2A Multivariable Calculus 2013

Ex Sheet 1

Tutorial Exercises

T1 State the type of surface given by each of the following equations in *three dimensional space*.

(a) $4x + 5y - 2z = 20$, (b) $x^2 + y^2 = 1$, (c) $x^2 + y^2 + z^2 - 2x = 10$,

(d) $x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$, (e) $25x^2 + 4y^2 + z^2 = 100$, (f) $x^2 + y^2 + z^2 = 16$.

T2 Sketch the part of the plane $3x + 2y + z = 3$, that lies in the first octant $(= \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}.)$

T3 Match the graphs in Figure 1 with its corresponding contour-maps (cross-sections) from Figure 2. Give reasons for your choices.

Lecture 1

- **Key Points:**
  - key examples of surfaces
    - planes
    - ellipsoids
    - cylinders
  - cross sections and level curves
- **Read:**
  - Stewart Section 12.6 (p854)
  - Stewart Section 14.1 (p902)
- **Textbook Exercises:**
  - Exercises 14.1 (p913) Qs 23–70

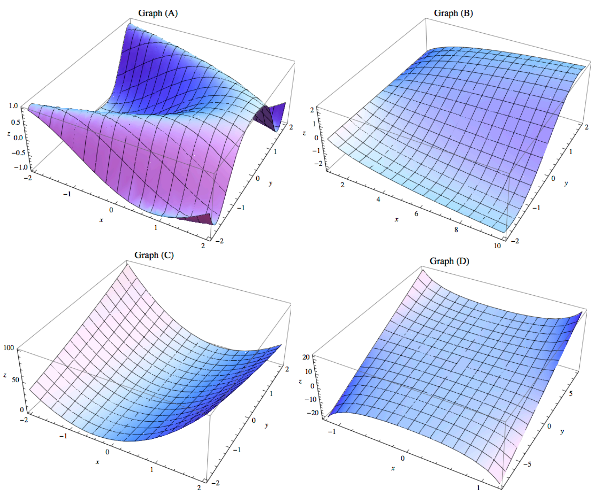

Figure 1: Cross sections of 4 graphs (See question T3).

T4 Find all partial derivatives of the functions

(a) $f(x, y) = x\cos(xy + x)$, (b) $g(s, t) = \dfrac{st}{s + t}$, (c) $r(u, v) = (uv + v)^3$,

(d) $h(x, y, z) = \dfrac{yz + zx + xy}{xyz}$, (e) $q(x, y, z) = xe^{-(x^2+y^2)}$.

Can you find a way to rewrite $h(x, y, z)$ in (d) so that calculating its partial derivatives is very easy?

Lecture 2

- **Key Points:**
  - definition of a partial derivative
  - calculation of partial derivatives
- **Read:**
  - Stewart 14.3 (pp924–930, up to Higher Derivatives)
- **Textbook Exercises:**
  - Exercises 14.3 (p936) Qs 15–44 and Qs 47–52

Further Exercises

F1 Complete the square in each of the following expressions

(a) $x^2 + y^2 + z^2 + 2 = 2(x + y + z)$, (b) $z = \sqrt{2x + 2y - x^2 - y^2 - 1}$.

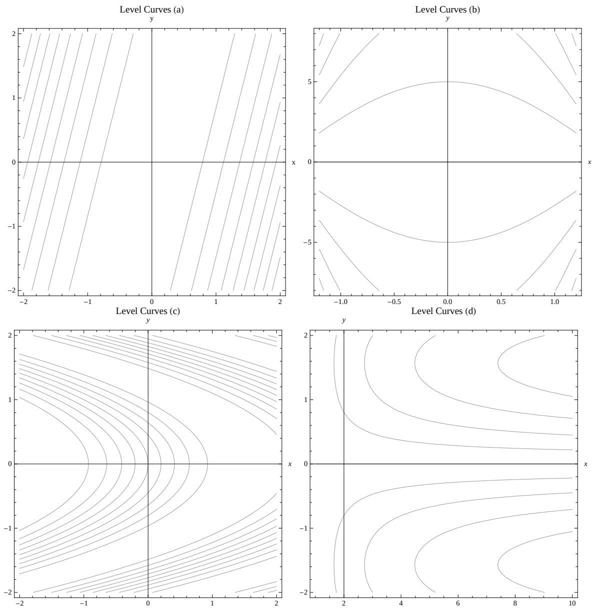

Figure 2: Cross sections of 4 graphs (See question T3).

and hence describe and sketch the surfaces they represent.

F2 By considering the level curves and cross sections $x = 0$ and $y = 0$, sketch the surfaces

$$\text{(a) } x^2 + y^2 - z^2 = 0, \quad \text{(b) } z = x^2 + y^2, \quad \text{(c) } 2x^2 + y^2 + z^2 = 1.$$

Which surface is the paraboloid and which is the ellipsoid?

F3 Sketch the region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $x - y + z = 1$ and $z = 2$.

F4 Find $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ where

$$\text{(a) } \phi(x, y) = g(x + y), \quad \text{(b) } \phi(x, y) = f(x)g(y),$$

where f and g are differentiable functions of one variable.

F5 Let $u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy$. Show that

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = 4(x^2 + y^2).$$

F6 Let $f(x, y, z) = \frac{xyz}{r^2}$, where $r^2 = x^2 + y^2 + z^2$. Prove that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = f.$$

F7 Let $f(x, y) = xy^2 \sin(\frac{x}{y})$. Prove that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f.$$