Another Application of Line Integrals

Work done is one application of the line integral. Another application is circulation in fluid dynamics:

Definition

Let $\underline{v}(x,y) = (u(x,y), v(x,y), 0)$ be the velocity of a fluid at point (x,y). (This represents a two-dimensional, steady flow.) The circulation of $\underline{v}$ around a simple closed curve C is the line integral

$$\boxed{\int_C \underline{v} \cdot d\underline{r}}$$

Interpretation

- This has units velocity × length.
- It is the average velocity of the flow in the direction of the curve × the length of the curve.

Application

Circulation appears in classical aerofoil theory in the Kutta-Joukowski Theorem:

"The lift force on an aircraft wing is proportional to the circulation of air around it."

An application of Green's Theorem

Green's Theorem can be used to prove the following:

Theorem (A special case of the Kelvin Circulation Theorem.) The circulation of a two-dimensional, steady, irrotational flow around any simple closed curve C is zero.

Proof

$\underline{v} = (u, v, 0)$ is irrotational and so

$$\underline{0} = \operatorname{curl} \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & 0 \end{vmatrix} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \underline{k} .$$

Therefore $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$.

By Green's Theorem the circulation is

$$\int_C \underline{v} \cdot d\underline{r} = \iint_D \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)}_{=0} dx\,dy = 0. \qquad \square$$

Remark

In this theorem we are assuming that the region enclosed by the curve C is completely filled with fluid.