

Tutorial Exercises

T1 Evaluate the following using beta functions:

- (a) $\int_0^{\pi/2} \sin^3 x \cos^2 x \, dx$, (b) $\int_0^{\pi/2} \sin^7 x \cos^3 x \, dx$, (c) $\int_0^{\pi} \cos^5 x \, dx$,
 (d) $\int_0^{\pi/2} \sin^4 x \cos^2 x \, dx$, (e) $\int_0^{\pi} \sin^5 x \, dx$, (f) $\int_0^{\pi} \sin^2 x \cos^3 x \, dx$,
 (g) $\int_0^{2\pi} \sin^3 x \cos^3 x \, dx$, (h) $\int_0^{2\pi} \sin^4 x \cos^3 x \, dx$,

Solution

(a) $\frac{2}{15}$, (b) $\frac{1}{40}$, (c) 0, (d) $\frac{\pi}{32}$, (e) $\frac{16}{15}$, (f) 0, (g) 0, (h) 0.

T2 Given the change of variables

$$u = \frac{1}{3}(x + y) \quad v = \frac{1}{3}(x - 2y)$$

express x and y in terms of u and v .

Solution

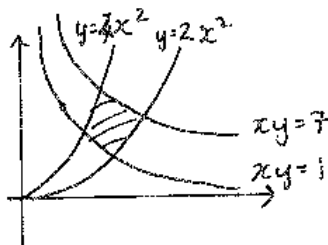
$$x = 2u + v, \quad y = u - v.$$

T3 By making a suitable change of variables, evaluate

$$\iint xy \, dx \, dy$$

over the region enclosed by the two hyperbolas $xy = 1$ and $xy = 7$ and the two parabolas $y = 2x^2$ and $y = 4x^2$.

Solution



Let $u = xy$ and $v = y/x^2$, so $1 \leq u \leq 7$ and $2 \leq v \leq 4$. The Jacobian is

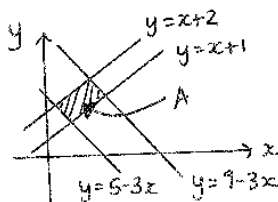
$$\frac{\partial(u, v)}{\partial(x, y)} = (y) \cdot (1/x^2) - x \cdot (-2y/x^3) = 3y/x^2,$$

and so $\frac{\partial(x, y)}{\partial(u, v)} = x^2/3y$. Hence the integral is

$$\begin{aligned} I &= \int_1^7 du \int_2^4 xy \cdot \left| \frac{x^2}{3y} \right| dv = \frac{1}{3} \int_1^7 du \int_2^4 \frac{u}{v} dv \\ &= \frac{1}{3} \int_1^7 u du \int_2^4 \frac{1}{v} dv = \frac{1}{3} \left[\frac{u^2}{2} \right]_1^7 [\log v]_2^4 \\ &= \frac{1}{6} \cdot 48 \cdot (\log 4 - \log 2) = 8 \log 2. \end{aligned}$$

T4 Use double integration and an appropriate change of variables to find the area of the parallelogram enclosed by the lines $y = x + 1$, $y = x + 2$, $y = 5 - 3x$, $y = 9 - 3x$

Solution



Let $u = y - x$ and $v = y + 3x$, so $1 \leq u \leq 2$ and $5 \leq v \leq 9$. The Jacobian is

$$\frac{\partial(u, v)}{\partial(x, y)} = (-1) \cdot (1) - 1 \cdot (3) = -4,$$

and so $\frac{\partial(x, y)}{\partial(u, v)} = -1/4$. Hence this area, A , is

$$\iint_A dA = \int_1^2 du \int_5^9 1 \cdot \left| \frac{-1}{4} \right| dv = \frac{1}{4} [u]_1^2 [v]_5^9 = 1.$$

T5 Evaluate

$$\iiint yz^2 \, dx dy dz$$

throughout the cube given by $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution

$$I = \int_0^1 dx \int_0^1 dy \int_0^1 yz^2 \, dz = [x]_0^1 \left[\frac{1}{2} y^2 \right]_0^1 \left[\frac{1}{3} z^3 \right]_0^1 = \frac{1}{6}.$$

F1 Evaluate the following using beta functions:

- (a) $\int_0^{\pi/2} \cos^6 x \, dx$, (b) $\int_0^{\pi/2} \sin^2 x \, dx$, (c) $\int_0^{\pi/2} \sin x \cos x \, dx$,
 (d) $\int_0^{\pi/2} \sin^7 x \, dx$, (e) $\int_0^{\pi} \sin^6 x \, dx$, (f) $\int_0^{\pi} \sin^2 x \cos^3 x \, dx$,
 (g) $\int_0^{\pi} \cos^6 x \, dx$, (h) $\int_0^{\pi} \sin^3 x \cos^2 x \, dx$, (i) $\int_0^{\pi} \sin^4 x \cos^4 x \, dx$,
 (j) $\int_0^{2\pi} \cos^4 x \, dx$, (k) $\int_0^{2\pi} \sin^3 x \cos^2 x \, dx$, (l) $\int_0^{2\pi} \sin^2 x \cos^6 x \, dx$,

Solution

(a) $\frac{5\pi}{32}$, (b) $\frac{\pi}{4}$, (c) $\frac{1}{2}$, (d) $\frac{16}{35}$, (e) $\frac{5\pi}{16}$, (f) 0, (g) $\frac{5\pi}{16}$, (h) $\frac{4}{15}$, (i) $\frac{3\pi}{128}$, (j) $\frac{3\pi}{4}$, (k) 0, (l) $\frac{5\pi}{64}$,

F2 Evaluate the integral

$$\int_0^3 dx \int_{x/4}^{x/4+2} \left(\frac{x+y}{4} \right) dy,$$

using the change of variables $u = \frac{x}{4}$, $v = \frac{x+y}{2}$.

Solution

Upon rearranging we have $x = 4u$ and $y = 2v - 4u$. Since $0 \leq x \leq 3$, we then have $0 \leq u \leq 3/4$. Similarly, given $x/4 \leq y \leq x/4 + 2$ we obtain $\frac{5}{2}u \leq v \leq \frac{5}{2}u + 1$. The Jacobian is

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{8},$$

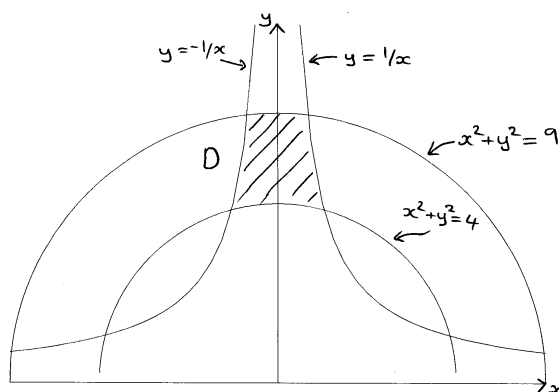
and so $\frac{\partial(x, y)}{\partial(u, v)} = 8$. Hence the integral is

$$I = \int_0^{\frac{3}{4}} du \int_{\frac{5}{2}u}^{\frac{5}{2}u+1} \frac{v}{2} \cdot |8| \, dv = 2 \int_0^{\frac{3}{4}} 1 + 5u \, du = \frac{69}{16}$$

F3 Evaluate

$$\iint_D x^4 - y^4 \, dx \, dy$$

where D is the region illustrated below.



Solution

Let $u = x^2 + y^2$ and $v = xy$. The Jacobian is

$$\frac{\partial(u, v)}{\partial(x, y)} = (2x) \cdot x - (2y) \cdot y = 2(x^2 - y^2),$$

and so $\frac{\partial(x, y)}{\partial(u, v)} = 1/(2(x^2 - y^2))$. Now, since in D $y > |x|$ and hence $x^2 - y^2 < 0$,

$$(x^4 - y^4) \left| \frac{1}{2(x^2 - y^2)} \right| = -\frac{x^2 + y^2}{2} = -\frac{u}{2},$$

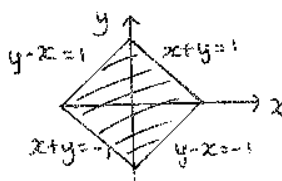
hence the integral is

$$-\iint_D \frac{1}{2} u \, du \, dv = -\int_4^9 \left(\int_{-1}^1 \frac{1}{2} u \, dv \right) du = -\int_4^9 u \, du = -\frac{65}{2}.$$

F4 By making a suitable change of variables, evaluate

$$\iint x^2 \, dx \, dy$$

over the square enclosed by the lines $x + y = -1$, $x + y = 1$, $y - x = -1$, $y - x = 1$.

Solution

Let $u = x + y$ and $v = y - x$, so $-1 \leq u \leq 1$ and $-1 \leq v \leq 1$. Solving for x and y in terms of u and v we get $x = (u - v)/2$. The Jacobian is

$$\frac{\partial(u, v)}{\partial(x, y)} = (1) \cdot (1) - 1 \cdot (-1) = 2,$$

and so $\frac{\partial(x, y)}{\partial(u, v)} = 1/2$. Hence the integral is

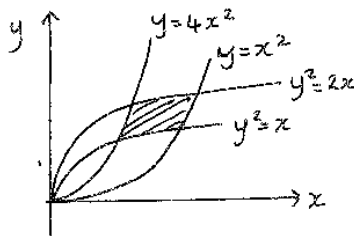
$$\begin{aligned} I &= \int_{-1}^1 du \int_{-1}^1 x^2 \cdot \left| \frac{1}{2} \right| dv = \frac{1}{2} \int_{-1}^1 du \int_{-1}^1 \left(\frac{1}{2} (u - v) \right)^2 dv \\ &= \frac{1}{8} \int_{-1}^1 du \int_{-1}^1 (u^2 - 2uv + v^2) dv = \frac{1}{8} \int_{-1}^1 \left[u^2 v - uv^2 + \frac{v^3}{3} \right]_{-1}^1 du \\ &= \frac{1}{8} \int_{-1}^1 \left(2u^2 - \frac{2}{3} \right) du = \frac{1}{8} \left[\frac{2u^3}{3} + \frac{2u}{3} \right]_{-1}^1 = \frac{1}{3}. \end{aligned}$$

F5 By making a suitable change of variables, evaluate

$$\iint \frac{y^2}{x} \, dx \, dy$$

over the region in the first quadrant enclosed by the four parabolas $y^2 = x$, $y^2 = 2x$, $y = x^2$, $y = 4x^2$

Solution



Let $u = y^2/x$ and $v = y/x^2$, so $1 \leq u \leq 2$ and $1 \leq v \leq 4$. The Jacobian is

$$\frac{\partial(u, v)}{\partial(x, y)} = (-y^2/x^2) \cdot (1/x^2) - (2y/x) \cdot (-2y/x^3) = 3y^2/x^4,$$

and so $\frac{\partial(x, y)}{\partial(u, v)} = x^4/(3y^2)$. Hence the integral is

$$I = \int_1^2 du \int_1^4 \frac{y^2}{x} \cdot \left| \frac{x^4}{3y^2} \right| dv = \frac{1}{3} \int_1^2 du \int_1^4 x^3 dv.$$

Now, $y = vx^2$, substituting this into the expression for u gives, $u = v^2x^4/x = v^2x^3$. Hence, $x^3 = u/v^2$. Thus,

$$I = \frac{1}{3} \int_1^2 du \int_1^4 \frac{u}{v^2} dv = \frac{1}{3} \left[\frac{u^2}{2} \right]_1^2 \left[-\frac{1}{v} \right]_1^4 = \frac{3}{8}.$$

F6 Evaluate $\iint_R (x^2 + y^2) dA$, where R is the parallelogram with vertices $(0, 0)$, $(2, 0)$, $(3, 1)$ and $(1, 1)$.

Solution

The parallelogram is bounded by the lines $y = 0$, $y = 1$, $y = x$ and $y = x - 2$. Letting $u = x - y$ and $v = y$ the domain can be described by $0 \leq u \leq 2$ and $0 \leq v \leq 1$.

Jacobian is

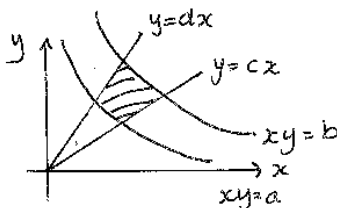
$$\frac{\partial(u, v)}{\partial(x, y)} = (1) \cdot (1) - 0 \cdot (-1) = 1,$$

and so $\frac{\partial(x, y)}{\partial(u, v)} = 1$. Hence the integral is

$$\iint_R dA = \int_0^1 dv \int_0^2 (u+v)^2 + v^2 |1| du = \int_0^1 \left[\frac{1}{3}(u+v)^3 + v^2 u \right]_0^2 dv = 6.$$

F7 Show that the area of the region in the first quadrant enclosed by the two hyperbolas $xy = a$, $xy = b$ and the two lines $y = cx$, $y = dx$, where $b > a > 0$ and $d > c > 0$ is

$$\frac{1}{2}(b-a) \log \left(\frac{d}{c} \right).$$

Solution

Let $u = xy$ and $v = y/x$, so $a \leq u \leq b$ and $c \leq v \leq d$. The Jacobian is

$$\frac{\partial(u, v)}{\partial(x, y)} = (y) \cdot (1/x) - x \cdot (-y/x^2) = 2y/x,$$

and so $\frac{\partial(x, y)}{\partial(u, v)} = x/(2y)$. Hence this area, A , is

$$\iint_A dA = \int_a^b du \int_c^d 1 \cdot \left| \frac{x}{2y} \right| dv = \frac{1}{2} [u]_a^b [\log v]_c^d = \frac{1}{2} (b - a) \log(d/c).$$

F8 Use change of variables to evaluate

$$\iint x^4 + y^4 \, dx dy$$

over the interior of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution

Let $x = au$ and $y = bv$, so the ellipse becomes $u^2 + v^2 \leq 1$ and so $0 \leq u \leq 1$ and $0 \leq v \leq 1$. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = (a) \cdot (b) - 0 \cdot (0) = ab,$$

Hence the integral is

$$\begin{aligned} I &= \int \int_{u^2+v^2 \leq 1} (a^4 u^4 + b^4 v^4) ab \, du dv \\ &= ab(a^4 + b^4) \int_0^{2\pi} d\theta \int_0^1 r^5 \cos^4 \theta \, dr, \quad (\text{By symmetry } \int \int_{\text{circle}} u^4 \, du dv = \int \int_{\text{circle}} v^4 \, du dv) \\ &= ab(a^4 + b^4) \int_0^{2\pi} \cos^4 \theta \, d\theta \left[\frac{r^6}{6} \right]_0^1 \\ &= ab(a^4 + b^4) 4 \frac{3.1}{4.2} \frac{\pi}{2} \frac{1}{6} = \frac{\pi ab}{8} (a^4 + b^4). \end{aligned}$$

F9 Evaluate

$$\int_3^5 dx \int_1^4 dy \int_1^2 xy \, dz.$$

Solution

$$I = \int_5^3 dx \int_1^4 [xyz]_1^2 dy = \int_5^3 \left[\frac{1}{2}xy^2\right]_1^4 dx = \left[\frac{15}{4}x^2\right]_3^5 = 60.$$