Chapter 3

Differentiation of vectors: solutions

Example 3.1 Find the gradient of the scalar field $f(x, y, z) = x^2y + x\cosh yz$. (Recall from 1S/1Y that $\cosh x = \frac{e^x+e^{-x}}{2}$ is the hyperbolic cosine and the hyperbolic sine is given by $\sinh x = \frac{e^x-e^{-x}}{2}$.)

Solution : We have

$$\frac{\partial f}{\partial x} = 2xy + \cosh yz, \quad \frac{\partial f}{\partial y} = x^2 + xz\sinh yz, \quad \frac{\partial f}{\partial z} = xy\sinh yz.$$

Therefore,

$$\operatorname{grad} f = (2xy + \cosh yz, x^2 + xz\sinh yz, xy\sinh yz).$$

□

Example 3.2 Let $\mathbf{r} = (x, y, z)$ so that $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. Show that

$$\nabla(r^n) = nr^{n-2}\mathbf{r},$$

for any integer n and deduce the values of $\operatorname{grad}(r)$, $\operatorname{grad}(r^2)$ and $\operatorname{grad}(1/r)$.

Solution : We have

$$\begin{aligned}\frac{\partial}{\partial x}r^n &= \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{n/2} \\ &= 2x\,\frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1} \\ &= nxr^{n-2}.\end{aligned}$$

Then, using the symmetry of r with respect to x, y and z, we get

$$\frac{\partial}{\partial y}r^n = nyr^{n-2}, \quad \frac{\partial}{\partial z}r^n = nzr^{n-2},$$

and thus

$$\nabla(r^n) = \left(\frac{\partial}{\partial x}(r^n), \frac{\partial}{\partial y}(r^n), \frac{\partial}{\partial z}(r^n)\right) = (nxr^{n-2}, nyr^{n-2}, nzr^{n-2}) = nr^{n-2}\mathbf{r}.$$

Hence

$$\begin{aligned}\operatorname{grad}(r) &= \nabla(r) = 1r^{1-2}\mathbf{r} = \frac{\mathbf{r}}{r}, \\ \operatorname{grad}(r^2) &= 2r^{2-2}\mathbf{r} = 2\mathbf{r},\end{aligned}$$

and

$$\operatorname{grad}(1/r) = \nabla(r^{-1}) = (-1)r^{-1-2}\mathbf{r} = -\mathbf{r}/r^3.$$

□

Example 3.3 Determine $\text{grad}(\mathbf{c}\cdot\mathbf{r})$, when c is a constant (vector).

Solution : Let $\mathbf{c} = (c_1, c_2, c_3)$ so that

$$\begin{aligned}\text{grad}(\mathbf{c}\cdot\mathbf{r}) &= \text{grad}(c_1x + c_2y + c_2z)\\ &= \left(\frac{\partial(c_1x + c_2y + c_2z)}{\partial x}, \frac{\partial(c_1x + c_2y + c_2z)}{\partial y}, \frac{\partial(c_1x + c_2y + c_2z)}{\partial z}\right)\\ &= (c_1, c_2, c_3) = \mathbf{c}.\end{aligned}$$

□

Example 3.4 Find the directional derivative of $f = x^2yz^3$ at the point $P(3, -2, -1)$ in the direction of the vector $(1, 2, 2)$.

Solution : The *unit* vector with the same direction as $(1, 2, 2)$ is

$$\mathbf{u} = \frac{(1,2,2)}{\sqrt{1^2+2^2+2^2}} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right).$$

Hence the required directional derivative is

$$\begin{aligned}\mathbf{u}\cdot\nabla f &= \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\cdot(2xyz^3, x^2z^3, 3x^2yz^2)\\ &= \frac{1}{3}(2xyz^3 + 2x^2z^3 + 6x^2yz^2).\end{aligned}$$

At the point P, this gives

$$\frac{\partial f}{\partial \mathbf{u}}(3, -2, -1) = \frac{1}{3}(12 - 18 - 108) = -38.$$

□

Example 3.5 Consider $f = \ln(xy + z^3)$ at the point $P(1, 1, 1)$. In what direction does f have the maximal rate of change? What is this maximal rate of change?

Solution : The theorem above states that f increases the fastest in the direction of the gradient vector at the point $P(1, 1, 1)$.

$$\nabla f = \left(\frac{y}{xy+z^3}, \frac{x}{xy+z^3}, \frac{3z^2}{xy+z^3}\right),$$

hence $\nabla f(1, 1, 1) = (1/2, 1/2, 3/2)$ is the direction of the maximal rate of change. The maximal rate of change is

$$|\nabla f(1,1,1)| = \frac{\sqrt{11}}{2}.$$

□

Example 3.6 Show that the divergence of $\mathbf{F} = (x - y^2, z, z^3)$ is positive at all points in $\mathbb{R}^3$.

Solution : We have

$$\operatorname{div}\mathbf{F} = \frac{\partial(x-y^2)}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial(z^3)}{\partial z} = 1 + 0 + 3z^2 = 1 + 3z^2.$$

Hence for every (x, y, z), $\operatorname{div}\mathbf{F} \geq 1 > 0$. □

Example 3.7 Find the values of n for which $\nabla^2(r^n) = 0$.

Solution : We have $r = \sqrt{x^2+y^2+z^2}$ and so from Example 3.2,

$$\frac{\partial(r^n)}{\partial x} = nxr^{n-2}.$$

Therefore,

$$\begin{aligned}\frac{\partial^2(r^n)}{\partial x^2} &= nr^{n-2} + nx\,(n-2)xr^{n-4} \\ &= nr^{n-4}\big(r^2 + (n-2)x^2\big),\end{aligned}$$

and because of the symmetry in r with respect to x, y and z, we also have

$$\frac{\partial^2(r^n)}{\partial y^2} = nr^{n-4}\big(r^2 + (n-2)y^2\big), \quad \frac{\partial^2(r^n)}{\partial z^2} = nr^{n-4}\big(r^2 + (n-2)z^2\big).$$

Taking the sum of these we get

$$\begin{aligned}\nabla^2(r^n) &= nr^{n-4}\big(3r^2 + (n-2)(x^2+y^2+z^2)\big) \\ &= n(n+1)r^{n-2}.\end{aligned}$$

Hence $\nabla^2(r^n) = 0$ if and only if $n = 0$ or $n = -1$. □

Example 3.8 Determine $\operatorname{curl}\mathbf{F}$ when $\mathbf{F} = (x^2y, xy^2 + z, xy)$.

Solution : We have

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & xy^2+z & xy \end{vmatrix} \\ &= (x-1)\mathbf{i} + (0-y)\mathbf{j} + (y^2-x^2)\mathbf{k} \\ &= (x-1, -y, y^2-x^2).\end{aligned}$$

□

Example 3.9 If $\mathbf{c}$ is a constant vector, find $\operatorname{curl}(\mathbf{c}\times\mathbf{r})$.

Solution : We have $\mathbf{r} = (x, y, z)$ and let $\mathbf{c} = (c_1, c_2, c_3)$. First, we calculate

$$\mathbf{c} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & c_3 \\ x & y & z \end{vmatrix} = (c_2 z - c_3 y, c_3 x - c_1 z, c_1 y - c_2 x).$$

Then,

$$\begin{aligned} \operatorname{curl}(\mathbf{c} \times \mathbf{r}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ c_2 z - c_3 y & c_3 x - c_1 z & c_1 y - c_2 x \end{vmatrix} \\ &= \big(c_1 - (-c_1)\big)\mathbf{i} + \big(c_2 - (-c_2)\big)\mathbf{j} + \big(c_3 - (-c_3)\big)\mathbf{k} \\ &= 2\mathbf{c}. \end{aligned}$$

□

Example 3.10 Prove the identities

(i) $\operatorname{curl} \operatorname{grad} f = 0$, (ii) $\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl} \mathbf{F} + \operatorname{grad} f \times \mathbf{F}$, (iii) $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + (\operatorname{grad} f) \cdot \mathbf{F}$

Solution : We have (i)

$$\begin{aligned} \operatorname{curl} \operatorname{grad} f &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} \\ &= \big((f_z)_y - (f_y)_z\big)\mathbf{i} + \big((f_x)_z - (f_z)_x\big)\mathbf{j} + \big((f_y)_x - (f_x)_y\big)\mathbf{k} \\ &= (0, 0, 0) = \mathbf{0}, \end{aligned}$$

and (ii),

$$\begin{aligned} \operatorname{curl}(f\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix} \\ &= \big((fF_3)_y - (fF_2)_z\big)\mathbf{i} + \big((fF_1)_z - (fF_3)_x\big)\mathbf{j} + \big((fF_2)_x - (fF_1)_y\big)\mathbf{k} \\ &= f\left[\big((F_3)_y - (F_2)_z\big)\mathbf{i} + \big((F_1)_z - (F_3)_x\big)\mathbf{j} + \big((F_2)_x - (F_1)_y\big)\mathbf{k}\right] \\ &\qquad + (f_y F_3 - f_z F_2)\mathbf{i} + (f_z F_1 - f_x F_3)\mathbf{j} + (f_x F_2 - f_y F_1)\mathbf{k} \\ &= f \operatorname{curl} \mathbf{F} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= f \operatorname{curl} \mathbf{F} + \operatorname{grad} f \times \mathbf{F}, \end{aligned}$$

as required. (iii) Left as an exercise. □

Example 3.11 Let $\mathbf{r} = (x, y, z)$ denote a position vector with length $r = \sqrt{x^2 + y^2 + z^2}$ and $\mathbf{c}$ is a constant (vector). Determine

$$\text{(i) } \operatorname{div}(r^n(\mathbf{c} \times \mathbf{r})), \quad \text{(ii) } \operatorname{curl}(r^n(\mathbf{c} \times \mathbf{r})).$$

Solution :

$$\mathbf{c} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & c_3 \\ x & y & z \end{vmatrix} = (c_2 z - c_3 y, c_3 x - c_1 z, c_1 y - c_2 x).$$

(i) Using the identity $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \operatorname{grad}(f) \cdot \mathbf{F}$, and setting $f = r^n$ and $\mathbf{F} = \mathbf{c} \times \mathbf{r}$ gives

$$\begin{aligned} \operatorname{div}(r^n(\mathbf{c} \times \mathbf{r})) &= r^n \operatorname{div}(\mathbf{c} \times \mathbf{r}) + \operatorname{grad}(r^n) \cdot (\mathbf{c} \times \mathbf{r}) \\ &= 0 + nr^{n-2}\mathbf{r} \cdot (\mathbf{c} \times \mathbf{r}) \\ &= 0. \end{aligned}$$

This uses the result from Example 3.2, and the fact that $(\mathbf{c} \times \mathbf{r})$ is perpendicular to $\mathbf{r}$.

(ii), using the identity $\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl} \mathbf{F} + \operatorname{grad}(f) \times \mathbf{F}$ gives

$$\begin{aligned} \operatorname{curl}(r^n(\mathbf{c} \times \mathbf{r})) &= r^n \operatorname{curl}((\mathbf{c} \times \mathbf{r})) + \operatorname{grad}(r^n) \times (\mathbf{c} \times \mathbf{r}) \\ &= r^n 2\mathbf{c} + nr^{n-2}\mathbf{r} \times (\mathbf{c} \times \mathbf{r}) \quad \text{(by Ex 3.9 and 3.2)} \\ &= 2r^n\mathbf{c} + nr^{n-2}((\mathbf{r} \cdot \mathbf{r})\mathbf{c} - (\mathbf{r} \cdot \mathbf{c})\mathbf{r}) \quad \text{(by the vector tripple product)} \\ &= (2+n)r^n\mathbf{c} - nr^{n-2}(\mathbf{r} \cdot \mathbf{c})\mathbf{r} \end{aligned}$$

□