Mathematics 2A—Multivariate Calculus (2013/14)

C. A Cobbold

October 24, 2013

Teaching arrangements

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  - no tutorial this week

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- **This week**: Lectures on Tuesday and Thursday
  - no tutorial this week
- **Other weeks**: Lectures on Tuesday and Thursday and a tutorial on Monday
  - students come to tutorials *every other week*. Go to MyCampus for information on which tutorial group you are in and which weeks you have a tutorial.

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- Bring lecture notes to tutorials!
- Tutorials are an important resource and opportunity for getting feedback. Be proactive and ask tutors to look at your work and ask them questions.

Other arrangements

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- **Recommended course book:** James Stewart, Multivariable Calculus Internation Edition, (Seventh Edition),Brooks Cole /Cengage .

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- Only the Chapter 1 lecture notes will be given out in class. You need to download the notes for Chapters 2,3 and 4 from Moodle yourself in advance.

Syllabus of *2A - Multivariate Calculus*

General theme: differentiation and integration of functions of several variables and its applications.

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- extend to functions of two variables,
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- chain rule for partial derivatives.

Functions of one variable

For example, volume V of a sphere is a function of one variable, its radius r,

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We write $V = f(r)$, where the *rule* is $f(r) = \frac{4}{3}\pi r^3$.

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- the maximal domain of f is $\mathbb{R}$.

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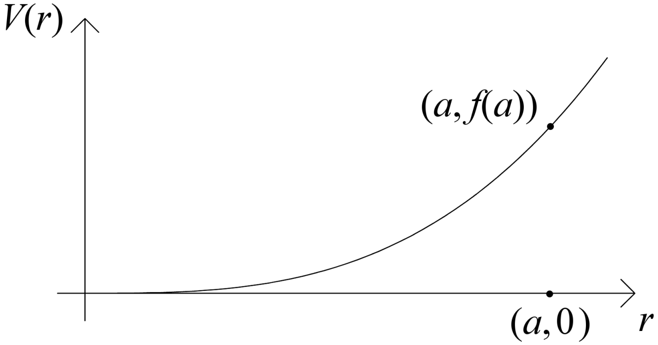

Functions of two variables

Example

Volume V of a cylinder depends on *two* dimensions, the radius r and the height h - $V = f(r, h)$, where $f(r, h) = \pi r^2 h$ defines a *function of two variables*.

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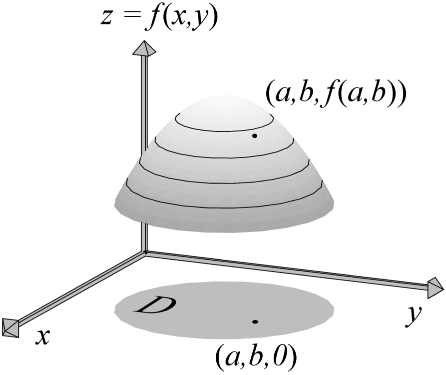
Domain

Subset D of of $\mathbb{R}^2$, i.e., a region in a plane.
If not specified, the maximal domain is assumed.

Functions of two variables

Graph

The set of points $(a, b, c) \in \mathbb{R}^3$ where $(a, b) \in D$ and $c = f(a, b)$ - a *surface*.



Visualisation of surfaces - Spheres

- Radius r, centre (a, b, c) - points (x, y, z) a distance r from (a, b, c). Pythagoras's theorem $\implies$

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2,$$

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- $+$ means "northern" hemisphere
  $-$ means "southern" hemisphere.

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$$(x + \tfrac{1}{2}\alpha)^2 + (y + \tfrac{1}{2}\beta)^2 + (z + \tfrac{1}{2}\gamma)^2 = \frac{1}{4}(\alpha^2 + \beta^2 + \gamma^2) - \delta,$$

- sphere if and only if $\frac{1}{4}(\alpha^2 + \beta^2 + \gamma^2) - \delta > 0$.

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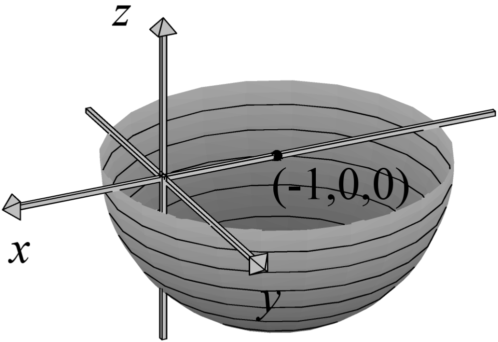
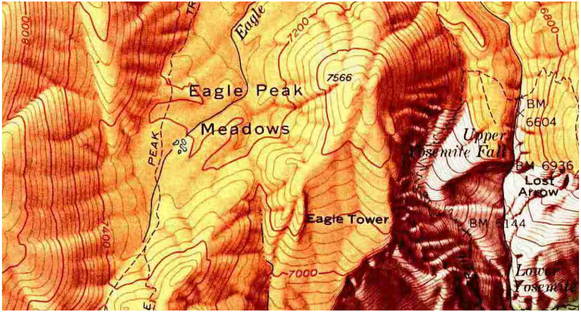


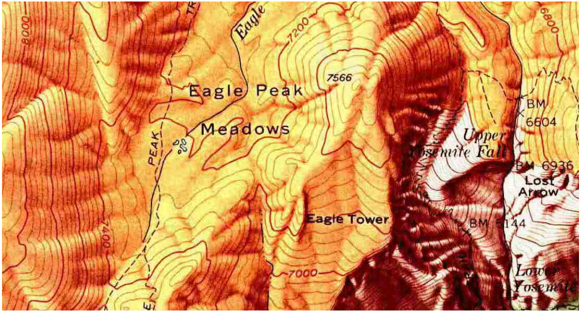
Visualisation of surfaces - Cross-sections

- For a surface $z = f(x, y)$ the set of points satisfying, $f(x, y) = c$, is a *level curve* or *contour*,



Visualisation of surfaces - Cross-sections

- For a surface $z = f(x, y)$ the set of points satisfying, $f(x, y) = c$, is a *level curve* or *contour*,
- think of $z = f(x, y)$ as part of the surface of the earth - each level curve represents a particular contour line on its map.



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- each point in D lies on one level curve.

Visualisation of surfaces - Cross-sections

Example 2

By considering the level curves and the cross-sections $x = 0$ and $y = 0$, obtain a sketch of $z = \sqrt{x^2 + y^2}$.

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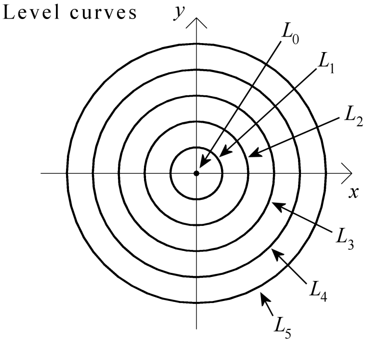

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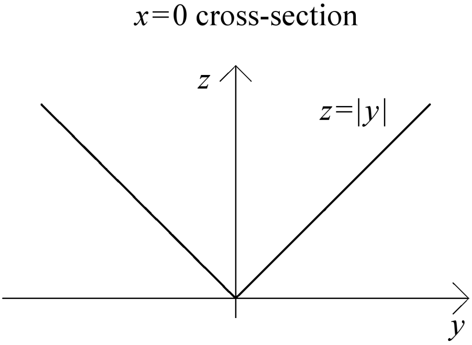

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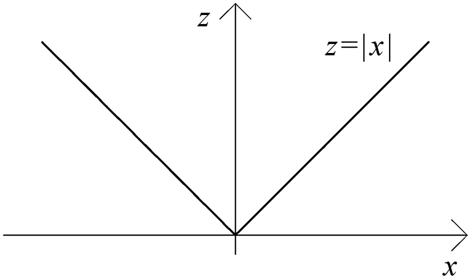

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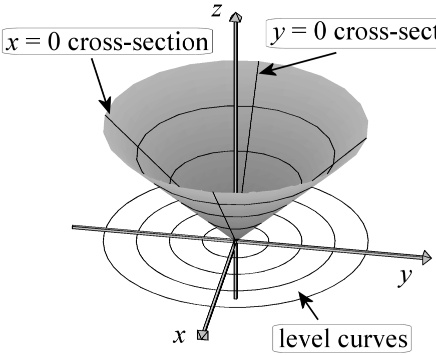

Visulatisation of surfaces - Ellipsoid

- An **ellipsoid** of radius r_1 in the x-direction, r_2 in the y-direction and r_3 in the z-direction, with centre (a, b, c) is defined by

$$\frac{(x-a)^2}{(r_1)^2} + \frac{(y-b)^2}{(r_2)^2} + \frac{(z-c)^2}{(r_3)^2} = 1.$$

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- When $r_1 = r_2 = r_3$ we recover the equation for the sphere.

Visualisation of surfaces - Planes

- Recall: a plane with normal vector $\mathbf{n} = (\alpha, \beta, \gamma)$ has equation $\alpha x + \beta y + \gamma z = \delta$,

Visualisation of surfaces - Planes

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- the graph of $f(x, y) = ax + by + c$ is the plane $z = ax + by + c$ with normal $(a, b, -1)$ passing through the point $(0, 0, c)$.

Visualisation of surfaces - Planes

Example 3

Sketch the part of the surface $2x + y + 4z = 1$ where $x, y, z \geq 0$.

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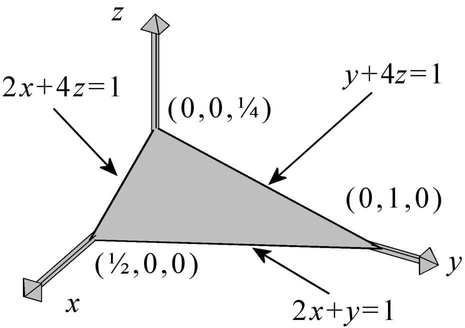

Visualisation of surfaces - Circular cylinder

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- Generalisable to cylinders centred at (a, b, c), cylinders lying parallel to the x or y axes and cyclinders with ellispes as cross sections.

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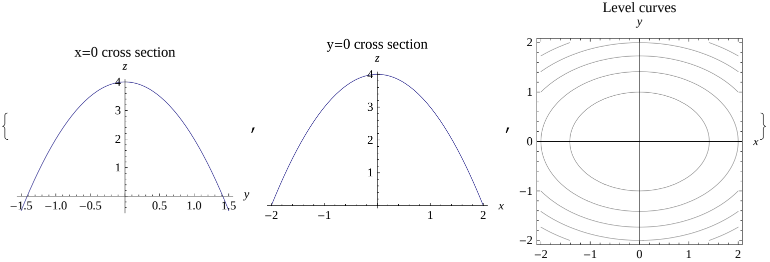

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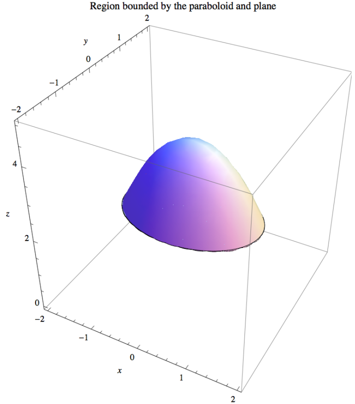

Ordinary derivative

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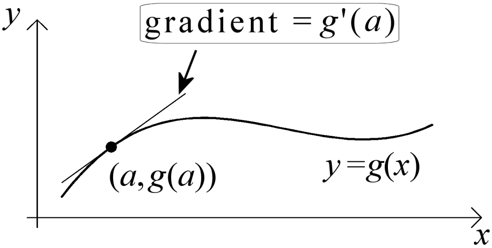

Partial derivatives

- On surface $z = f(x, y)$, there is no single meaning of gradient,

Partial derivatives

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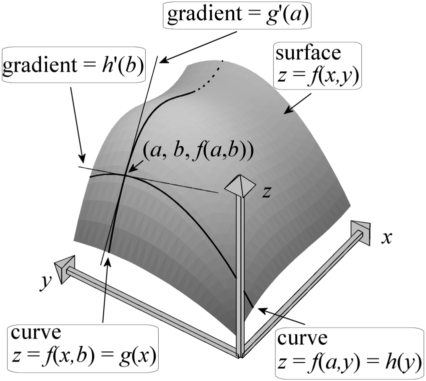
- straight down a mountain side gradient may be very large and traversing the mountain the gradient is much less,
- necessary to define *two* gradients on cross-section of the surface in the x and y directions.

Partial derivatives

Taking cross-sections $x = a$ and $y = b$ we get the graphs of two functions of *one* variable - $z = f(x, b) = g(x)$ and $z = f(a, y) = h(y)$



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- for a function of $x_1, x_2, \dots, x_n$

$$\frac{\partial f}{\partial x_i} = \text{derivative w.r.t. } x_i \text{ with all other variables constant.}$$

Partial derivatives

- Important to distinguish notation used for ordinary and partial derivatives.

$$\text{Ordinary derivative : } \frac{df}{dx}, \quad \text{partial derivative : } \frac{\partial f}{\partial x},$$

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$$\text{Ordinary derivative} : \frac{df}{dx}, \quad \text{partial derivative} : \frac{\partial f}{\partial x},$$

- subscript notation for partial derivatives

$$\frac{\partial f}{\partial x} \equiv f_x, \text{ and } \frac{\partial f}{\partial y} \equiv f_y,$$

Partial derivatives

Example 5

Find f_x, f_y and z_x where

$$\text{(a) } f(x,y) = x^3y^2 + x, \quad \text{(b) } z(x,y) = \sin^{-1}\left(\frac{x}{x+y}\right) \text{ and } x, y > 0.$$

Partial derivatives

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$$\text{(a) } f(x,y) = x^3y^2+x, \quad \text{(b)} z(x,y) = \sin^{-1}\left(\frac{x}{x+y}\right) \text{ and } x, y > 0.$$

[$\sin^{-1} u$ is the inverse sine function and *not* the reciprocal $1/\sin u$. Domain of $\sin^{-1}$ is $[-1,1]$ and $x/(x+y)$ lies in this domain.]

Partial derivatives

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Answer

(a) $f_x = 3x^2y^2 + 1$, $f_y = 2x^3y$.

(b) $z_x = \dfrac{y}{x+y}\dfrac{1}{\sqrt{2xy+y^2}}$.

Partial derivatives

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Chain rule

Recall from Level-1:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

We used

$$\frac{\partial}{\partial x} f(g(x,y)) = f'(g(x,y))g_x(x,y).$$

Partial derivatives

Example 6

Find z_x where z is defined implicitly as a function of x and y by the equation

$$x^4 + 2y^2 + z^3 - 2x^2yz = 1$$

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Answer

$$z_x = \frac{4x^3 - 4xyz}{2x^2y - 3z^2}$$

Partial derivatives

Example 7

For $r \in \mathbb{R}^+$, let $u = f(r)$ where $r^2 = x^2 + y^2 + z^2$. Show that

$$xu_x + yu_y + zu_z = rf'(r).$$

Higher order derivatives

Let u be a function of $x, y, \ldots$ then u_x and u_y are functions of $x, y, \ldots$ and so may define

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$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(u_y) = u_{yx}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(u_y) = u_{yy},$$

etc.

Higher order derivatives

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  Let u be a function of x, y such u_{xy} and u_{yx} exist and are continuous at a point (a, b). Then,

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- Also for functions of more variables and higher order derivatives - e.g. if $u = u(x, y, z)$ then

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$$u_{xyxz} = u_{zyxx} = u_{yxxz} = \cdots = u_{xxyz},$$

- in 2A, we assume the order of taking partial derivatives is unimportant.

Higher order derivatives

Example 8

Determine all second order derivatives of $u = \sin xy$ and verify that $u_{xy} = u_{yx}$.

Higher order derivatives

Example 8

Determine all second order derivatives of $u = \sin xy$ and verify that $u_{xy} = u_{yx}$.

Answers

$$\begin{aligned}
u_{xx} &= -y^2 \sin xy, \\
u_{xy} &= \cos xy - yx \sin xy, \\
u_{yx} &= \cos xy - xy \sin xy, \\
u_{yy} &= -x^2 \sin xy.
\end{aligned}$$

Higher order derivatives

Example 9

Let $u = f(x/y)$, where f is an arbitrary (twice differentiable, with continuous second derivative) function of one variable. Show that

$$xu_x + yu_y = 0,$$

and *deduce* that

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0.$$

Two variable chain rule

- Chain rule for functions of one variable - used to find derivative of $F(x) = f(u(x))$ -

$$\frac{dF}{dx} = \frac{du}{dx}\frac{df}{du} = u'(x)f'(u(x)).$$

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- extend this technique to functions of several variables
- **Theorem**
  Let $F(x, y) = f\big(u(x, y), v(x, y)\big)$. Then

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial f}{\partial v} \qquad \text{and} \qquad \frac{\partial F}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial f}{\partial v}.$$

  This is called the *chain rule for functions of two variables*.

Two variable chain rule

- Observe the pattern

$$\frac{\partial F}{\partial x} = \frac{\boxed{\partial u}}{\partial x}\frac{\partial f}{\boxed{\partial u}} + \frac{\boxed{\partial v}}{\partial x}\frac{\partial f}{\boxed{\partial v}},$$

Two variable chain rule

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$$\frac{\partial F}{\partial x} = \frac{\boxed{\partial u}}{\partial x}\frac{\partial f}{\boxed{\partial u}} + \frac{\boxed{\partial v}}{\partial x}\frac{\partial f}{\boxed{\partial v}},$$

- extends in an obvious way to functions of any number of variables - if $F(x, y, z) = f\left(u(x, y, z), v(x, y, z), w(x, y, z)\right)$ then

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial f}{\partial v} + \frac{\partial w}{\partial x}\frac{\partial f}{\partial w}.$$

Two variable chain rule - Special cases

- if $F(x, y) = f(u(x, y))$ then

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Partial derivatives are written as ordinary derivatives when used on functions of one variable.

Two variable chain rule

Example 10

Let $w = u^2 + v^2$ where $u = \sin\theta$ and $v = \cos\phi$. Use the chain rule to calculate w_θ and w_ϕ in terms of θ and ϕ.

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Let $w = u^2 + v^2$ where $u = \sin\theta$ and $v = \cos\phi$. Use the chain rule to calculate w_θ and w_ϕ in terms of θ and ϕ.

Answer

$$w_\theta = \sin 2\theta, \quad w_\phi = -\sin 2\phi.$$

Examples of ODEs and PDEs

- An *ordinary differential equation* (ODE) is a relationship between a function of one variable and its derivatives

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- An *ordinary differential equation* (ODE) is a relationship between a function of one variable and its derivatives
- e.g Newton's law of cooling states that
  "the rate of change of temperature of an object is proportional to the temperature difference between it and its surroundings"
- in mathematical terms this is the differential equation

$$\frac{dT}{dt} = k(T - T_0),$$

where $T(t)$ is the temperature, T_0 the temperature of the surroundings and k a constant

Examples of ODEs and PDEs

- A *partial differential equation* (PDE) is a relationship between a function of more than one variable and its partial derivatives

Examples of ODEs and PDEs

- A *partial differential equation* (PDE) is a relationship between a function of more than one variable and its partial derivatives
- e.g. the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where $u(x, t)$ is the displacement (from a rest position) of the point x at time t and c is the wave speed.

Definitions

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- the *general solution* includes all possible solutions—includes arbitrary constants (ODE) or arbitrary functions (PDE)
- a solution without arbitrary constants/functions is called a *particular solution*. This may be found by giving extra conditions in the form of initial or boundary conditions.

First order PDEs

Example 11

Find the general solution of the PDE,

$$\frac{\partial f}{\partial x} = x^2 + y + 9,$$

where f is a function of two independent variables x and y.

First order PDEs

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Answers

Solution is

$$\frac{1}{3}x^3 + xy + 9x + A(y)$$

where A is an arbitrary function.

First order PDEs

Example 12

Find the general solution of the PDE,

$$\frac{\partial^2 f}{\partial x \partial y} = 2x,$$

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First order PDEs

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Find the general solution of the PDE,

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where f is a function of two independent variables x and y.

Answers

Solution is

$$x^2 y + A(y) + B(x),$$

where A and B are arbitrary functions.

Solving PDEs using change of variable

- In this section we solve some first and second order PDEs by using a change of independent variables to write it in a simpler form

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$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}.$$

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$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}.$$

- in fact, for any expression E (e.g. a derivative of z)

$$\frac{\partial}{\partial x}(E) = \frac{\partial u}{\partial x}\frac{\partial}{\partial u}(E) + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}(E) \tag{1}$$

this is used when we consider second order PDEs.

First order PDEs

Example 13

By changing variables from (x, y) to (u, v), where $u = xy$, $v = x/y$, solve the PDE

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2x^2 \sin(xy).$$

First order PDEs

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$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2x^2 \sin(xy).$$

Answers

Solution is

$$z = -\frac{x}{y}\cos(xy) + A(x/y),$$

where A is an arbitrary function.

First order PDEs

Example 14

By changing variables from (x, y) to (u, v), where $u = x^3/y$, $v = x$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of partial derivatives with respect to u and v. Hence, solve the PDE

$$x\frac{\partial f}{\partial x} + 3y\frac{\partial f}{\partial y} = \frac{6x^5}{y}.$$

First order PDEs

Example 14

By changing variables from (x, y) to (u, v), where $u = x^3/y$, $v = x$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of partial derivatives with respect to u and v. Hence, solve the PDE

$$x\frac{\partial f}{\partial x} + 3y\frac{\partial f}{\partial y} = \frac{6x^5}{y}.$$

Answers

Solution is

$$f = \frac{3x^5}{y} + A(x^3/y),$$

where A is an arbitrary function.

Chapter 2: Double and triple integration

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Area under curves

- In first year definite integrals arise as "areas under curves".

Area under curves

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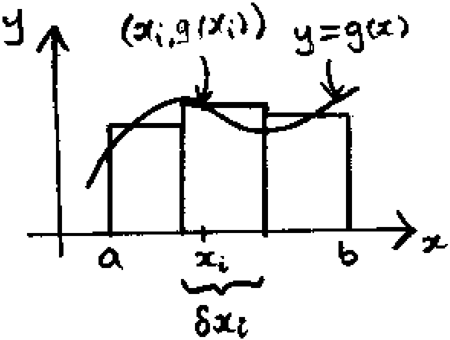
- We approximate the area under the curve the sum of areas of rectangles (called a *Riemann sum*)



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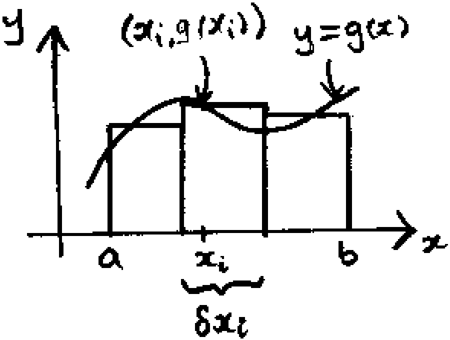

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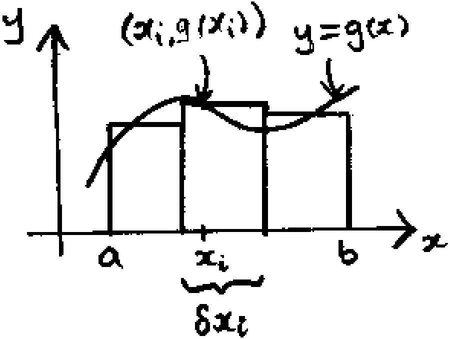


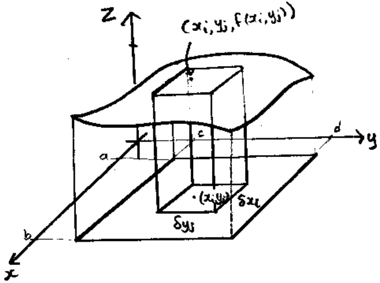
$$\int_a^b g(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} g(x_i)\delta x_i.$$

Double integration on rectangular domains

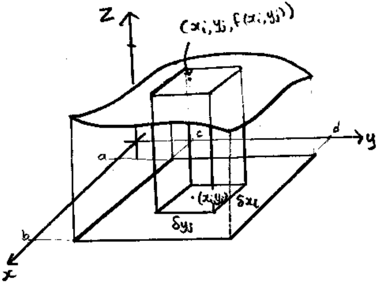
- Similarly, the "volume under a surface" $z = f(x, y)$ on the set $D \subset \mathbb{R}^2$ is approximated by the sum of the volumes of cuboids.

Double integration on rectangular domains

- Similarly, the "volume under a surface" $z = f(x, y)$ on the set $D \subset \mathbb{R}^2$ is approximated by the sum of the volumes of cuboids.
- Divide $R = [a, b] \times [c, d]$ into subrectangles of area $\delta A_{ij} = \delta x_i \delta y_j$ and the cuboid above this has height $f(x_i, y_j)$.



Double integration on rectangular domains



- The whole volume is approximated by

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dxdy = \lim_{M,N\to\infty} \sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i, y_j)\delta A_{ij}.$$

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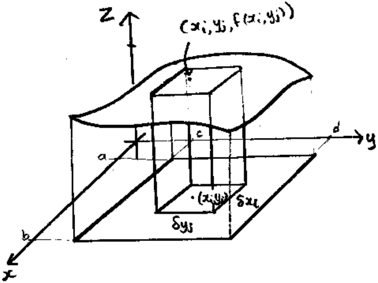

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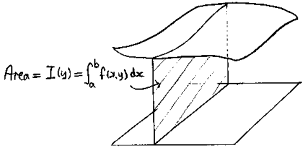
- If the limit as $M, N \to \infty$ exists we say that f is *integrable over* R and $dA = dxdy$ is called the *area element*.

Double integration on rectangular domains

The solid under the curve is made up of slices with y fixed



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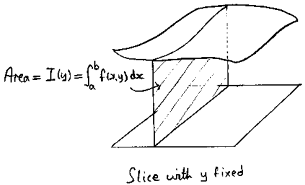

The area under the curve in such a cross section is

$$I(y) = \int_a^b f(x,y)\,dx,$$

where y is fixed in the integrand.

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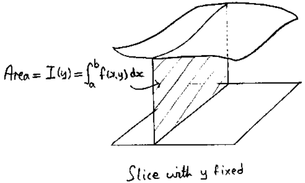

The area under the curve in such a cross section is

$$I(y) = \int_a^b f(x, y)\, dx,$$

where y is fixed in the integrand. The volume under the surface is then

$$\iint_R f(x, y)\, dxdy = \int_c^d I(y)\, dy = \int_c^d \left(\int_a^b f(x, y)\, dx \right)\, dy.$$

Double integration on rectangular domains

Instead, summing the areas of cross sections of the solid with x fixed, we have

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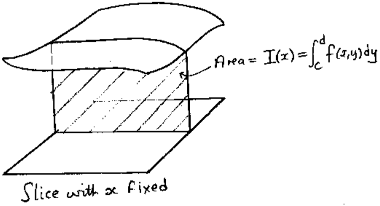

Notation:

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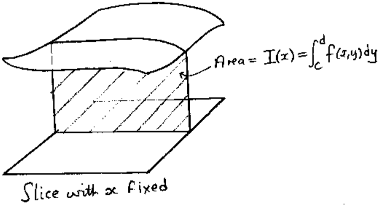

Notation:

$$\int_a^b dx \int_c^d f(x,y)\,dy \quad \text{for} \quad \int_a^b \left(\int_c^d f(x,y)\,dy \right)\,dx.$$

Double integration on rectangular domains

Example 1

Evaluate $$\iint_R x^2 + y^2 \, dxdy$$

where R is $[1,3] \times [2,4]$.

Double integration on rectangular domains

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Evaluate $$\iint_R x^2 + y^2 \, dxdy$$

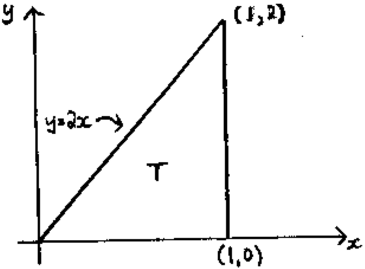
where R is $[1,3] \times [2,4]$.

Answer

$$\frac{164}{3}$$

Double integration on regular domains

Consider a more complicated domain T which is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,2)$.



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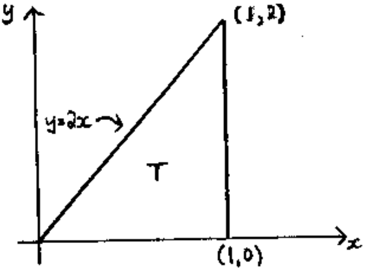
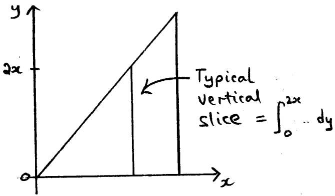


The domain T is bounded by the lines $y = 0$, $x = 1$ and $y = 2x$.

Double integration on regular domains

To evaluate a double integral over T we could split T into a collection of **vertical slices**,



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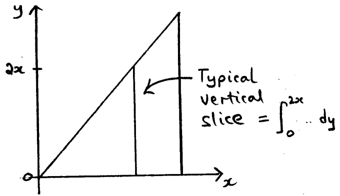


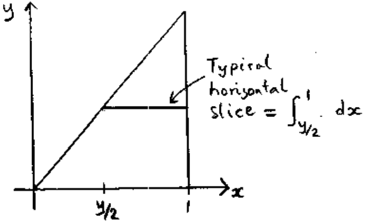
integrate with respect to y and then integrate the result with respect to x.

$$\iint_T f(x,y)\,dxdy = \int_0^1 dx \int_0^{2x} f(x,y)\,dy.$$

Notice that the limits in the first integral *depend on* x.

Double integration on regular domains

Alternatively, looking at horizontal slices, with end-points $x = \frac{1}{2}y$, $x = 1$, and summing these from $y = 0$ to $y = 2$.



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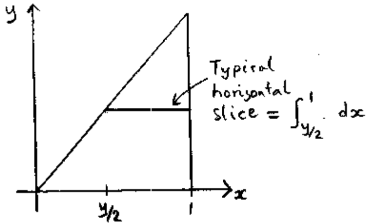

Thus the integral is also

$$\iint_T f(x, y)\, dxdy = \int_0^2 dy \int_{\frac{1}{2}y}^{1} f(x, y)\, dx.$$

Double integration on regular domains

Definition

Let D be a domain in the x, y-plane. D is said to be

- Type I (*y-simple*) if it is bounded by lines $x = a$, $x = b$ and curves $y = g(x)$, $y = h(x)$, the intersection of any vertical line $x = c$, where $c \in [a, b]$, is an interval or a single point,

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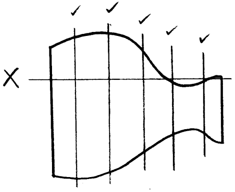
- *regular* if it the union of finitely many disjoint type I and type II domains. Every type I and type II domain is regular.

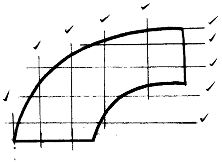
.

Double integration on regular domains

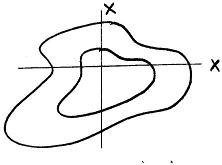
Example



Type I and not type II



Type I and type II

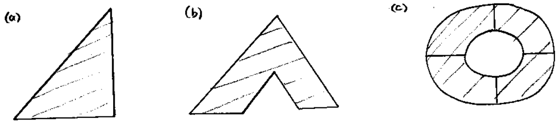


neither type I or type II

Double integration on regular domains

Example 2

State whether each of the domains shown below are type I and/or type II or regular.



Double integration on regular domains

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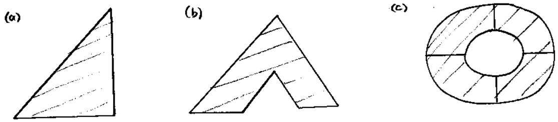

Answers

(a) Both, (b) Type I only, (c) Neither.

Double integration on regular domains

Theorem

If D is the type I domain defined by $g(x) \leq y \leq h(x)$ where $a \leq x \leq b$ then

$$\iint_D f(x,y)\,dxdy = \int_a^b dx \int_{g(x)}^{h(x)} f(x,y)\,dy.$$

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If D is the type II domain defined by $g(y) \leq x \leq h(y)$ where $a \leq y \leq b$ then

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$$\iint_D f(x, y)\, dxdy = \int_a^b dy \int_{g(y)}^{h(y)} f(x, y)\, dx.$$

The *inner integral* may have a limit depending on the other variable but the *outer integral*) has constant limits.

Double integration on regular domains

Example 3

Evaluate

$$\iint_D xy^2 \, dxdy,$$

where D is the region in the first quadrant bounded by the curve $y = 4x^2$, the x axis and the line $x = 1$.

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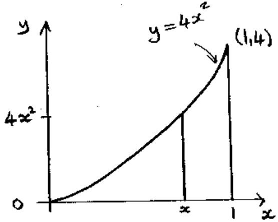

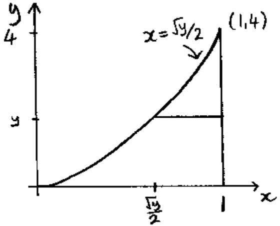

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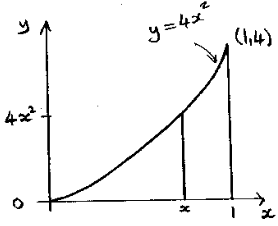

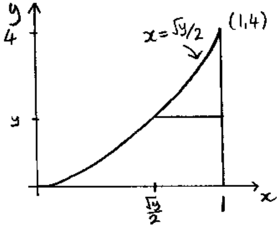

$$I = \frac{8}{3}$$

Double integration on regular domains

Example 4

Evaluate

$$I = \iint_D 3x^2 + y^2 \, dxdy,$$

where D is the triangle with vertices $(0,0)$, $(1,1)$ and $(2,1)$.

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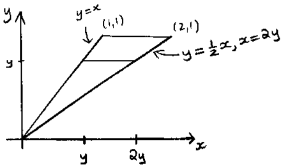

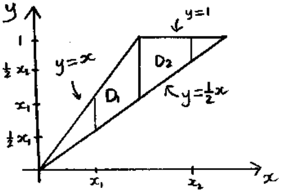

Double integration on regular domains

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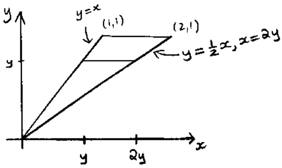
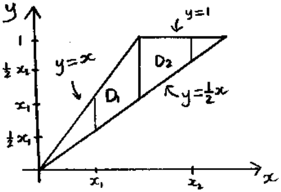
Answer





$$I = 2.$$

Double integration on regular domains

Example 5

Evaluate

$$I = \int_0^1 dx \int_{\sqrt{x}}^1 \frac{e^{y^2}}{\sqrt{x}} \, dy.$$

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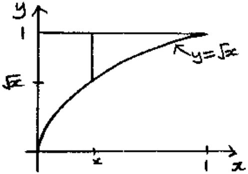

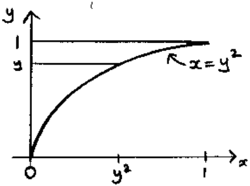

Double integration on regular domains

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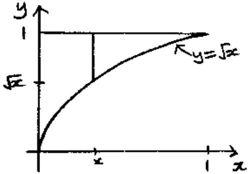

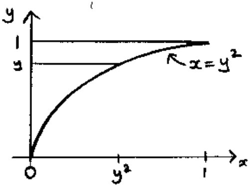

$$I = e - 1.$$

Double integration on regular domains

Example 6

Find the volume of the tetrahedron T, bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

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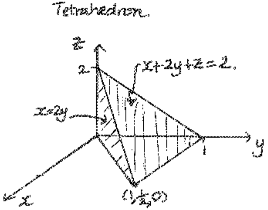

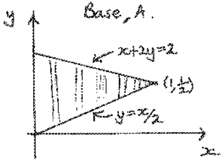

Double integration on regular domains

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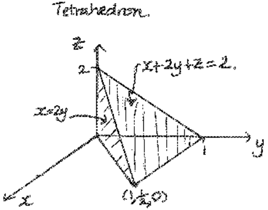

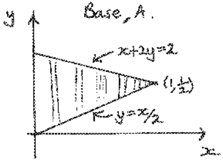

$$T = \frac{1}{3}.$$

Double integration in polar coordinates

The position of a point (x, y) on the cartesian plane can be specified by r, θ which are

Polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \theta \in [0, 2\pi), \quad r \geq 0.$$

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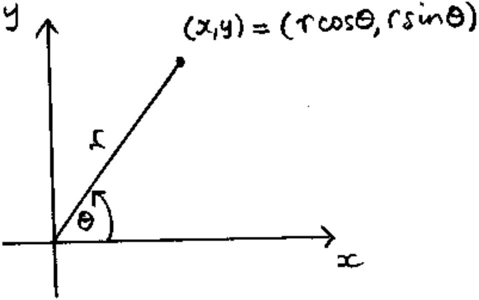

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In cartesian coordinates, the area of an elementary rectangle using in the Riemann sum is $\delta A = \delta x \delta y$. In polar coordinates, the area element has area $\delta A \approx r \delta r \delta \theta$.

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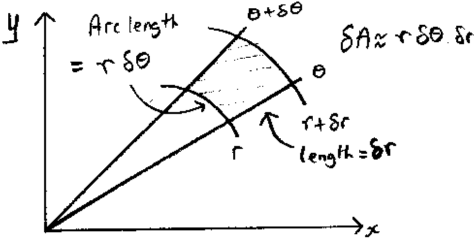

For this reason in polar coordinates, $dA = r dr d\theta$, i.e.,

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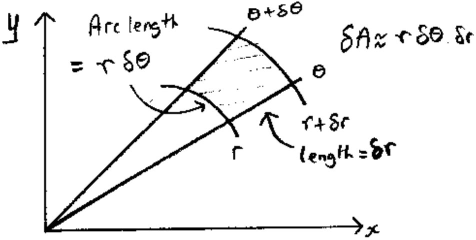

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When either the domain is circular or the integrand is written in terms of $x^2 + y^2\ (= r^2)$, use polar coordinates.

Double integration in polar coordinates

Example 7

Use polar coordinates to evaluate

$$I = \iint_D x + y \, dxdy,$$

where D is part of the annulus between circles of radius 1 and 2, centre $(0, 0)$ lying in upper half plane.

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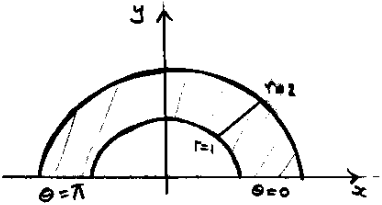

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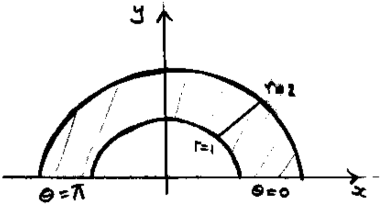

$$I = \frac{14}{3}.$$

Double integration in polar coordinates

Example 8

Evaluate

$$I = \iint_D y \, dxdy,$$

where D is the part of the disk of radius $a \, (> 0)$ and centre $(a, 0)$ lying in the first quadrant.

Double integration in polar coordinates

Example 8

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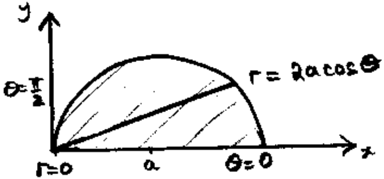
Answer



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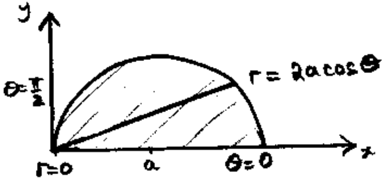

$$I = \frac{2a^3}{3}.$$

Beta and Gamma functions

Beta functions can help us easily integrate functions that involve powers of cosine and sine.

- *Beta function:*

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \qquad p > 0 \text{ and } q > 0.$$

A particularly useful form is,

$$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}(y)\cos^{2q-1}(y)\,dy\ .$$

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- *Gamma function:*

  $$\Gamma(k) = \int_0^{\infty} x^{k-1}e^{-x}\; dx, \quad k > 0.$$

Properties of Beta and Gamma functions

1. $\Gamma(1) = 1$, $\Gamma(2) = 1$, $\Gamma(3) = 2$ and in general $\Gamma(n) = (n-1)!$ for every positive integer n.

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4. $\Gamma(1/2) = \sqrt{\pi}$.
5. $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Properties of Beta and Gamma functions

Result

From the properties of Gamma functions we can derive the following result:

Property of Beta functions

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} K$$

where $K = 1$ unless m and n are both even in which case $K = \pi/2$.

Properties of Beta and Gamma functions

Result

From the properties of Gamma functions we can derive the following result:

Property of Beta functions

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} K$$

where $K = 1$ unless m and n are both even in which case $K = \pi/2$.

In the special cases, $m = 0$ or $m = 1$ none of the numerator factors involving m appear.

Properties of Beta and Gamma functions

Result

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For example,

$$\int_0^{\pi/2} \sin^3 x \cos^6 x \, dx = \frac{2.5.3.1}{9.7.5.3.1} = \frac{2}{63}.$$

Simplifying sine and cosine integrals

Properties of the graphs of sine and cosine seen in 1S/X simplify the integral before applying Beta functions.

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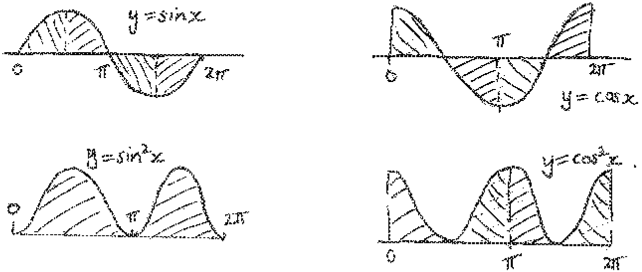

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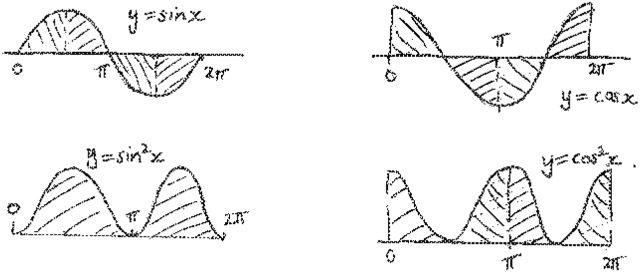

We deduce

$$\int_0^{\pi} \sin x \, dx = 2\int_0^{\pi/2} \sin x \, dx; \quad \int_0^{2\pi} \sin x \, dx = 0; \quad \int_0^{\pi} \cos x \, dx = 0;$$

$$\int_0^{\pi} \sin^2 x \, dx = 2\int_0^{\pi/2} \sin^2 x \, dx; \quad \int_0^{\pi} \cos^2 x \, dx = 2\int_0^{\pi/2} \cos^2 x \, dx \quad \ldots \text{et}$$

Beta functions

Example 9

Evaluate:

(a) $I = \int_0^{\pi} \sin^3 x \cos^4 x \; dx$, (b) $I = \int_0^{\pi} \sin^3 x \cos^5 x \; dx$,

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Beta functions

Example 9

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$$\text{(c) } I = \int_0^{2\pi} \sin^2 x \cos^4 x \; dx.$$

Answers

(a) $I = \frac{4}{35}$, (b) $I = 0$, (c) $I = \frac{\pi}{8}$.

Change of variables in double integration

Definition

Consider a change of variables x, y to u, v. So $x = x(u, v)$ and $y = y(u, v)$. The *Jacobian* $\dfrac{\partial(u, v)}{\partial(x, y)}$ is the determinant

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

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If the change of variables is invertible then the Jacobian is nonzero and

$$\frac{\partial(x, y)}{\partial(u, v)} = 1 \bigg/ \frac{\partial(u, v)}{\partial(x, y)} \, .$$

Change of variables in double integration

Theorem

Let the change of variables x, y to u, v be invertible on the domain D. Then

$$\iint_D f(x,y)\, dxdy = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv,$$

where D is the domain in the xy-plane and S is the corresponding domain in the uv-plane.

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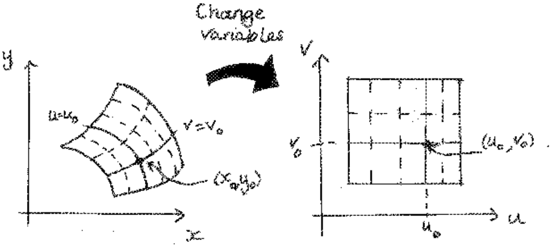
Often it is convenient to use

$$\frac{\partial(x,y)}{\partial(u,v)} = 1 \left/ \frac{\partial(u,v)}{\partial(x,y)}\right.$$

Change of variables in double integration

- The idea here is to choose variables u, v in which the domain is simply described, preferably with constant limits, e.g.



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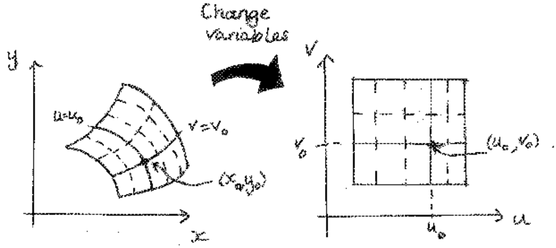

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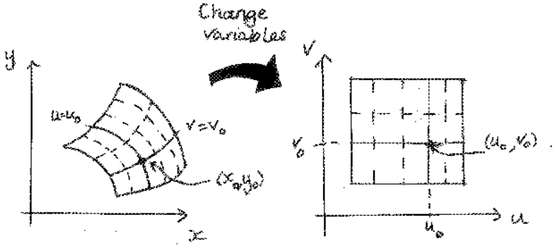

The lines $u = u_0$ and $v = v_0$ in the uv-plane get mapped to curves $x = x(u_0, v)$, $y = y(u_0, v)$ and $x = x(u, v_0)$, $y = y(u, v_0)$ in the xy-plane.

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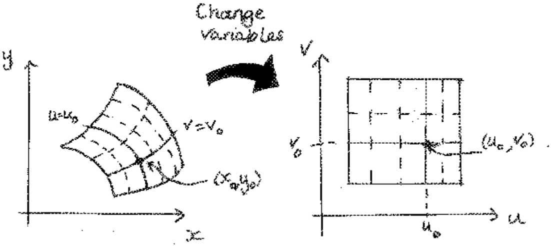

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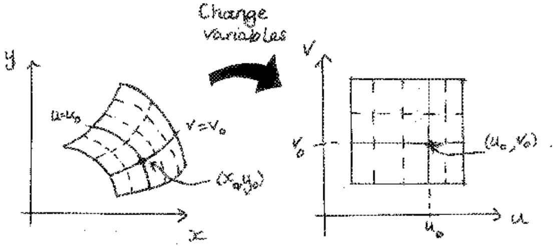

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Change of variables in double integration

- Let $\mathbf{r} = (x, y)$. The tangent vector at (x_0, y_0) to the curve $(x(u, v_0), y(u, v_0))$ is $\mathbf{r}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right)$.

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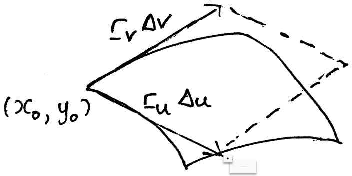
$$\Delta u\, \mathbf{r}_u \times \Delta v\, \mathbf{r}_v = \Delta u \Delta v (x_u y_v - x_v y_u)\mathbf{k} = \Delta u \Delta v \frac{\partial(x, y)}{\partial(u, v)}\mathbf{k}.$$



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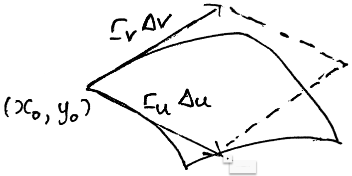

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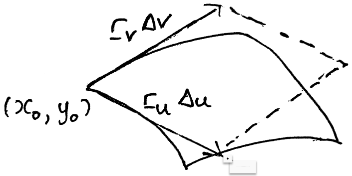

Change of variables in double integration

- Summing the elements that make up the region D

$$\iint_D f(x,y)\,dxdy = \lim_{N,\,M\to\infty} \sum_{i=1}^{N}\sum_{j=1}^{M} f(x_i,y_j)\Delta A$$
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- For the change to polar coordinates,

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giving the result

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Change of variables in double integration

Example 10

By making a suitable change of variables, evaluate

$$\iint_D x + 3y \, dxdy,$$

where D is the region bounded by the lines

$$y = x - 1, \quad y = x + 1, \quad y = -x - 1, \quad y = -x + 3.$$

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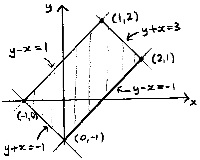

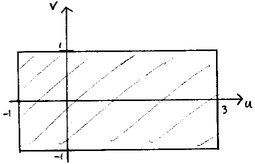

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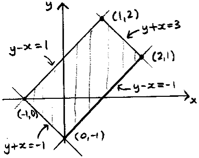

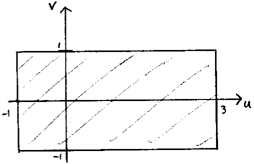

$I = 8.$

Change of variables in double integration

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Find the area bounded by the curves $y = e^x$, $y = 2e^x$, $y = e^{-x}$ and $y = 2e^{-x}$.

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$$\iint_A 1\, dxdy.$$

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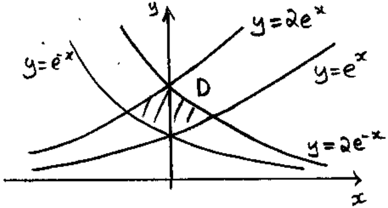

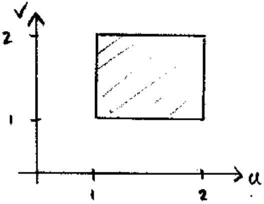

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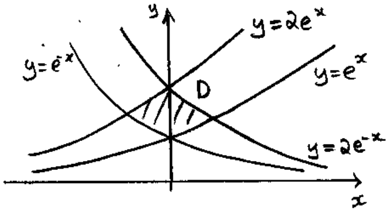

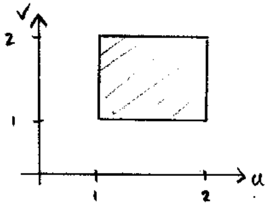

$$Area = 2(3 - 2\sqrt{2}).$$

Triple integration

Define triple integrals for functions of three variables.

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$$\iint_R f(x, y)\, dxdy = \lim_{N,\, M \to \infty} \sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i, y_j) \delta x_i \delta y_j.$$

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For a triple integral instead of summing over an area $\delta A_{ij} = \delta x_i \delta y_j$, we sum over a volume $\delta V_{ijk} = \delta x_i \delta y_j \delta z_k$ which leads us to

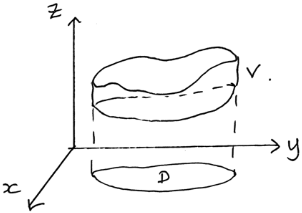
$$\iiint_V f(x,y,z)\,dxdydz = \lim_{N,\,M,\,L\to\infty} \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{k=1}^{L} f(x_i,y_j,z_k)\delta x_i \delta y_j \delta z_k.$$

Triple integration

- If V lies between two continuous functions of x and y then

$$\iiint_V f(x,y,z)\,dxdydz = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dxdy$$

where D is the projection of V onto the xy plane.



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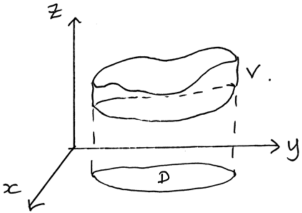

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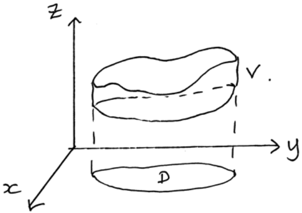

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Triple integration

In general if V lies between two continuous functions of x and y then

Triple integral

$$\iiint_V f(x,y,z)\,dxdydz = \underbrace{\int_a^b}_{\text{Constants}} dx \underbrace{\int_{h_1(x)}^{h_2(x)}}_{\text{Curves}} dy \underbrace{\int_{g_1(x,y)}^{g_2(x,y)}}_{\text{Surfaces}} f(x,y,z)\,dz.$$

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Example 12

Evaluate

$$I = \iiint_V z\, dxdydz,$$

where V is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

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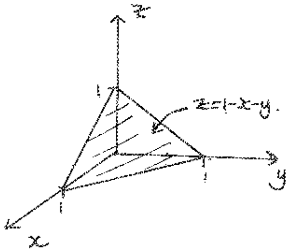
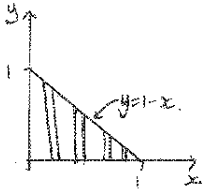
Answer





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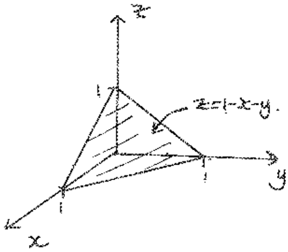

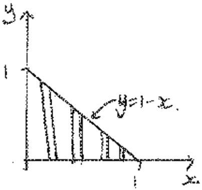

$$I = \frac{1}{24}.$$

Triple integration

Example 13

Set up (but do not evaluate) the integral for the volume of the solid that lies below the paraboloid $z = 9 - x^2 - y^2$ and above the plane $z = 5$.

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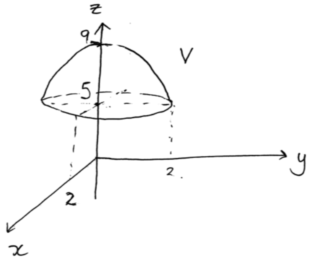

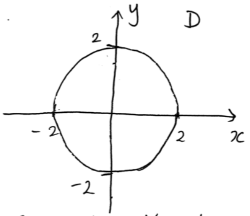

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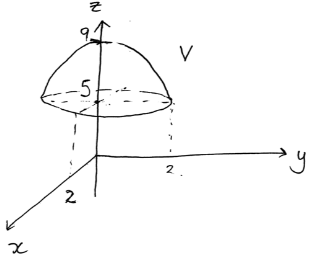

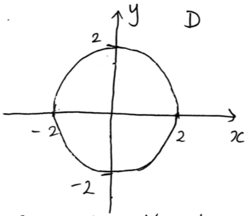

Projection of V onto the xy-plane.

Triple integration in spherical coordinates

The position of a point (x, y, z) in cartesian coordinates can be specified by $\rho,\ \theta,\ \phi$ which are

Spherical coordinates

$$x = \rho \sin\phi \cos\theta, \quad y = \rho\, \sin\phi\, \sin\theta, \quad z = \rho\cos\phi.$$

$\theta \in [0, 2\pi)$, $\phi \in [0, \pi)$, $\rho \geq 0$.

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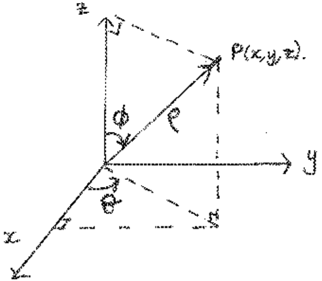

Triple integration in spherical coordinates

In cartesian coordinates, the volume of an elementary cuboid used in the Riemann sum is $\delta V = \delta x\, \delta y\, \delta z$. In spherical coordinates, the volume element is $\delta V \approx \rho^2 \sin\phi\, \delta\theta\, \delta\phi\, \delta\rho$.

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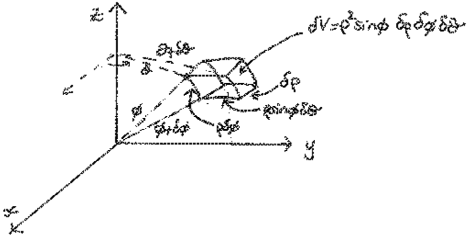

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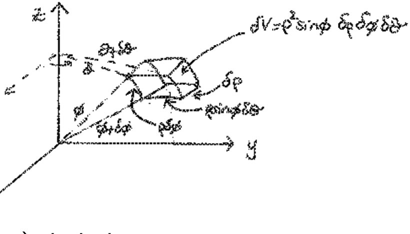

$$\iiint_V f(x, y, z)\, dxdydz =$$

$$\iiint_V f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \; d\theta d\phi d\rho.$$

When either the domain is spherical or the integrand is written in terms of $x^2 + y^2 + z^2\ (= \rho^2)$, use spherical coordinates.

Triple integration in spherical coordinates

Example 14

Use spherical coordinates to evaluate

$$I = \iiint_B \exp((x^2 + y^2 + z^2)^{3/2})\, dxdydz,$$

where B is the unit ball, $B = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$.

Triple integration in spherical coordinates

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Answer

$$I = \frac{4}{3}\pi(e - 1).$$

Triple integration in spherical coordinates

Example 15

Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

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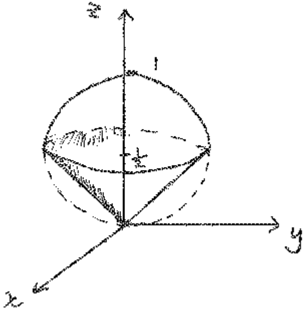

Triple integration in spherical coordinates

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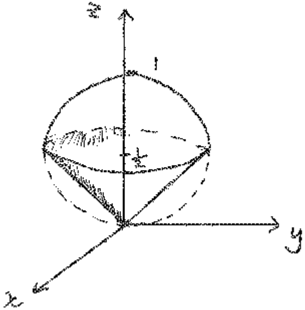
Answer



$$V = \frac{\pi}{8}.$$

Chapter 3: Differentiation of vectors

- Scalar- and vector-valued functions

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- Scalar- and vector-valued functions
- vector and scalar fields

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- Scalar- and vector-valued functions
- vector and scalar fields
- types of derivative—grad, div and curl

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- they are *vector-valued* functions—the result is a 2- or 3-vector
- examples include velocity as a function of time and direction of the Earth's magnetic field.

Parametric equations of curves

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- position as a function of time is one example. We will revisit parametric equations in Chapter 4.

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- A function $f(\mathbf{r})$ where $\mathbf{r} = (x, y)$ or (x, y, z) (mapping a vector to a scalar) is called a *scalar field*.

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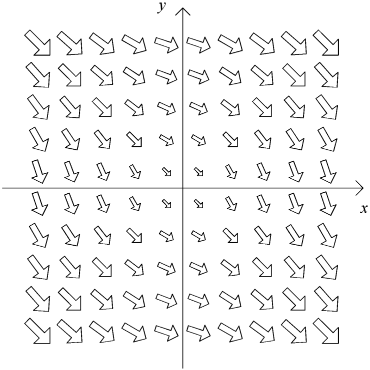
A typical vector field



e.g. velocity at different points in a fluid.

Different types of derivative

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Name of product	Formula	Type of result	Derivative
Scalar multiplication	$\alpha\mathbf{u}$	Vector	∇f
Scalar or dot product	$\mathbf{u}\cdot\mathbf{v}$	Scalar	$\nabla\cdot\mathbf{F}$
Vector or cross product	$\mathbf{u}\times\mathbf{v}$	Vector	$\nabla\times\mathbf{F}$

Gradient of a scalar field

The *gradient* of a scalar field f is

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Answer

$$\mathrm{grad}\, f = (2xy + \cosh yz, x^2 + xz \sinh yz, xy \sinh yz).$$

Gradient of a scalar field

Example 2

Let $\mathbf{r} = (x, y, z)$ so that $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. Show that

$$\nabla(r^n) = nr^{n-2}\mathbf{r},$$

for any integer n and deduce the values of $\mathrm{grad}(r)$, $\mathrm{grad}(r^2)$ and $\mathrm{grad}(1/r)$.

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Answers

$$\begin{aligned}\mathrm{grad}(r) &= \frac{\mathbf{r}}{r},\\ \mathrm{grad}(r^2) &= 2\mathbf{r},\\ \mathrm{grad}(1/r) &= -\frac{\mathbf{r}}{r^3}.\end{aligned}$$

Gradient of a scalar field

Example 3

Determine $\mathrm{grad}(\mathbf{c} \cdot \mathbf{r})$, when $\mathbf{c}$ is a constant (vector).

Gradient of a scalar field

Example 3

Determine $\operatorname{grad}(\mathbf{c} \cdot \mathbf{r})$, when $\mathbf{c}$ is a constant (vector).

Answer

$\operatorname{grad}(\mathbf{c} \cdot \mathbf{r}) = \mathbf{c}$.

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- Important: only valid for *unit* vectors $\mathbf{u}$
- Partial derivatives are directional derivatives, e.g.

$$\frac{\partial f}{\partial \mathbf{i}} = \frac{\partial f}{\partial x}.$$

Directional derivative

Example 4

Find the directional derivative of $f = x^2yz^3$ at the point $P(3, -2, -1)$ in the direction of the vector $(1, 2, 2)$.

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Answer

$$\frac{\partial f}{\partial \mathbf{u}}(3, -2, -1) = -38.$$

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Directional derivative

Example 5

Consider $f = \ln(xy + z^3)$ at the point $P(1, 1, 1)$. In what direction does f have the maximal rate of change? What is this maximal rate of change?

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Answer

Direction is $(1/2, 1/2, 3/2)$. Maximal rate of change is

$$|\nabla f(1,1,1)| = \frac{\sqrt{11}}{2}.$$

Divergence of a vector field

- The *divergence* of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is the *scalar* obtained as the "scalar product" of ∇ and $\mathbf{F}$,

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- so called, because it measures the tendency of a vector field to diverge (positive divergence) or converge (negative divergence)
- a vector field is said to be *incompressible* (or *solenoidal*) if its divergence is zero.

Divergence of a vector field

- Consider $\mathbf{F} = (x, y, 0)$, $\mathbf{G} = (x, -y, 0)$ and $\mathbf{H} = (-x, -y, 0)$

Divergence of a vector field

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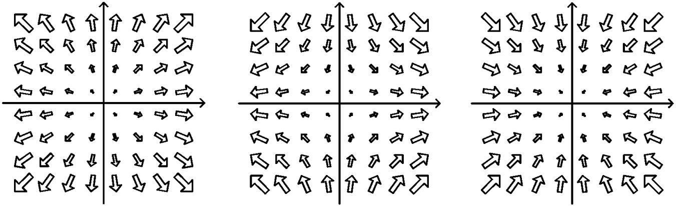
- 

F, positive divergence **G**, incompressible **H**, negative divergence

Divergence of a vector field

Example 6

Show that the divergence of $\mathbf{F} = (x - y^2, z, z^3)$ is positive at all points in $\mathbb{R}^3$.

Laplacian

- A particular example of divergence is the *Laplacian* of a scalar field f

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- Can be extended in a natural way to the Laplacian of a vector field $\mathbf{F} = (F_1, F_2, F_3)$,

$$\boxed{\nabla^2 \mathbf{F} = (\nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3)}\,.$$

Laplacian

Example 7

Find the values of n for which $\nabla^2(r^n) = 0$.

Laplacian

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Find the values of n for which $\nabla^2(r^n) = 0$.

Answer

$\nabla^2(r^n) = 0$ if and only if $n = 0$ or $n = -1$.

Curl of a vector field

- The *curl* of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is the *vector* obtained as the "vector product" of ∇ and $\mathbf{F}$

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$$\boxed{\operatorname{curl}\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}}\,.$$

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- can be calculated using a 3×3 determinant,

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

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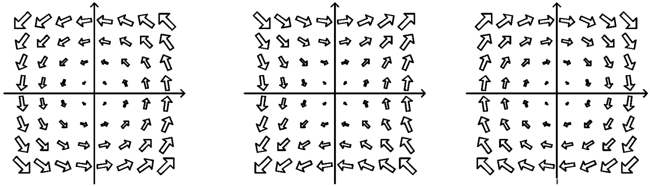
- 

F, anticlockwise rotation **G**, irrotational **H**, clockwise rotation

Curl of a vector field

Example 8

Determine $\operatorname{curl}\mathbf{F}$ when $\mathbf{F} = (x^2y, xy^2 + z, xy)$.

Curl of a vector field

Example 8

Determine $\operatorname{curl} \mathbf{F}$ when $\mathbf{F} = (x^2y, xy^2 + z, xy)$.

Answer

$\operatorname{curl} \mathbf{F} = (x - 1, -y, y^2 - x^2)$.

Curl of a vector field

Example 9

If **c** is a constant vector, find $\mathrm{curl}(\mathbf{c} \times \mathbf{r})$.

Curl of a vector field

Example 9

If $\mathbf{c}$ is a constant vector, find $\mathrm{curl}(\mathbf{c} \times \mathbf{r})$.

Answer

$\mathrm{curl}(\mathbf{c} \times \mathbf{r}) = 2\mathbf{c}$.

Nabla identities

Analogues involving div, grad and curl of the elementary rules of differentiation such as linearity $(f+g)'(x) = f'(x) + g'(x)$ the product rule $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$.

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$$\boxed{\begin{aligned} \mathrm{grad}(f+g) &= \mathrm{grad}\, f + \mathrm{grad}\, g & \mathrm{grad}(fg) &= f(\mathrm{grad}\, g) + (\mathrm{grad}\, f)g, \\ \mathrm{div}(\mathbf{F}+\mathbf{G}) &= \mathrm{div}\,\mathbf{F} + \mathrm{div}\,\mathbf{G} & \mathrm{div}(f\mathbf{F}) &= f\,\mathrm{div}\,\mathbf{F} + \mathrm{grad}\, f\cdot\mathbf{F}, \\ \mathrm{curl}(\mathbf{F}+\mathbf{G}) &= \mathrm{curl}\,\mathbf{F} + \mathrm{curl}\,\mathbf{G} & \mathrm{curl}(f\mathbf{F}) &= f\,\mathrm{curl}\,\mathbf{F} + \mathrm{grad}\, f\times\mathbf{F}, \\ \mathrm{curl}\,\mathrm{grad}\, f &= \mathbf{0}, & \mathrm{div}\,\mathrm{curl}\,\mathbf{F} &= 0. \end{aligned}}$$

Nabla identities

- Note the special cases

$$\mathrm{grad}(cf) = c\,\mathrm{grad}\,f, \quad \mathrm{div}(c\mathbf{F}) = c\,\mathrm{div}\,\mathbf{F}, \quad \mathrm{curl}(c\mathbf{F}) = c\,\mathrm{curl}\,\mathbf{F},$$

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  when c is a (scalar) constant.
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- e.g.

$$\begin{aligned} \operatorname{curl}(f\mathbf{F}) &= \nabla \times (f\mathbf{F}) \\ &= f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F} \\ &= f \operatorname{curl} \mathbf{F} + \operatorname{grad} f \times \mathbf{F}. \end{aligned}$$

Nabla identities

Example 10

Prove the identities

$$\text{(i)}\ \operatorname{curl}\operatorname{grad} f = 0, \qquad \text{(ii)}\ \operatorname{curl}(f\mathbf{F}) = f \operatorname{curl}\mathbf{F} + \operatorname{grad} f \times \mathbf{F}$$

$$\text{(iii)}\ \operatorname{div}(f\mathbf{F}) = f \operatorname{div}\mathbf{F} + (\operatorname{grad} f)\cdot\mathbf{F}.$$

Nabla identities

Example 11

Let $\mathbf{c}$ be a constant vector and $\mathbf{r} = (x, y, z)$ so that $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. Determine

(i) $\operatorname{div}(r^n(\mathbf{c} \times \mathbf{r}))$, (ii) $\operatorname{curl}(r^n(\mathbf{c} \times \mathbf{r}))$.

Nabla identities

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$$\text{(i)}\ \operatorname{div}(r^n(\mathbf{c} \times \mathbf{r})), \quad \text{(ii)}\ \operatorname{curl}(r^n(\mathbf{c} \times \mathbf{r})).$$

Answers

(i) $\operatorname{div}(r^n(\mathbf{c} \times \mathbf{r})) = 0$,
(ii) $\operatorname{curl}(r^n(\mathbf{c} \times \mathbf{r})) = (n+2)r^n\mathbf{c} - n(\mathbf{r} \cdot \mathbf{c})r^{n-2}\mathbf{r}$.