Why is $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ normal to the surface $f(x,y,z)=0$?

Answer:

If we can implicitly write z as a function of x, and y

$f(x, y, z(x,y)) = 0$. (1)

So $\underline{r} = (x, y, z(x,y))$ describes points in the plane.

$\underline{r}_x = (1, 0, \partial z/\partial x)$ and $\underline{r}_y = (0, 1, \partial z/\partial y)$ are tangent vectors to the plane.

By differentiating (1) w.r.t. x: implicitly:

$\frac{\partial f}{\partial x} \cdot 1 + 0 + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$ so $\partial z/\partial x = -f_x/f_z$.

Similarly $\partial z/\partial y = -f_y/f_z$.

So $\underline{r}_x = (1, 0, -f_x/f_z)$ and $\underline{r}_y = (0, 1, -f_y/f_z)$

$\underline{r}_x \times \underline{r}_y$ is perpendicular to the tangent vectors.

$$\underline{r}_x \times \underline{r}_y = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -f_x/f_z \\ 0 & 1 & -f_y/f_z \end{vmatrix} = \underline{i}\frac{f_x}{f_z} + \underline{j}\frac{f_y}{f_z} + \underline{k}$$

Multiplying by f_z gives:

(f_x, f_y, f_z) which is also perpendicular to the plane as required.