2A Multivariable Calculus 2013

Ex Sheet 1

Tutorial Exercises

T1 State the type of surface given by each of the following equations in *three dimensional space*.

(a) $4x + 5y - 2z = 20$, (b) $x^2 + y^2 = 1$, (c) $x^2 + y^2 + z^2 - 2x = 10$,

(d) $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$, (e) $25x^2 + 4y^2 + z^2 = 100$, (f) $x^2 + y^2 + z^2 = 16$.

Solution

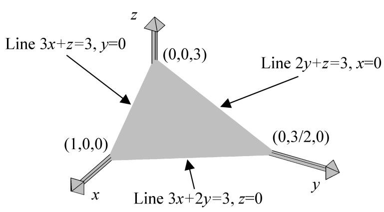
(a) Plane, (b) Cylinder, (c) Sphere centre $(1,0,0)$, radius $\sqrt{11}$, as seen by completing the square in the x terms, $(x^2 - 2x + 1) + y^2 + z^2 = 10 + 1$ giving $(x-1)^2 + y^2 + z^2 = 11$, (d) Ellipsoid, (e) Ellipsoid – Divide the equation by 100 to reduce to standard form:

$$\frac{x^2}{4} + \frac{y^2}{25} + \frac{z^2}{100} = 1$$

(f) Sphere.

T2 Sketch the part of the plane $3x + 2y + z = 3$, that lies in the first octant $(= \{(x,y,z) : x \geq 0, y \geq 0, z \geq 0\}.)$

Solution



T3 Match the graphs in Figure 1 with its corresponding contour-maps (cross-sections) from Figure 2. Give reasons for your choices.

Solution

A-(c), B-(d) , C-(a), D-(b).

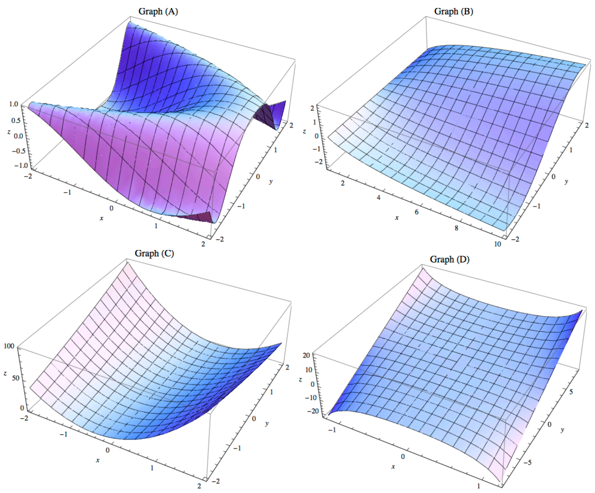


Figure 1: Cross sections of 4 graphs (See question T3).

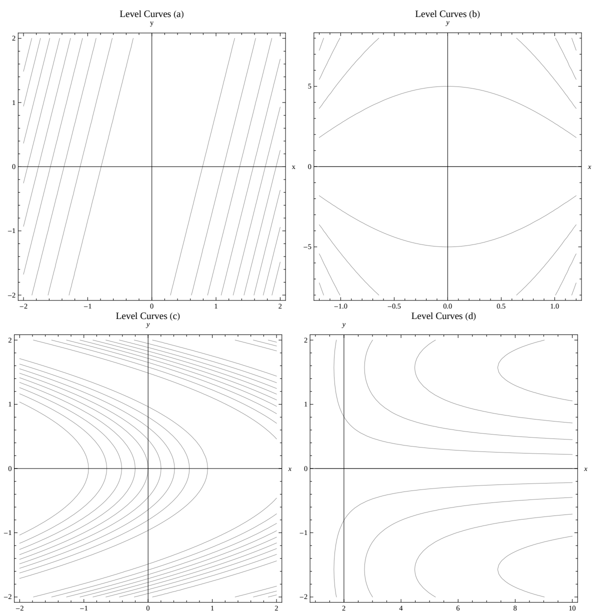


Figure 2: Cross sections of 4 graphs (See question T3).

T4 Find all partial derivatives of the functions

(a) $f(x,y) = x\cos(xy+x)$, (b) $g(s,t) = \dfrac{st}{s+t}$, (c) $r(u,v) = (uv+v)^3$,

(d) $h(x,y,z) = \dfrac{yz+zx+xy}{xyz}$, (e) $q(x,y,z) = xe^{-(x^2+y^2)}$.

Can you find a way to rewrite $h(x,y,z)$ in (d) so that calculating its partial derivatives is very easy?

Solution

(a) $f_x = \cos(xy+x) - x(y+1)\sin(xy+x)$, $f_y = -x^2\sin(xy+x)$,

(b) $g_s = \dfrac{t(s+t)-st.1}{(s+t)^2} = \dfrac{t^2}{(s+t)^2}$, $g_t = \dfrac{s^2}{(s+t)^2}$.

(c) $r_u = 3v(uv+v)^2$, $r_v = 3(u+1)(uv+v)^2$.

(d) $h = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$. Hence $h_x = -\dfrac{1}{x^2}$, $h_y = -\dfrac{1}{y^2}$ and $h_z = -\dfrac{1}{z^2}$.

(e) $q_x = (1-2x^2)e^{-(x^2+y^2)}$, $q_y = -2xye^{-(x^2+y^2)}$ and $q_z = 0$.

Further Exercises

F1 Complete the square in each of the following expressions

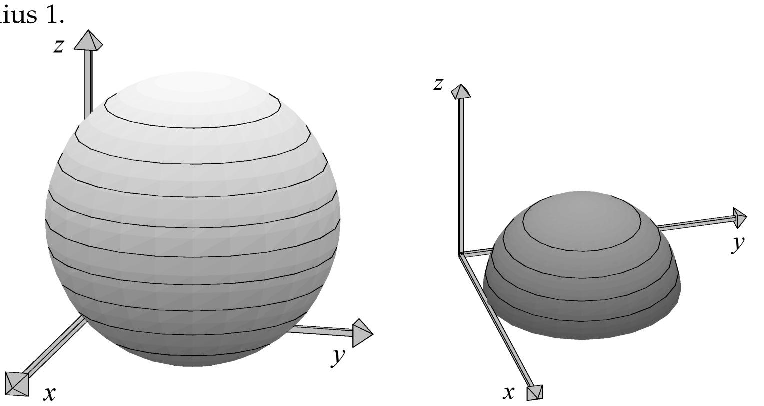
(a) $x^2+y^2+z^2+2 = 2(x+y+z)$, (b) $z = \sqrt{2x+2y-x^2-y^2-1}$.

and hence describe and sketch the surfaces they represent.

Solution

(a) We have $x^2-2x+y^2-2y+z^2-2z+2=0$ and so completing the square gives $(x-1)^2+(y-1)^2+(z-1)^2=1$, the sphere with centre $(1,1,1)$ and radius 1.

(b) We have $z \geq 0$ and $(x-1)^2+(y-1)^2+z^2=1$, the upper hemisphere with centre $(1,1,0)$ and radius 1.



Sphere, centre (1,1,1), radius 1 Hemisphere, centre (1,1,0), radius 1

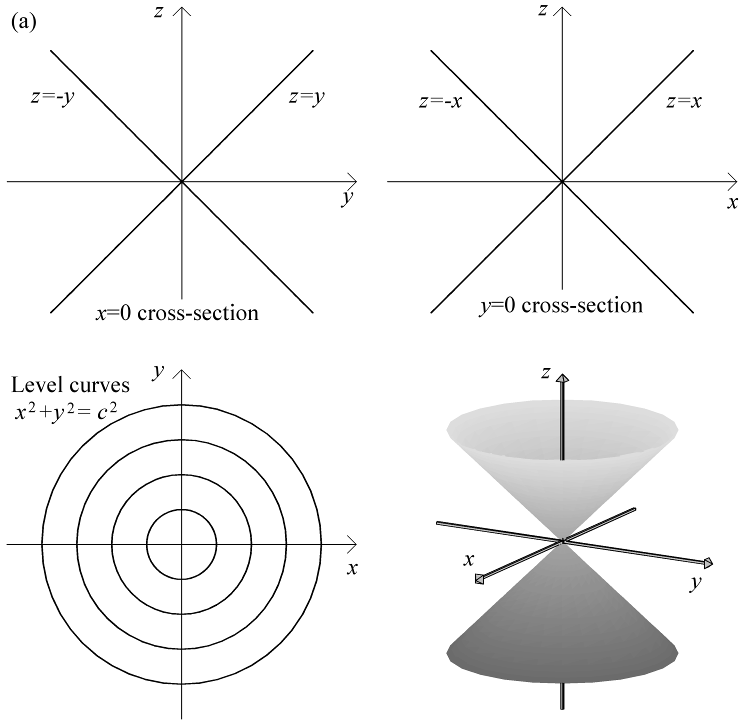
F2 By considering the level curves and cross sections $x=0$ and $y=0$, sketch the surfaces

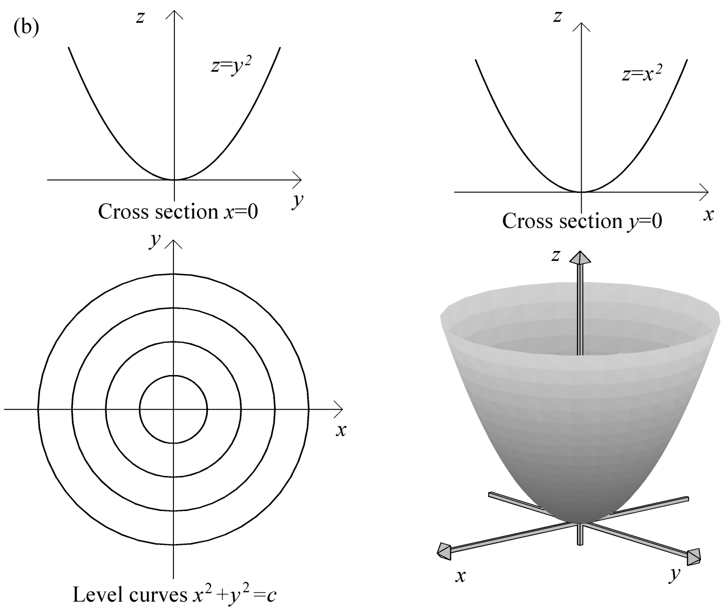
(a) $x^2+y^2-z^2=0$, (b) $z=x^2+y^2$, (c) $2x^2+y^2+z^2=1$.

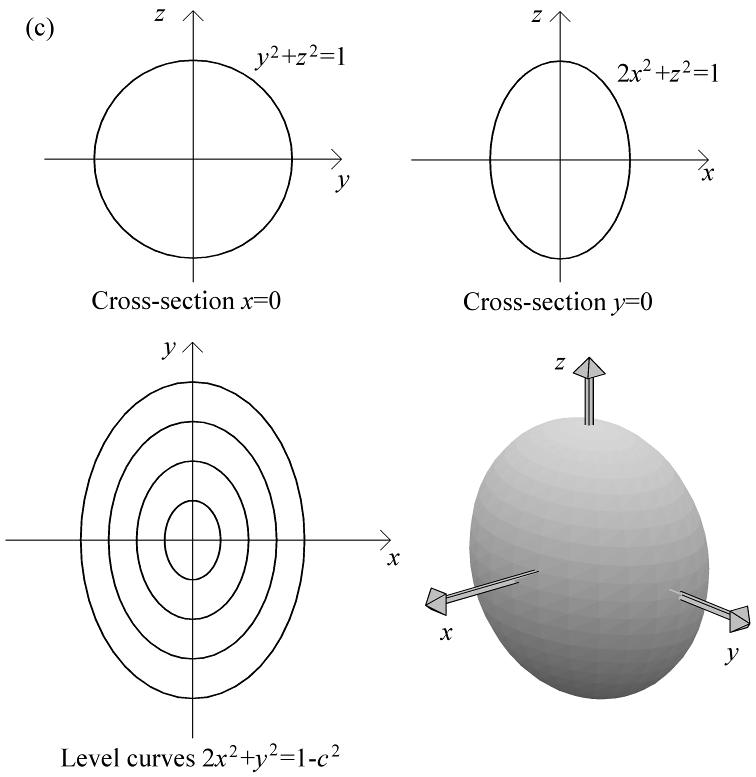
Which surface is the paraboloid and which is the ellipsoid?

Solution

(a) Cross section $x = 0$: $z = \pm y$; Cross section $y = 0$: $z = \pm x$; Level curves: $x^2 + y^2 = c^2$, so that $L_0 = \{(0,0)\}$ or L_c is the circle centre $(0,0)$, radius $|c|$ for $c \neq 0$.





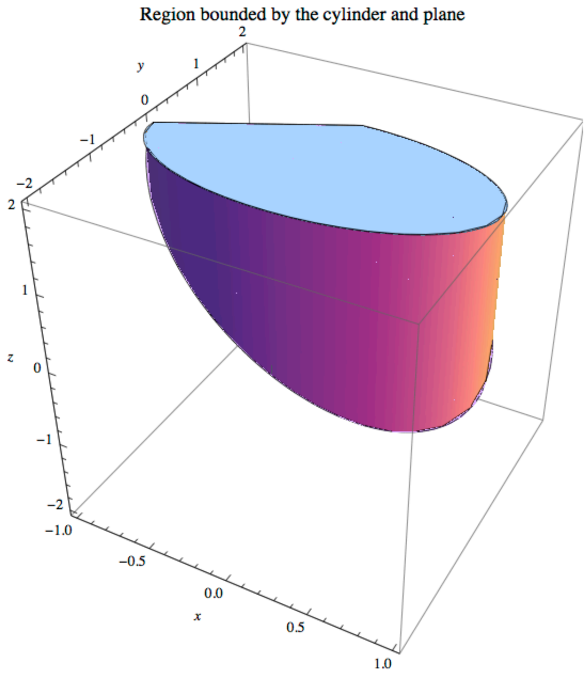


(b) Cross section $x = 0$: $z = y^2$ (parabola); Cross section $y = 0$: $z = x^2$ (parabola); Level curves: $x^2 + y^2 = c$, so that $L_c = \emptyset$ for $c < 0$ or $L_0 = \{(0,0)\}$ or L_c is the circle centre $(0,0)$, radius $\sqrt{c}$ for $c > 0$. This is paraboloid.

(c) Cross section $x = 0$: $y^2 + z^2 = 1$ (circle); Cross section $y = 0$: $2x^2 + z^2 = 1$ (ellipse); Level curves: $2x^2 + y^2 = 1 - c^2$, so that $L_{\pm 1} = \{(0,0)\}$, L_c is an ellipse for $|c| \leq 1$ or $L_c = \emptyset$ for $|c| > 1$. This is an ellipsoid.

F3 Sketch the region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $x - y + z = 1$ and $z = 2$.

Solution



F4 Find $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ where

$$\text{(a) } \phi(x,y) = g(x+y), \quad \text{(b) } \phi(x,y) = f(x)g(y),$$

where f and g are differentiable functions of one variable.

Solution

(a) $\frac{\partial \phi}{\partial x} = g'(x+y)$, $\frac{\partial \phi}{\partial y} = g'(x+y)$, (b) $\frac{\partial \phi}{\partial x} = f'(x)g(y)$, $\frac{\partial \phi}{\partial x} = f(x)g'(y)$

F5 Let $u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy$. Show that

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = 4(x^2+y^2).$$

Solution

$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = 2x \cdot 2x - 2y \cdot (-2y) = 4(x^2+y^2)$.

F6 Let $f(x,y,z) = \frac{xyz}{r^2}$, where $r^2 = x^2+y^2+z^2$. Prove that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = f.$$

Solution

Since $r^2 = x^2 + y^2 + z^2$, we have $2r \cdot \frac{\partial r}{\partial x} = 2x$, so $\frac{\partial r}{\partial x} = x/r$. Similarly, $\frac{\partial r}{\partial y} = y/r$ and $\frac{\partial r}{\partial z} = z/r$. Then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(xyz\frac{1}{r^2}\right) = yz\frac{1}{r^2} + xyz\left(\frac{-2}{r^3}\right)\frac{\partial r}{\partial x} = \frac{yz}{r^2} - \frac{2xyz}{r^3}\frac{x}{r} = \frac{yz}{r^2} - \frac{2x^2yz}{r^4}.$$

Similarly, by symmetry, we have

$$\frac{\partial f}{\partial y} = \frac{xz}{r^2} - \frac{2xy^2z}{r^4}, \quad \frac{\partial f}{\partial z} = \frac{xy}{r^2} - \frac{2xyz^2}{r^4}.$$

So,

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} &= \frac{xyz}{r^2} - \frac{2x^3yz}{r^4} + \frac{xyz}{r^2} - \frac{2xy^3z}{r^4} + \frac{xyz}{r^2} - \frac{2xyz^3}{r^4} \\ &= \frac{3xyz}{r^2} - \frac{2xyzr^2}{r^4} = \frac{3xyz}{r^2} - \frac{2xyz}{r^2} = \frac{xyz}{r^2} = f \end{aligned}$$

as required.

F7 Let $f(x,y) = xy^2\sin(\frac{x}{y})$. Prove that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f.$$

Solution

$$\frac{\partial f}{\partial x} = y^2\sin\left(\frac{x}{y}\right) + xy^2\cos\left(\frac{x}{y}\right)\cdot\frac{1}{y} = y^2\sin\left(\frac{x}{y}\right) + yx\cos\left(\frac{x}{y}\right).$$

$$\frac{\partial f}{\partial y} = 2xy\sin\left(\frac{x}{y}\right) - xy^2\cos\left(\frac{x}{y}\right)\frac{x}{y^2} = 2xy\sin\left(\frac{x}{y}\right) - x^2\cos\left(\frac{x}{y}\right).$$

So,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = xy^2\sin\left(\frac{x}{y}\right) + yx^2\cos\left(\frac{x}{y}\right) + 2xy^2\sin\left(\frac{x}{y}\right) - yx^2\cos\left(\frac{x}{y}\right) = 3xy^2\sin\left(\frac{x}{y}\right) = 3f.$$