2A Multivariable Calculus 2013

Ex Sheet 9

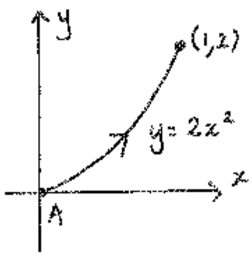
Tutorial Exercises

T1 Evaluate

$$\int_P xy^2\,dx + x^4y\,dy,$$

where P is the arc of the parabola $y = 2x^2$ from $A(0,0)$ to $B(1,2)$. (a) by parametrising the curve, (b) using x, y coordinates.

Solution



(a) *Parameterise P*

$$x = t, \quad y = 2t^2 \quad 0 \le t \le 1.$$

$$I = \int_0^1 t.4t^4\frac{dx}{dt} + t^4.2t^2\frac{dy}{dt}\,dt = \int_0^1 4t^5 + 2t^6.4t\,dt = \left[\frac{4t^6}{6} + t^8\right]_0^1 = \frac{5}{3}.$$

(b) *Integrate directly*

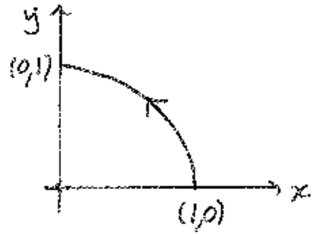
$$I = \int_0^1 x(2x^2)^2\,dx + \int_0^2 \left(\frac{y}{2}\right)^2 y\,dy = \left[\frac{4}{6}x^6\right]_0^1 + \left[\frac{1}{16}y^4\right]_0^2 = \frac{4}{6} + \frac{16}{16} = 5/3.$$

T2 The curve C consists of the part of the circle $x^2 + y^2 = 1$ in the first quadrant starting at $(1,0)$ and ending at $(0,1)$. Evaluate

$$\int_C 3xy^2\,dx + x^2y\,dy,$$

(a) by parametrising the curve, (b) using x, y coordinates.

Solution



(a) *Parameterise P*

$$x = \cos t, \quad y = \sin t \quad 0 \le t \le \pi/2.$$

$$I = \int_0^{\pi/2} 3\cos t \sin^2 t \frac{dx}{dt} + \cos^2 t \sin t \frac{dy}{dt}\, dt = \int_0^{\pi/2} -3\cos t \sin^3 t + \cos^3 t \sin t\, dt = -3.\frac{1.2}{4.2} + \frac{2.1}{4.2} = -1/2.$$

(b) *Integrate directly*

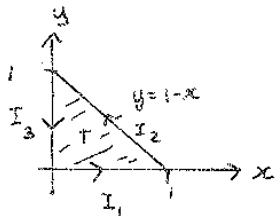
$$I = \int_1^0 3x(1-x^2)\, dx + \int_0^1 (1-y^2)y\, dy = \left[\frac{3x^2}{2} - \frac{3x^4}{4}\right]_1^0 + \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = -\frac{3}{4} + \frac{1}{4} = -1/2.$$

T3 T is the perimeter of the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$ taken in the anticlockwise direction. Evaluate

$$\int_T xy\, dx \; + \; 6(1+x)\, dy,$$

(a) by Green's Theorem, (b) directly (if you want to test yourself).

Solution



(a) *By Green's Theorem,*

$$I = \int_0^1 dx \int_0^{1-x} \frac{\partial(6+6x)}{\partial x} - \frac{\partial(xy)}{\partial y}\, dy = \int_0^1 dx \int_0^{1-x} (6-x)\, dy = \int_0^1 (6-x)\, [y]_0^{1-x}\, dx$$
$$= \int_0^1 6 - 7x + x^2\, dx = \left[6x - \frac{7x^2}{2} + \frac{x^3}{3}\right]_0^1 = \frac{17}{6}.$$

(b) *Directly,* We label the integrals according to the graph above. Notice $dy = 0$ on I_1 and $dx = 0$ for I_3.

$$I_1 = \int_0^1 x.0\, dx + 0 = 0$$

$$I_2 = \int_1^0 x(1-x)\,dx + \int_0^1 6(1+1-y)\,dy = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_1^0 + \left[12y - 3y^2\right]_0^1 = \frac{53}{6}.$$

$$I_3 = 0 + \int_1^0 6(1+0)\,dy = [6y]_1^0 = -6.$$

So the total integral is $I_1 + I_2 + I_3 = 17/6$.

T4 Verify that the vector function

$$\mathbf{F} = (2x + 3yz^2, 3xz^2, 6xyz)$$

is conservative and find a potential function for it, i.e. find a scalar function ϕ for which $\mathbf{F} = \operatorname{grad}\phi$. Evaluate

$$\int_C \mathbf{F}\cdot\mathbf{dr}$$

where C is the straight line segment joining $(1,2,5)$ to $(0,6,6)$.

Solution

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+3yz^2 & 3xz^2 & 6xyz \end{vmatrix} = (6xz - 6xz, -(6yz - 6yz), 3z^2 - 3z^2) = \mathbf{0}.$$

Therefore $\mathbf{F}$ is irrotational and also conservative. Let ϕ be the potential function for $\mathbf{F}$, so $\operatorname{grad}\phi = \mathbf{F}$. Then

$$(1)\ \frac{\partial\phi}{\partial x} = 2x + 3yz^2,\ (2)\ \frac{\partial\phi}{\partial y} = 3xz^2,\ (3)\ \frac{\partial\phi}{\partial z} = 6xyz.$$

Integrating (1) w.r.t x gives

$$\phi = x^2 + 3xyz^2 + A(y,z),$$

where A is an arbitrary function. Substituting this in (2) gives

$$3xz^2 + \frac{\partial A}{\partial y} = 3xz^2, \text{ i.e. } \frac{\partial A}{\partial y} = 0.$$

Thus $A(y,z) = B(z)$ and hence $\phi = x^2 + 3xyz^2 + B(z)$.

Substituting this into (3) gives

$$6xyz + B'(z) = 6xyz, \text{ i.e. } B'(z) = 0.$$

Thus $B(z) = C$, where C is a constant. Choosing this constant to be zero gives the potential function $\phi = x^2 + 3xyz^2$.

So,

$$I = \int_C \mathbf{F}\cdot\mathbf{dr} = \phi(0,6,6) - \phi(1,2,5) = 0 - (1 + 150) = -151.$$

T5 Show that the vector function

$$\mathbf{F} = (3x^2 + 2y^2, 4xy + z^2 - 2z, 2yz - 2y)$$

is conservative and find a potential function for it. Find the work done when **F** moves along any curve from the point $(1, 0, 9)$ and $(2, 2, 0)$.

Solution

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2+2y^2 & 4xy+z^2-2z & 2yz-2y \end{vmatrix} = (2z-2-(2z-2), -(0-0), 4y-4y) = \mathbf{0}.$$

Therefore **F** is irrotational and also conservative. Let ϕ be the potential function for **F**, so $\operatorname{grad}\phi = \mathbf{F}$. Then

$$(1)\ \frac{\partial\phi}{\partial x} = 3x^2+2y^2,\ (2)\ \frac{\partial\phi}{\partial y} = 4xy+z^2-2z,\ (3)\ \frac{\partial\phi}{\partial z} = 2yz-2y.$$

Integrating (1) w.r.t x gives

$$\phi = x^3 + 2xy^2 + A(y, z),$$

where A is an arbitrary function. Substituting this in (2) gives

$$4xy + \frac{\partial A}{\partial y} = 4xy + z^2 - 2z,\ \text{i.e.}\ \frac{\partial A}{\partial y} = z^2 - 2z.$$

Thus $A(y, z) = yz^2 - 2yz + B(z)$ and hence $\phi = x^3 + 2xy^2 + yz^2 - 2yz + B(z)$.

Substituting this into (3) gives

$$2yz - 2y + B'(z) = 2yz - 2y,\ \text{i.e.}\ B'(z) = 0.$$

Thus $B(z) = C$, where C is a constant. Choosing this constant to be zero gives the potential function $\phi = x^3 + 2xy^2 + yz^2 - 2yz$.

So,

$$I = \int_C \mathbf{F}\cdot d\mathbf{r} = \phi(2,2,0) - \phi(1,0,9) = (8+16+0-0) - (1+0+0-0) = 23.$$

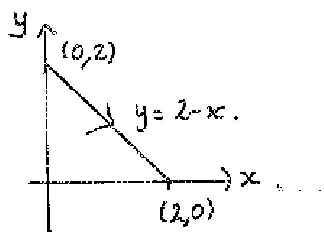
Further Exercises

F1 Evaluate

$$\int_L 4y\,dx + 3xy\,dy,$$

where L is the straight line segment from $A(0, 2)$ to $B(2, 0)$. (a) by parametrising the curve, (b) using x, y coordinates.

Solution



(a) *Parameterise P*

$$x = t, \quad y = 2 - t \quad 0 \le t \le 2.$$

$$I = \int_0^2 4(2-t)\frac{dx}{dt} + 3t(2-t)\frac{dy}{dt}\,dt = \int_0^2 8 - 4t - 6t + 3t^2\,dt = \int_0^2 8 - 10t + 3t^2\,dt = \left[8t - 5t^2 + t^3\right]_0^2 = 4.$$

(b) *Integrate directly*

$$I = \int_0^2 4(2-x)\,dx + \int_2^0 3(2-y)y\,dy = \left[8x - 2x^2\right]_0^2 + \left[3y^2 - y^3\right]_2^0 = 8 - 4 = 4.$$

F2 Use Green's Theorem to evaluate

$$\int_K 2xy^3\,dx \;+\; 3x^2\,dy,$$

where K is the perimeter of the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ in the anticlockwise direction.

Solution

By Green's Theorem,

$$I = \int\int_A \frac{\partial(3x^2)}{\partial x} - \frac{\partial(2xy^3)}{\partial y}\,dxdy = \int_0^1 dx \int_0^1 6x(1-y^2)\,dy = \int_0^1 6x\left[y - \frac{y^3}{3}\right]_0^1 dx = \int_0^1 4x\,dx = \left[2x^2\right]_0^1 = 2.$$

F3 Evaluate

$$\int_C y^3\,dx \;+\; 4xy^2\,dy,$$

where C is the circle $x^2 + y^2 = a^2$, where $a > 0$, in the anticlockwise direction (a) by Green's Theorem, (b) by parameterising the curve.

Solution

(a) *By Green's Theorem,*

$$I = \int\int_A 4y^2 - 3y^2\,dxdy = \int\int_A y^2\,dxdy = \int_0^{2\pi} d\theta \int_0^a r^3 \sin^2\theta\,dr = \int_0^{2\pi} \sin^2\theta\,d\theta \left[\frac{r^4}{4}\right]_0^a = 4.\frac{1}{2}.\frac{\pi}{2}.\frac{a^4}{4} = \frac{a^4\pi}{4}.$$

(b) *Directly*

$$x = a\cos t, \quad y = a\sin t \quad 0 \le t \le 2\pi\,.$$

$$I = \int_0^{2\pi} a^3\sin^3 t\frac{dx}{dt} + 4a^3\cos t\sin^2 t\frac{dy}{dt}\,dt = \int_0^{2\pi} -a^4\sin^4 t + 4a^4\cos^2 t\sin^2 t\,dt$$
$$= a^4(-4.\frac{3.1}{4.2}.\frac{\pi}{2} + 4.4.\frac{1.1}{4.2}.\frac{\pi}{2}) = \frac{\pi a^4}{4}.$$

F4 Use Green's Theorem to evaluate

$$\int_E (5x - 4y)\,dx \;+\; (x + 2y)\,dy,$$

where E is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ in the anticlockwise direction. (Recall the area of the standard ellipse is πab, this can be calculated by evaluating a double integral using the change of variables $u = x/a$ and $v = y/b$.) Also evaluate this integral by parameterising the curve keeping in mind that the standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has the parametric equations $x = a\cos t$, $y = b\sin t$ $(0 \le t \le 2\pi)$.

Solution

By Green's Theorem,

$$I = \int\int_A \frac{\partial(x+2y)}{\partial x} - \frac{\partial(5x-4y)}{\partial y}\,dxdy = \int\int_A 5\,dxdy = 5 \times \text{ Area of the ellipse} = 5\pi ab = 30\pi,$$

since A is an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with $a = 2$ and $b = 3$.

Alternatively, let

$$x = 2\cos t, \quad y = 3\sin t \quad 0 \le t \le 2\pi\,.$$

$$\begin{aligned} I &= \int_0^{2\pi} (10\cos t - 12\sin t)\frac{dx}{dt} + (2\cos t + 6\sin t)\frac{dy}{dt}\,dt \\ &= 24\int_0^{2\pi} \sin^2 t\,dt + 6\int_0^{2\pi} \cos^2 t\,dt - 2\int_0^{2\pi} \sin t\cos t\,dt = 23.4.\frac{1}{2}\frac{\pi}{2} + 6.4.\frac{1}{2}.\frac{\pi}{2} - 2.0 = 30\pi. \end{aligned}$$

By applying beta functions and noting that the last integral is zero because $\sin t\cos t$ makes a positive contribution to the integral in quadrants 1 and 3 and a negative contribution in quadrants 2 and 4, thus overall the integral of $\sin t\cos t$ is 0.

F5 Determine which of the following vector fields are conservative. For those which are conservative, find a potential.

a) $\mathbf{F} = (yz^2, xz^2, 2xyz)$,

b) $\mathbf{G} = (x^3y + z, yz, x + y + z^2)$,

c) $\mathbf{H} = \left(\frac{2xz}{1+x^2+y^2}, \frac{2yz}{1+x^2+y^2}, \log(1+x^2+y^2)\right)$,

d) $\mathbf{K}(x,y,z) = (2x+6y, 6x+6y+5z, 5y-8z-3)$

e) $\mathbf{G}(x,y,z) = (2x+yz^2+3z, 8y+xz^2, 2xyz+3x+6z)$.

Solution

(a) First compute the curl;

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2 & 2xyz \end{vmatrix} = (2xz - 2xz, 2yz - 2yz, z^2 - z^2) = \mathbf{0}.$$

Since $\mathbf{F}$ is defined everywhere and $\operatorname{curl}\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is conservative.

If $\operatorname{grad}\phi = \mathbf{F}$, then

$$(1)\ \frac{\partial \phi}{\partial x} = yz^2,\ (2)\ \frac{\partial \phi}{\partial y} = xz^2,\ (3)\ \frac{\partial \phi}{\partial z} = 2xyz$$

Integrating (1) with respect to x, we get

$$\phi = xyz^2 + A(y,z),$$

where A is an arbitrary function. Substituting this into (2) gives

$$xz^2 + \frac{\partial A}{\partial y} = xz^2, \quad \text{i.e. } \frac{\partial A}{\partial y} = 0.$$

Therefore $A(y,z) = B(z)$ and so $\phi = xyz^2 + B(z)$. Substituting this into (3) gives

$$2xyz + B'(z) = 2xyz, \quad (\text{i.e. } B'(z) = 0,$$

so that $B(z) = C$, a constant. Thus, $\phi(x,y,z) = xyz^2$ (choosing $C = 0$) gives a potential function.

(b) $\operatorname{curl}\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^3y+z & yz & x+y+z^2 \end{vmatrix} = (1-y)\mathbf{i} + \cdots \neq \mathbf{0}$. Therefore, $\mathbf{G}$ is not conservative.

(c) We have

$$\begin{aligned} \operatorname{curl}\mathbf{H} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{2xz}{1+x^2+y^2} & \dfrac{2yz}{1+x^2+y^2} & \log(1+x^2+y^2) \end{vmatrix} \\ &= \left(\frac{2y}{1+x^2+y^2} - \frac{2y}{1+x^2+y^2}\right)\mathbf{i} + \left(\frac{2x}{1+x^2+y^2} - \frac{2x}{1+x^2+y^2}\right)\mathbf{j} \\ &\quad + \left(\frac{4xyz}{(1+x^2+y^2)^2} - \frac{4xyz}{(1+x^2+y^2)^2}\right)\mathbf{k} = \mathbf{0}. \end{aligned}$$

Since $1+x^2+y^2 > 0$, $\mathbf{H}$ is defined everywhere and $\operatorname{curl}\mathbf{H} = \mathbf{0}$, $\mathbf{H}$ is conservative.

Let $\operatorname{grad}\phi = \mathbf{H}$. Then

$$(1)\ \frac{\partial \phi}{\partial x} = \frac{2xz}{1+x^2+y^2},\ (2)\ \frac{\partial \phi}{\partial y} = \frac{2yz}{1+x^2+y^2},\ (3)\ \frac{\partial \phi}{\partial z} = \log(1+x^2+y^2).$$

Integrating (1) w.r.t x gives

$$\phi = z\log(1+x^2+y^2) + A(y,z),$$

where A is an arbitrary function. Substituting this in (2) gives

$$\frac{2yz}{1+x^2+y^2} + \frac{\partial A}{\partial y} = \frac{2yz}{1+x^2+y^2}, \ \text{i.e. } \frac{\partial A}{\partial y} = 0.$$

Thus $A(y,z) = B(z)$ and hence $\phi = z\log(1+x^2+y^2) + B(z)$.

Substituting this into (3) gives

$$\log(1+x^2+y^2) + B'(z) = \log(1+x^2+y^2), \ \text{i.e. } B'(z) = 0.$$

Thus B is a constant. Choosing this constant to be zero gives the potential function $\phi = z\log(1+x^2+y^2)$.

(d)

$$\operatorname{curl}\mathbf{K} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+6y & 6x+6y+5z & 5y-8z-3 \end{vmatrix} = (5-5, 0, 6-6) = \mathbf{0}.$$

$\mathbf{K}$ is defined everywhere and $\operatorname{curl}\mathbf{K} = \mathbf{0}$, $\mathbf{K}$ is conservative. Let ϕ be the potential function for $\mathbf{K}$, so $\operatorname{grad}\phi = \mathbf{K}$. Then

$$(1)\ \frac{\partial\phi}{\partial x} = 2x+6y,\ (2)\ \frac{\partial\phi}{\partial y} = 6x+6y+5z,\ (3)\ \frac{\partial\phi}{\partial z} = 5y-8z-3.$$

Integrating (1) w.r.t x gives

$$\phi = x^2+6xy+A(y,z),$$

where A is an arbitrary function. Substituting this in (2) gives

$$6x+\frac{\partial A}{\partial y} = 6x+6y+5z,\ \text{i.e.}\ \frac{\partial A}{\partial y} = 6y+5z.$$

Thus $A(y,z) = 3y^2+5yz+B(z)$ and hence $\phi = x^2+6xy+3y^2+5yz+B(z)$.

Substituting this into (3) gives

$$5y+B'(z) = 5y-8z-3,\ \text{i.e.}\ B'(z) = -8z-3.$$

Thus $B(z) = -4z^2-3z+C$, where C is a constant. Choosing this constant to be zero gives the potential function $\phi = x^2+6xy+3y^2+5yz+-4z^2-3z$.

(e)

$$\operatorname{curl}\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x+yz^2+3z & 8y+xz^2 & 2xyz+3x+6z \end{vmatrix} = (2xz-2xz, -((2yz+3)-(2yz+3)), z^2-z^2) = \mathbf{0}.$$

$\mathbf{G}$ is defined everywhere and $\operatorname{curl}\mathbf{G} = \mathbf{0}$, $\mathbf{F}$ is conservative.

Let ϕ be the potential function for $\mathbf{G}$, so $\operatorname{grad}\phi = \mathbf{G}$. Then

$$(1)\ \frac{\partial\phi}{\partial x} = 2x+yz^2+3z,\ (2)\ \frac{\partial\phi}{\partial y} = 8y+xz^2,\ (3)\ \frac{\partial\phi}{\partial z} = 2xyz+3x+6z.$$

Integrating (1) w.r.t x gives

$$\phi = x^2+xyz^2+3xz+A(y,z),$$

where A is an arbitrary function. Substituting this in (2) gives

$$xz^2+\frac{\partial A}{\partial y} = 8y+xz^2,\ \text{i.e.}\ \frac{\partial A}{\partial y} = 8y.$$

Thus $A(y,z) = 4y^2+B(z)$ and hence $\phi = x^2+xyz^2+3xz+4y^2+B(z)$.

Substituting this into (3) gives

$$2xyz + 3x + B'(z) = 2xyz + 3x + 6z, \text{ i.e. } B'(z) = 6z.$$

Thus $B(z) = 3z^2 + C$, where C is a constant. Choosing this constant to be zero gives the potential function $\phi = x^2 + xyz^2 + 3xz + 4y^2 + 3z^2$.

Harder challenge problems[1]

[1] Only attempt these if you have been able to do all the other problems successfully.

F6 Evaluate

$$\int_C x^2y\,dx + (y + xy^2)\,dy,$$

where C is the boundary of the region enclosed between $y = x^2$ and $x = y^2$

Solution

By Green's Theorem,

$$I = \int\int_A \frac{\partial(y+xy^2)}{\partial x} - \frac{\partial(x^2y)}{\partial y}\,dxdy = \int\int_A y^2 - x^2\,dxdy = \int_{x=0}^{1} dx \int_{x}^{\sqrt{x}} y^2 - x^2\,dy = 0.$$