2A Multivariable Calculus 2013

Ex Sheet **2**

Tutorial Exercises

T1 Find all second order partial derivatives of

$$\text{(a) } z = x\log(1+y), \quad \text{(b) } z = \sin(xy), \quad \text{(c) } z = \left(\frac{x}{y}\right)^2.$$

Check in each case that $z_{xy} = z_{yx}$.

Solution

(a) $z_x = \log(1+y)$ and $z_y = x/(1+y)$. Therefore $z_{xx} = 0$, $z_{xy} = 1/(1+y)$, $z_{yx} = 1/(y+1)$ and $z_{yy} = -x/(1+y)^2$.

(b) $z_x = y\cos(xy)$ and $z_y = x\cos(xy)$. Therefore $z_{xx} = -y^2\sin(xy)$, $z_{xy} = \cos(xy) - xy\sin(xy)$, $z_{yx} = \cos(xy) - xy\sin(xy)$ and $z_{yy} = -x^2\sin(xy)$.

(c) $z_x = 2x/y^2$ and $z_y = -2x^2/y^3$. Therefore $z_{xx} = 2/y^2$, $z_{xy} = -4x/y^3$, $z_{yx} = -4x/y^3$ and $z_{yy} = 6x^2/y^4$.

T2 Let $\phi(x,y) = f(u)$, where $u = x^2y^3$ and f is a twice differentiable function of one variable. Show that

$$\frac{\partial\phi}{\partial x} = 2xy^3 f'(u) \quad \text{and} \quad \frac{\partial^2\phi}{\partial x^2} = 4x^2y^6 f''(u) + 2y^3 f'(u).$$

Find similar expressions for $\frac{\partial\phi}{\partial y}$ and $\frac{\partial^2\phi}{\partial y^2}$. Hence show that

$$9x^2\frac{\partial^2\phi}{\partial x^2} - 4y^2\frac{\partial^2\phi}{\partial y^2} + 3x\frac{\partial\phi}{\partial x} = 0.$$

Solution

By the chain rule $\frac{\partial\phi}{\partial x} = f'(u)\frac{\partial u}{\partial x}$. Therefore $\frac{\partial\phi}{\partial x} = 2xy^3 f'(u)$. Differentiating again with respect to x gives

$$\frac{\partial^2\phi}{\partial x^2} = 2y^3 f'(u) + 2xy^3 f''(u)2xy^3 = 2y^3 f'(u) + 4x^2y^6 f''(u).$$

Similarly, $\frac{\partial\phi}{\partial y} = f'(u)\frac{\partial u}{\partial y} = 3x^2y^2 f'(u)$. So,

$$\frac{\partial^2\phi}{\partial y^2} = 6x^2y f'(u) + 3x^2y^2 f''(u)3x^2y^2 = 6x^2y f'(u) + 9x^4y^4 f''(u).$$

So,

$$\begin{aligned} 9x^2\frac{\partial^2\phi}{\partial x^2} - 4y^2\frac{\partial^2\phi}{\partial y^2} + 3x\frac{\partial\phi}{\partial x} &= 9x^2(2y^3 f'(u) + 4x^2y^6 f''(u)) - 4y^2(6x^2y f'(u) + 9x^4y^4 f''(u)) + 6x^2y^3 f'(u) \\ &= 18x^2y^3 f'(u) + 36x^4y^6 f''(u) - 24x^2y^3 f'(u) - 36x^4y^6 f''(u) + 6x^2y^3 f'(u) = 0. \end{aligned}$$

T3 Let $z(x,y) = e^x g(y-4x)$, where g is an arbitrary twice differentiable function of one variable. Show that

$$\frac{\partial z}{\partial x} + 4\frac{\partial z}{\partial y} = z.$$

By taking suitable partial derivatives of this equation, show that

$$\frac{\partial^2 z}{\partial x^2} + 8\frac{\partial^2 z}{\partial x \partial y} + 16\frac{\partial^2 z}{\partial y^2} = z.$$

Solution

$z = e^x g(y-4x)$. The important thing to remember here is that g is a function, so $g(y-4x)$ is g applied to $y-4x$, thus to differentiate z with respect to x requires applying the product rule together with the chain rule (using $u = y-4x$.) So,

$$\frac{\partial z}{\partial x} = e^x g(y-4x) + e^x \frac{\partial}{\partial dx}\left(g(y-4x)\right) = e^x g(y-4x) - 4e^x g'(y-4x).$$

Similarly,

$$\frac{\partial z}{\partial y} = e^x g'(y-4x) \cdot 1 = e^x g'(y-4x).$$

Hence,

$$\frac{\partial z}{\partial x} + 4\frac{\partial z}{\partial y} = e^x g(y-4x) - 4e^x g'(y-4x) + 4e^x g'(y-4x) = e^x g(y-4x) = z.$$

So

$$\frac{\partial z}{\partial x} + 4\frac{\partial z}{\partial y} = z \tag{1}$$

Differentiating (1) with respect to x gives

$$\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial x} \tag{2}$$

Differentiating (1) with respect to y gives

$$\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y} \tag{3}$$

Now take ((2))+$(4 \times (3))$, which gives

$$\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial x \partial y} + 16\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial x} + 4\frac{\partial z}{\partial y} = z\,.$$

Rearranging and using (1)

$$\frac{\partial^2 z}{\partial x^2} + 8\frac{\partial^2 z}{\partial x \partial y} + 16\frac{\partial^2 z}{\partial y^2} = z.$$

T4 Let $z = z(x,y)$. Use the chain rule[1] for functions of two variables to find z_x and z_y when

(a) $z = \tan^{-1} r,$ where $r = \sqrt{x^2+y^2},$ (b) $z = \cos uv,$ where $u = xy,\ v = \dfrac{x}{y}.$

[1] In (b) you verify the answers you get by first expressing z directly in terms x and y.

Solution

(a) $z_x = r_x z_r = \dfrac{x}{\sqrt{x^2+y^2}}\dfrac{1}{1+r^2}$, $z_y = r_y z_r = \dfrac{y}{\sqrt{x^2+y^2}}\dfrac{1}{1+r^2}$.

(b) $z_x = u_x z_u + v_x z_v = y(-v\sin(uv)) + \dfrac{1}{y}(-u\sin(uv)) = -\left(yv + \dfrac{u}{y}\right)\sin(uv) = -2x\sin(x^2)$,
$z_y = x(-v\sin(uv)) - \dfrac{x}{y^2}(-u\sin(uv)) = -\left(xv - \dfrac{xu}{y^2}\right)\sin(uv) = 0.$

Since $z = \cos(x^2)$, $z_x = -2x\sin(x^2)$ and $z_y = 0$ in agreement with the above.

T5 Find the general solution of the PDE

$$\frac{\partial \phi}{\partial x} = y\cos(2x+y),$$

where ϕ is a function of two independent variables x and y.

Solution

$$\phi(x,y) = \frac{y}{2}\sin(2x+y) + A(y), \text{ where } A \text{ is an arbitrary function.}$$

Further Exercises

F1 Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, where
(a) $x\ln z + y = 3$, (b) $x^2y + y^2z = z^3$,

Solution

$$\text{(a) } \frac{\partial z}{\partial x} = -\frac{z}{x}\ln z, \quad \frac{\partial z}{\partial y} = -\frac{z}{x}, \text{ (b) } \frac{\partial z}{\partial x} = \frac{2xy}{3z^2-y^2}\ln z, \quad \frac{\partial z}{\partial y} = \frac{x^2+2yz}{3z^2-y^2}.$$

F2 Let $\phi(x,y) = f(r)$ where $r^2 = x^2+y^2$ and let f be a twice differentiable function of one variable. Show that

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = f''(r) + \frac{1}{r}f'(r).$$

Solution

Since $r^2 = x^2+y^2$, we have $2r \cdot \frac{\partial r}{\partial x} = 2x$, so $\frac{\partial r}{\partial x} = x/r$. Similarly, $\frac{\partial r}{\partial y} = y/r$. Thus

$$\frac{\partial \phi}{\partial x} = f'(r)\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}\,.$$

Using the product rule and chain rule we then obtain the second derivative,

$$\begin{aligned}\frac{\partial^2\phi}{\partial x^2} &= \frac{\partial}{\partial x}\left(f'(r)\frac{x}{r}\right) = f''(r)\frac{\partial r}{\partial x}\frac{x}{r} + f'(r)\frac{\partial}{\partial x}\left(x\frac{1}{r}\right) \\ &= \frac{x^2}{r^2}f''(r) + f'(r)\left(1\cdot\frac{1}{r} + x\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x}\right) = \frac{x^2}{r^2}f''(r) + \frac{1}{r}f'(r) - \frac{x^2}{r^3}f'(r)\,.\end{aligned}$$

Similarly,

$$\frac{\partial^2\phi}{\partial y^2} = \frac{y^2}{r^2}f''(r) + \frac{1}{r}f'(r) - \frac{y^2}{r^3}f'(r)\,.$$

Hence,

$$\begin{aligned}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= \left(\frac{x^2+y^2}{r^2}\right)f''(r) + \frac{2}{r}f'(r) - \frac{(x^2+y^2)}{r^3}f'(r) = \frac{r^2}{r^2}f''(r) + \frac{2}{r}f'(r) - \frac{r^2}{r^3}f'(r) \\ &= f''(r) + \frac{2}{r}f'(r) - \frac{1}{r}f'(r) = f''(r) + \frac{1}{r}f'(r)\,.\end{aligned}$$

F3 Find the value of n such that the function $2xy + x^n y^{2n}$ is a solution of the partial differential equation

$$2x^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2} + 18f = 36xy.$$

Solution

Let $f = 2xy + x^n y^{2n}$. Then,

$$\frac{\partial f}{\partial x} = 2y + nx^{n-1}y^{2n}, \quad \frac{\partial^2 f}{\partial x^2} = n(n-1)x^{n-2}y^{2n}$$

$$\frac{\partial f}{\partial y} = 2x + 2nx^n y^{2n-1}, \quad \frac{\partial^2 f}{\partial x^2} = 2n(2n-1)x^n y^{2n-2}.$$

Hence, f satisfies the PDE if and only if

$$n(n-1)2n^2x^{n-2}y^{2n} - 2n(n-1)x^n y^2 y^{2n-2} + 36xy + 18x^n y^{2n} = 36xy,$$

for all x and y. Collecting terms we obtain

$$(2n^2 - 2n - 4n^2 + 2n + 18)x^n y^{2n} = 0 \quad \text{and hence,} \quad -2n^2 + 18 = 0.$$

Factorising this quadratic and solving for n we have

$$(n-3)(n+3) = 0 \quad \text{so} \quad n = 3 \ \text{ or } \ n = -3.$$

F4 Let $f(x,y) = 3r^2 + 2\log r$, where $r^2 = x^2 + y^2$. Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 12.$$

By suitable partial differentiation of this equation, deduce that

$$\frac{\partial^4 f}{\partial x^4} + 2\frac{\partial^4 f}{\partial x^2\partial y^2} + \frac{\partial^4 f}{\partial y^4} = 0.$$

Solution

Since $r^2 = x^2 + y^2$, we have $2r \cdot \frac{\partial r}{\partial x} = 2x$, so $\frac{\partial r}{\partial x} = x/r$. Similarly, $\frac{\partial r}{\partial y} = y/r$. Then

$$\frac{\partial f}{\partial x} = 6r\frac{\partial r}{\partial x} + \frac{2}{r}\frac{\partial r}{\partial x} = 6r\frac{x}{r} + \frac{2}{r}\frac{x}{r} = 6x + \frac{2x}{r^2}\,.$$

Similarly, $\frac{\partial f}{\partial y} = 6y + \frac{2y}{r^2}$.

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= 6 + \frac{\partial}{\partial x}\left(2x\frac{1}{r}\right) = 6 + 2\frac{1}{r^2} + 2x\left(\frac{-2}{r^3}\right)\frac{\partial r}{\partial x}\\ &= 6 + \frac{2}{r^2} - \frac{4x}{r^3}\frac{x}{r} = 6 + \frac{2}{r^2} - \frac{4x^2}{r^4}\,.\end{aligned}$$

Similarly, (by symmetry) we can immediately say

$$\frac{\partial^2 f}{\partial x^2} = 6 + \frac{2}{r^2} - \frac{4y^2}{r^4}\,.$$

So,

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= 6 + \frac{2}{r^2} - \frac{4x^2}{r^4} + 6 + \frac{2}{r^2} - \frac{4y^2}{r^4}\\ &= 12 + \frac{4}{r^2} - \frac{4(x^2+y^2)}{r^4} = 12 + \frac{4}{r^2} - \frac{4}{r^2} = 12.\end{aligned}$$

Finally, we differentiate $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 12$ twice with respect to x to give

$$\frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial x^2 \partial y^2} = 0 \tag{4}$$

and twice with respect to y gives

$$\frac{\partial^4 f}{\partial y^4} + \frac{\partial^4 f}{\partial x^2 \partial y^2} = 0 \tag{5}$$

Adding equations (4) and (5) gives

$$\frac{\partial^4 f}{\partial x^4} + 2\frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4} = 0\,.$$

F5 Let $z(x, y) = (x + y)^2 + h(xy)$, where h is an arbitrary twice differentiable function of one variable. Show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2(x^2 - y^2).$$

Deduce that

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = 2(x^2 - y^2).$$

Solution

Let $z = (x+y)^2 + h(xy)$. Then,

$$\frac{\partial z}{\partial x} = 2(x+y)\cdot 1 + h'(xy)\cdot y = 2(x+y) + yh'(xy).$$

$$\frac{\partial z}{\partial y} = 2(x+y)\cdot 1 + h'(xy)\cdot x = 2(x+y) + xh'(xy).$$

So,

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x(x+y) + 2xyh'(xy) - 2y(x+y) - xyh'(xy) = 2x^2 + 2xy - 2xy + 2y^2 = 2(x^2 - y^2)\,.$$

So

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2(x^2 - y^2) \tag{6}$$

Differentiating (6) with respect to x gives

$$1\cdot\frac{\partial z}{\partial x} + x\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial x\partial y} = 4x \tag{7}$$

Differentiating (6) with respect to y gives

$$x\frac{\partial^2 z}{\partial x\partial y} - \frac{\partial z}{\partial y} - y\frac{\partial^2 z}{\partial y^2} = -4y \tag{8}$$

Now take $(x\times(7)) + (y\times(8))$, which gives

$$x\frac{\partial z}{\partial x} + x^2\frac{\partial^2 z}{\partial x^2} - xy\frac{\partial^2 z}{\partial x\partial y} + xy\frac{\partial^2 z}{\partial x\partial y} - y\frac{\partial z}{\partial y} - y^2\frac{\partial^2 z}{\partial y^2} = 4x^2 - 4y^2$$

Rearranging and using (6)

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = 4(x^2-y^2) - \left(x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right) = 4(x^2-y^2) - 2(x^2-y^2) = 2(x^2-y^2).$$

F6 Let $z = z(x,y)$. Use the chain rule for functions of two variables to find z_x and z_y when

(a) $z = f(u)$, where $u = \sin(x-y)$, (b) $z = \log(1+uv)$, where $u = x+y$, $v = x-y$,
(c) $z = \phi(u,v)$, where $u = e^x$, $v = e^y$, (d) $z = u^2+v^2$, where $u = a(x,y)$, $v = b(x,y)$.

Solution

(a) $z_x = u_x f'(u) = \cos(x-y)f'(u)$, $z_y = u_y f'(u) = -\cos(x-y)f'(u)$.

(b) $z_x = u_x \times v/(1+uv) + v_x \times u/(1+uv) = (v+u)/(1+uv)$, $z_y = u_y \times v/(1+uv) + v_y \times u/(1+uv) = (v-u)/(1+uv)$.

(c) $z_x = u_x\phi_u + v_x\phi_v = e^x\phi_u$, $z_y = u_y\phi_u + v_y\phi_v = e^y\phi_v$.

(d) $z_x = u_x \times 2u + v_x \times 2v = 2a_x u + 2b_x v$, $z_y = u_y \times 2u + v_y \times 2v = 2a_y u + 2b_y v$.

F7 Find the general solution of the PDE

$$\frac{\partial f}{\partial y} = xy\exp(y^2) + 4\log x,$$

where f is a function of two independent variables x and y.

Solution

$$f(x,y) = \frac{x}{2}\exp(y^2) + 4y\log x + A(x), \text{ where } A \text{ is an arbitrary function.}$$