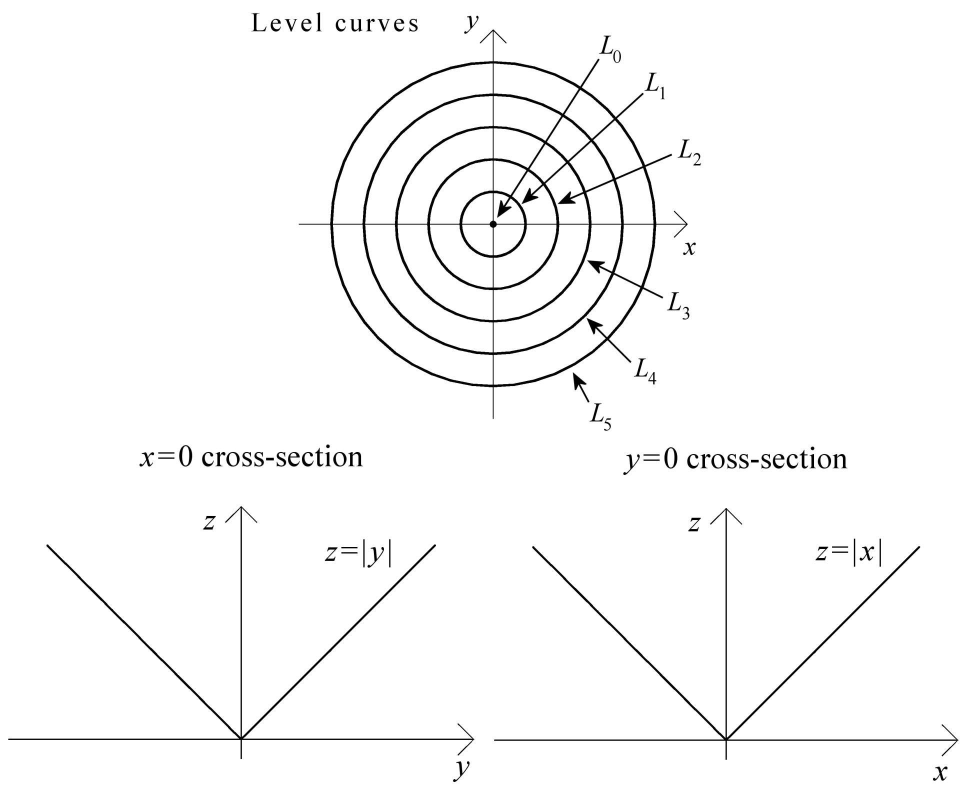

Figure 1.4: Cross sections

Putting this information together, we see that the surface defined by $z = \sqrt{x^2 + y^2}$ is a (circular) *cone* with vertex at $(0, 0)$ (Figure 1.5).

□

Example surfaces: Ellipsoid

An ellipsoid of radius r_1 in the x-direction, r_2 in the y-direction and r_3 in the z-direction, with centre (a, b, c) is defined by

$$\frac{(x-a)^2}{(r_1)^2} + \frac{(y-b)^2}{(r_2)^2} + \frac{(z-c)^2}{(r_3)^2} = 1.$$

It is called an ellipsoid because all the cross sections are ellipses. In the special case where $r_1 = r_2 = r_3$ we recover the equation for the sphere.

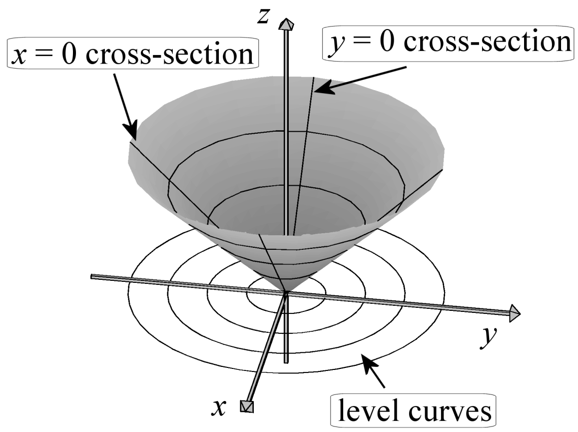

Figure 1.5: The cone $z = \sqrt{x^2 + y^2}$

Example surfaces: Planes

Recall that a plane with normal vector $\mathbf{n} = (\alpha, \beta, \gamma)$ has equation $\alpha x + \beta y + \gamma z = \delta$. In particular, the graph of $f(x, y) = ax + by + c$ is the plane $z = ax + by + c$ with normal $(a, b, -1)$ passing through the point $(0, 0, c)$. Observe that the cross-sections of a plane are either straight lines (or $\emptyset$.)

Example 1.3 Sketch the part of the surface

$$2x + y + 4z = 1,$$

where $x, y, z \geq 0$.

Solution : We consider the cross-section with the coordinate planes $x = 0$ (y, z-plane), $y = 0$ (x, z-plane) and $z = 0$ (x, y-plane).

The cross-section of $2x + y + 4z = 1$ with $x = 0$ is the line $y + 4z = 1$ (lying in the y, z-plane). This passed through the points $(0, 0, \frac{1}{4})$ and $(0, 1, 0)$. In a similar way we obtain the cross-section with the other coordinate planes; $2x + 4z = 1$ in the x, z-plane, passing through $(0, 0, \frac{1}{4})$ and $(\frac{1}{2}, 0, 0)$ and $2x + y = 1$ in the x, y-plane, passing through $(0, 1, 0)$ and $(\frac{1}{2}, 0, 0)$.

Answer: A sketch of the plane is shown in Figure 1.6. $\square$

Example surfaces: Circular Cylinder

A circular cylinder in $\mathbb{R}^3$ of radius r centred at the origin lying parallel to z-axis, is defined by

$$x^2 + y^2 = r^2.$$

Note this might look at first glance that this is the equation for a circle, it is not because the surface lies in $\mathbb{R}^3$, so although z does not appear in the equation it can take any value, so the surface looks like a circle for each height z, hence we obtain a cylinder. This is easily generalisable to cylinders centred at (a, b, c) and to cylinders lying parallel to the x or y axes. Other standard surfaces are shown in Table 1 of (Stewart (Ed. 7): Chapter 12, p854.)

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Example 1.4 Sketch the region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the plane $z = 2$.

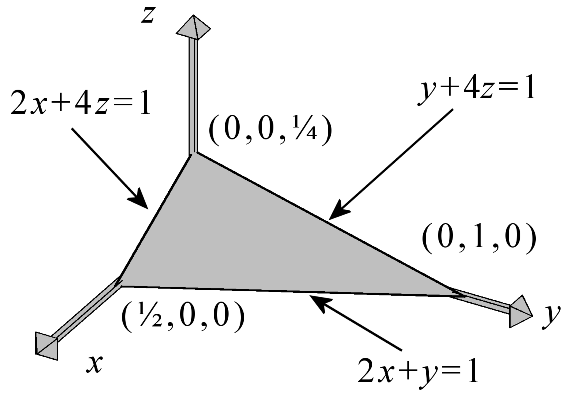

Figure 1.6: The plane $2x + y + 4z = 1$

Solution : The level curves of the paraboloid for $c > 4$ are $L_c = \emptyset$ (since $4 - x^2 - 2y^2 \leq 4$). For $c \leq 4$ are defined by the ellispse

$$L_c = \{(x, y) \in \mathbb{R}^2 : 4 - x^2 - 2y^2 = c\}.$$

In particular, the level curve where the plane insects the paraboloid is given by $L_2 = \{(x, y) \in \mathbb{R}^2 : 2 = x^2 + 2y^2\}$.

The cross section of the paraboloid when fixing $x = 0$ is the curve $z = 4 - 2y^2$ and fixing $y = 0$ gives the cross section $z = 4 - x^2$. These cross-sections are both parabolas and are illustrated in Figure 1.7.

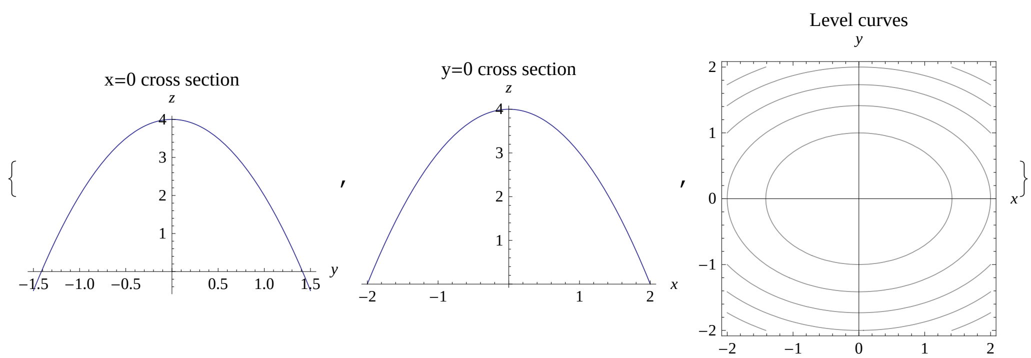

Figure 1.7: Cross sections and level curves of the paraboloid $z = 4 - x^2 - 2y^2$

Putting this information together, the region bounded by the parabolid and the plane is illustrated in Figure 1.8.

$\square$

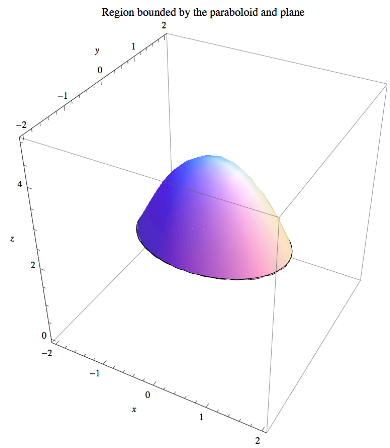

Figure 1.8: The region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the plane $z = 2$.

1.3 Partial derivatives

(Stewart (Ed. 7): Section 14.3, p924.)

In this section we want to generalise, to functions of several variables, the notion of *gradient* as it is understood for functions of one variable. Recall that if the limit

$$\lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

exists then this limit is called *derivative* of g at a. This is written as

$$\frac{dg}{dx}(a) \quad \text{or} \quad g'(a),$$

and is the gradient of the tangent to the graph of g at a point $(a, g(a))$.

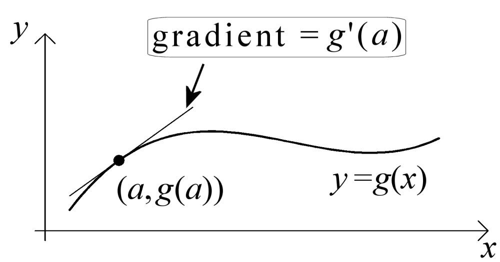

Figure 1.9: Derivative of a function of one variable

Now consider f, a function of two variables. On the surface $z = f(x, y)$, there is no single meaning of gradient. Imagine this surface to be a mountainside. When walking or skiing straight down the mountain the gradient may be very large but traversing the mountain the gradient is much less. Indeed by choosing a direction one may make the gradient have any value in between. For this reason it is necessary to define *two* gradients in terms of vertical cross-section of the surface in the x and y directions.

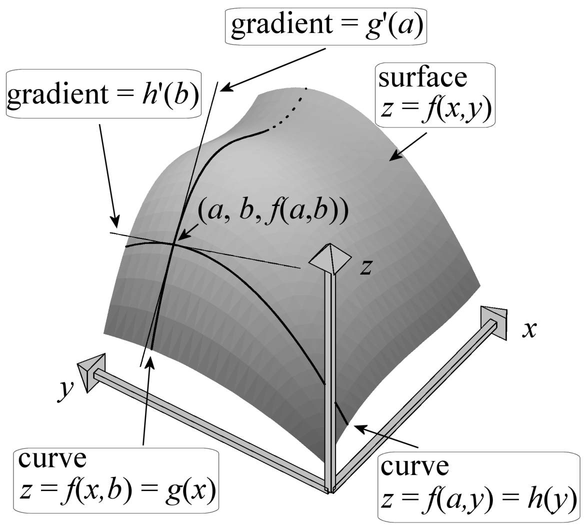

Figure 1.10: Gradients on cross-sections

As in Figure 1.10, consider a point $(a, b, f(a, b))$ on the surface. Taking the cross-sections $x = a$ and $y = b$ through this point we obtain the graphs of two functions of *one* variable; $z = f(x, b) = g(x)$ (say) and $z = f(a, y) = h(y)$ (say). For each of these functions we can (provided the derivatives exist) determine gradients called the *partial x and y derivatives of f at* (a, b) and written as

$$\frac{\partial f}{\partial x}(a, b) = \text{derivative of } f(x, y) \text{ w.r.t. } x \text{ with } y \text{ held constant, evaluated at } (x, y) = (a, b). \text{ This equals } g'(a).$$

and

$$\frac{\partial f}{\partial y}(a, b) = \text{derivative of } f(x, y) \text{ w.r.t. } y \text{ with } x \text{ held constant, evaluated at } (x, y) = (a, b). \text{ This equals } h'(b).$$

For a function f of n variables $x_1, x_2, \ldots, x_n$ we define n *partial derivatives*

$$\frac{\partial f}{\partial x_i} = \text{derivative of } f(x_1, \ldots, x_n) \text{ w.r.t. } x_i \text{ with all other variables held constant.}$$

Remarks

1. It is important to distinguish the notation used for partial derivatives $\dfrac{\partial f}{\partial x}$ from ordinary derivatives $\dfrac{df}{dx}$.

2. We also use subscript notation for partial derivatives. If $f = f(x, y)$ then we may write

$$\frac{\partial f}{\partial x} \equiv f_x \equiv f_1, \text{ and } \frac{\partial f}{\partial y} \equiv f_y \equiv f_2.$$

   In general, the notation f_n, where n is a positive integer, means the derivative of f with respect to its n-th argument, (with all other variables held constant). This notation is the direct analogue of the $'$ notation for ordinary derivatives. Recall we can use the chain rule to calculate

$$\frac{d}{dx} f(x^2) = f'(x^2) \frac{d}{dx}(x^2) = 2x f'(x^2).$$

   Below we carry out similar calculations involving partial derivatives.

3. Like ordinary derivatives, partial derivatives do not always exist at every point. In this module we will always assume that derivatives exist unless it is otherwise stated.
4. If $z = f(x, y)$ then the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ can be interpreted as the gradients of the tangent lines to the surface $z = f(x, y)$ in the directions parallel to the $x-$ and $y-$axes, respectively.

Formal definition of Partial Derivative

Suppose f is a suitably well behaved function of three variables x, y, z. Then at (a, b, c),

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(a+h, b, c) - f(a, b, c)}{h}.$$

This is by analogy with the definition of ordinary derivatives. Note how the y and z coordinates are unaffected.

Example 1.5 Find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial z}{\partial x}$ where

$$\text{(a) } f(x, y) = x^3y^2 + x, \quad \text{(b) } z(x, y) = \sin^{-1}\left(\frac{x}{x+y}\right) \text{ and } x, y > 0.$$

[Note that $\sin^{-1} u$ is the inverse sine function (sometimes written as $\arcsin u$), and *not* the reciprocal $1/\sin u$. The domain of $\sin^{-1}$ is $[-1, 1]$ and, since $x, y > 0$, $x/(x + y)$ lies in this domain.]

Solution : (a) To calculate the partial x derivative, we think of y as a constant and differentiate in the usual way with respect to x. Hence, we have

$$\frac{\partial f}{\partial x} = y^2 \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(x) = 3x^2y^2 + 1.$$

For the y derivative, we think of x as a constant and differentiate with respect to y;

$$\frac{\partial f}{\partial y} = x^3 \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(x) = 2x^3y.$$

Answer: $f_x = 3x^2y^2 + 1$ and $f_y = 2x^3y$.

(b) Let $u = x/(x + y)$. So, by the chain rule

$$\frac{\partial z}{\partial x} = \frac{d}{du}(\sin^{-1} u)\frac{\partial u}{\partial x}.$$

We have

$$\frac{d}{du}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} = \frac{1}{\sqrt{1 - \left(\dfrac{x}{x+y}\right)^2}} = \frac{|x+y|}{\sqrt{(x+y)^2 - x^2}} = \frac{x+y}{\sqrt{2xy + y^2}},$$

since $x, y > 0$. Also, by the quotient rule,

$$\frac{\partial u}{\partial x} = \frac{\dfrac{\partial}{\partial x}(x).(x+y) - x.\dfrac{\partial}{\partial x}(x+y)}{(x+y)^2} = \frac{y}{(x+y)^2}.$$

Hence

$$\frac{\partial z}{\partial x} = \frac{y}{x+y}\frac{1}{\sqrt{2xy + y^2}}.$$

Answer: $z_x = \frac{y}{x+y}\frac{1}{\sqrt{2xy+y^2}}$. □

Example 1.6 Find $\dfrac{\partial z}{\partial x}$ where z is defined implicitly as a function of x and y by the equation

$$x^4 + 2y^2 + z^3 - 2x^2yz = 1.$$

Solution : Differentiating implicitly with respect to x gives

$$4x^3 + 0 + 3z^2\frac{\partial z}{\partial x} - 4xyz - 2x^2y\frac{\partial z}{\partial x} = 0.$$

Rearranging to solve for $\frac{\partial z}{\partial x}$ yields

$$\frac{\partial z}{\partial x} = \frac{4x^3 - 4xyz}{2x^2y - 3z^2}.$$

□

Example 1.7 For $r \in \mathbb{R}^+$, let $u = f(r)$ where $r^2 = x^2 + y^2 + z^2$. Show that

$$xu_x + yu_y + zu_z = rf'(r).$$

Solution :

By the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} = f'(r)\frac{\partial r}{\partial x},$$

and similarly,

$$u_y = f'(r)\frac{\partial r}{\partial y}, \quad u_z = f'(r)\frac{\partial r}{\partial z}.$$

Now

$$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2 + y^2 + z^2) \qquad \text{i.e., } 2r\frac{\partial r}{\partial x} = 2x.$$

Therefore

$$\frac{\partial r}{\partial x} = \frac{x}{r},$$

and similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

Thus we have

$$xu_x + yu_y + zu_z = \frac{x^2}{r}f'(r) + \frac{y^2}{r}f'(r) + \frac{z^2}{r}f'(r) = \frac{(x^2 + y^2 + z^2)}{r}f'(r) = rf'(r),$$

as required. □

1.4 Higher order derivatives

(Stewart (Ed. 7): Section 14.3, p930.)

Let u be a function of several variables $x, y, \ldots$. Then u_x (if it exists) is also a function of the same variables and so may also have partial derivatives. We define

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_x) = u_{xx} = u_{11}, \qquad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}(u_x) = u_{xy} = u_{12},$$
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(u_y) = u_{yx} = u_{21}, \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(u_y) = u_{yy} = u_{22}, \quad \text{etc.}$$

In general, $u_{xyz\ldots}$ denotes the result of taking the x-derivative, then the y-derivative, then the z-derivative, $\ldots$ of u. The total number of partial derivatives taken is called the *order* of the derivative. For example, $u_{xxy} = u_{112}$ is a third order derivative.

There is no automatic guarantee that, for example, $u_{xy} = u_{yx}$ but the following theorem (the proof of which is omitted) states the conditions under which the order in which the derivatives are taken is unimportant.

Theorem (Clairaut's Theroem) *Let u be a function of x, y such u_{xy} and u_{yx} exist and are continuous at a point (a, b). [Roughly speaking, this means that there are no holes or jumps in the graphs of u_{xy} and u_{yx} at (a, b).] Then,*

$$u_{xy}(a, b) = u_{yx}(a, b).$$

Remarks

1. This result extends to functions of any number of variables and to third and higher order derivatives. For example, let u depend on three variables then, provided these derivatives exist and are continuous,

$$u_{1213} = u_{3211} = u_{2113} = \cdots = u_{1123}.$$

2. Unless otherwise stated, functions considered in this module will be assumed to have continuous partial derivatives of all orders. Hence the order in which we take partial derivatives will be unimportant.

Example 1.8 Determine all second order derivatives of $u = \sin xy$ and verify that $u_{xy} = u_{yx}$.

Solution : We have first derivatives

$$u_x = y \cos xy, \quad u_y = x \cos xy.$$

Hence, the second derivatives are

$$\begin{aligned}
u_{xx} &= \frac{\partial}{\partial x}(u_x) = y \frac{\partial}{\partial x}\big(\cos xy\big) = -y^2 \sin xy, \\
u_{xy} &= \frac{\partial}{\partial y}(u_x) = \frac{\partial}{\partial y}(y). \cos xy + y \frac{\partial}{\partial y}\big(\cos xy\big) = \cos xy - yx \sin xy, \\
u_{yx} &= \frac{\partial}{\partial x}(u_y) = \frac{\partial}{\partial x}(x). \cos xy + x \frac{\partial}{\partial x}\big(\cos xy\big) = \cos xy - xy \sin xy, \\
u_{yy} &= \frac{\partial}{\partial y}(u_y) = x \frac{\partial}{\partial y}\big(\cos xy\big) = -x^2 \sin xy.
\end{aligned}$$

Hence $u_{xy} = u_{yx} = \cos xy - xy \sin xy$ as required.

Answer: The second derivatives are $u_{xx} = -y^2 \sin xy$, $\quad u_{yy} = -x^2 \sin xy$, and $u_{xy} = u_{yx} = \cos xy - xy \sin xy$. □

Example 1.9 Let $u = f(x/y)$, where f is an arbitrary (twice differentiable, with continuous second derivative) function of one variable. Show that

$$xu_x + yu_y = 0,$$

and *deduce* that

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0.$$

Solution :

Using the chain rule, we have,

$$u_x = f'\left(\frac{x}{y}\right)\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}f'\left(\frac{x}{y}\right),$$
$$u_y = f'\left(\frac{x}{y}\right)\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = -\frac{x}{y^2}f'\left(\frac{x}{y}\right).$$

So,

$$xu_x + yu_y = x\frac{1}{y}f'\left(\frac{x}{y}\right) - y\frac{x}{y^2}f'\left(\frac{x}{y}\right) = 0.$$

[Although we *could* proceed by calculating u_{xx}, u_{xy} and u_{yy} and taking the appropriate combination, it is much less work to deduce the final part as indicated below.]

Since $xu_x + yu_y = 0$, its x- and y-derivatives must also equal 0. Hence

$$xu_{xx} + u_x + yu_{yx} = 0, \tag{1}$$

and

$$xu_{xy} + yu_{yy} + u_y = 0. \tag{2}$$

Taking $x \times (1) + y \times (2)$ [the need to have the correct coefficient for u_{xx} and u_{yy} dictates the choice of this combination of (1) and (2)] we get

$$x^2 u_{xx} + xu_x + xyu_{yx} + yxu_{xy} + y^2 u_{yy} + yu_y = 0.$$

Since $u_{xy} = u_{yx}$ and $xu_x + yu_y = 0$, we get

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0.$$

as required.

□

1.5 The chain rule for functions of several variables

(Stewart (Ed. 7): Section 14.5, p948.)

We have already made extensive use of the chain rule for functions of one variable. This is used to find the derivative of a *composition* of functions; if $F(x) = f(u(x))$ then

$$\frac{dF}{dx} = \frac{du}{dx}\frac{df}{du} = u'(x)f'(u(x)).$$

We now want to extend this technique to functions of several variables.

Theorem *Let $F(x,y) = f\big(u(x,y), v(x,y)\big)$. Then*

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial f}{\partial v} \qquad \textit{and} \qquad \frac{\partial F}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial f}{\partial v}.$$

This is called the chain rule for functions of two variables.

Remarks

1. Observe the pattern
$$\frac{\partial F}{\partial x} = \frac{\boxed{\partial u}}{\partial x}\frac{\partial f}{\boxed{\partial u}} + \frac{\boxed{\partial v}}{\partial x}\frac{\partial f}{\boxed{\partial v}},$$
[all terms on the right have ∂f on top and ∂x on bottom and ∂u or ∂v which "cancels".]

2. The chain rule is extended in an obvious way to functions of any number of variables. For example, if $F(x, y, z) = f\big(u(x, y, z), v(x, y, z), w(x, y, z)\big)$ then
$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial f}{\partial v} + \frac{\partial w}{\partial x}\frac{\partial f}{\partial w}.$$

3. There are two special cases of this formula. First, the one variable chain rule that we used above; if $F(x, y) = f(u(x, y))$ then
$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x}\frac{df}{du}.$$
Second, if $F(x) = f(u(x), v(x))$ then
$$\frac{dF}{dx} = \frac{du}{dx}\frac{\partial f}{\partial u} + \frac{dv}{dx}\frac{\partial f}{\partial v}.$$
Notice that the partial derivatives in the formula become ordinary derivatives wherever the function being differentiated is a function of only one variable.

Example 1.10 Let $w = u^2 + v^2$ where $u = \sin\theta$ and $v = \cos\phi$. Use the chain rule to calculate w_θ and w_ϕ in terms of θ and ϕ.

Solution :

Using the chain rule, we have
$$w_\theta = \frac{\partial u}{\partial \theta}\frac{\partial w}{\partial u} + \frac{\partial v}{\partial \theta}\frac{\partial w}{\partial v} = \cos\theta.2u + 0.2v = 2\cos\theta\sin\theta = \sin 2\theta,$$
and
$$w_\phi = \frac{\partial u}{\partial \phi}\frac{\partial w}{\partial u} + \frac{\partial v}{\partial \phi}\frac{\partial w}{\partial v} = 0.2u + (-\sin\phi).2v = -2\sin\phi\cos\phi = -2\sin 2\phi.$$
Answer: $w_\theta = \sin 2\theta$ and $w_\phi = -\sin 2\phi$. □

1.6 Partial differential equations

A differential equation is a relation between an unknown function and its derivatives. Such equations are extremely important in all branches of science; mathematics, physics, chemistry, biochemistry, economics,...

Typical examples are

- Newton's law of cooling which states that

  the rate of change of temperature of an object is proportional to the temperature difference between it and that of its surroundings.

This is formulated in mathematical terms as the differential equation

$$\frac{dT}{dt} = k(T - T_0),$$

where $T(t)$ is the temperature of the body at time t, T_0 the temperature of the surroundings (a constant) and k a constant of proportionality,

- the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the displacement (from a rest position) of the point x at time t and c is the wave speed.

The first example has unknown function T depending on one variable t and the relation involves the first order (ordinary) derivative $\frac{dT}{dt}$. This is a *ordinary differential equation*, abbreviated to ODE.

The second example has unknown function u depending on two variables x and t and the relation involves the second order partial derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial t^2}$. This is a *partial differential equation*, abbreviated to PDE.

The *order* of a differential equation is the order of the highest derivative that appears in the relation.

The unknown function is called the *dependent variable* and the variable or variables on which it depends are the *independent variables*.

A solution of a differential equation is an expression for the dependent variable in terms of the independent one(s) which satisfies the relation. The *general solution* includes all possible solutions and typically includes arbitrary functions (in the case of a PDE.) A solution without arbitrary functions is called a *particular solution*. Often we find a particular solution to a differential equation by giving extra conditions in the form of initial or boundary conditions.

Example 1.11 Find the general solution of the PDE,

$$\frac{\partial f}{\partial x} = x^2 + y + 9,$$

where f is a function of two independent variables x and y.

Solution : Integrating with respect to x and treating y as fixed gives

$$f = \int_{y \text{ fixed}} x^2 + y + 9 dx = \frac{x^3}{3} + xy + 9x + A(y)\,,$$

where A is an arbitrary function depending on the fixed variable y. □

Example 1.12 Find the general solution of the PDE,

$$\frac{\partial^2 f}{\partial x \partial y} = 2x,$$

where f is a function of two independent variables x and y.

Solution : The PDE can be expressed as

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2x,$$

Integrating with respect to x and treating y as fixed gives

$$\frac{\partial f}{\partial y} = \int_{y \text{ fixed}} 2x dx = x^2 + A(y)$$

where A is an arbitrary function depending on the fixed variable y. Integrating with respect to y, holding x fixed then gives

$$f = \int_{x \text{ fixed}} x^2 + A(y)dy = x^2 y + \int_{x \text{ fixed}} A(y)dy + B(x) = x^2 y + C(y) + B(x).$$

B is an arbitrary function of the fixed variable x. Since A was an arbitrary function of y its intergral is also and arbitrary function of y so let's call this function $C(y) = \int_{x \text{ fixed}} A(y)dy$. □

Solutions to PDEs by change of variable

In this section certain first order PDEs will be solved by means of a change of variables. Although there is a theory which may be used to *determine* the appropriate change of variable (see Mathematics 3H Mathematical Methods), in this module the change of variable will always be given.

We will make a change of independent variables from x, y to u, v (say). If $z = f(x, y)$ and we introduce new variables $u = u(x, y)$, $v = v(x, y)$ then the chain rule gives

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}.$$

More generally, this shows that for *any* expression $*$ that is to be thought of as a function of x, y or of u, v,

$$\frac{\partial}{\partial x}(*) = \frac{\partial u}{\partial x}\frac{\partial}{\partial u}(*) + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}(*). \tag{1}$$

This general form of the chain rule is useful when calculating second order derivatives.

Example 1.13 By changing variables from (x, y) to (u, v), where $u = xy$, $v = x/y$, solve the PDE

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2x^2 \sin(xy).$$

Solution By the chain rule,

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v} = y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v},$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}.$$

Therefore,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right) + y\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right) = 2xy\frac{\partial z}{\partial u}.$$

Inverting the change of variables we have

$$x = \sqrt{uv}, \quad y = \sqrt{\frac{u}{v}},$$

and so, after the change of variable the PDE becomes,

$$2u\frac{\partial z}{\partial u} = 2uv\sin u,$$

i.e.,

$$\frac{\partial z}{\partial u} = v\sin u.$$

Then

$$z = \int_{v \text{ fixed}} v\sin u\, du = -v\cos u + A(v),$$

and in terms of x and y this is Answer: $z = -\frac{x}{y}\cos(xy) + A\left(\frac{x}{y}\right)$, where A is an arbitrary function. □

Example 1.14 By changing variables from (x, y) to (u, v), where $u = x^3/y$, $v = x$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of parital derivatives with respect to u and v. Hence, solve the PDE

$$x\frac{\partial f}{\partial x} + 3y\frac{\partial f}{\partial y} = \frac{6x^5}{y}.$$

Solution By the chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial f}{\partial v} = \frac{3x^2}{y}\frac{\partial f}{\partial u} + 1\frac{\partial f}{\partial v},$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial f}{\partial v} = \frac{-x^3}{y^2}\frac{\partial f}{\partial u} - 0\frac{\partial f}{\partial v}.$$

Therefore,

$$x\frac{\partial f}{\partial x} + 3y\frac{\partial f}{\partial y} = x\left(\frac{3x^2}{y}\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) + y\left(\frac{-x^3}{y}\frac{\partial f}{\partial u}\right) = x\frac{\partial f}{\partial v}.$$

Inverting the change of variables we have

$$x = v, \quad y = \frac{v^3}{u},$$

and so, after the change of variable the PDE becomes,

$$v\frac{\partial f}{\partial v} = 6uv^2,$$

i.e.,

$$\frac{\partial f}{\partial u} = 6uv.$$

Then

$$f = \int_{u \text{ fixed}} 6uv\, dv = 3v^2u + A(u),$$

and in terms of x and y this is $f = 3\frac{x^5}{y} + A\left(\frac{x^3}{y}\right)$, where A is an arbitrary function.

Answer: $f = \frac{3x^5}{y} + A\left(\frac{x^3}{y}\right)$, where A is an arbitrary function. □