Chapter 1

Partial differentiation

Example 1.1 Sketch the graph of $f(x,y) = -\sqrt{1-2x-x^2-y^2}$.

Solution : Let $z = f(x,y)$. Completing the square, we have

$$z^2 = 1 - 2x - x^2 - y^2 = 2 - (x+1)^2 - y^2,$$

i.e. $(x+1)^2 + y^2 + z^2 = 2$. This is the sphere with centre $(-1,0,0)$ and radius $\sqrt{2}$. The part given by $z = -\sqrt{1-2x-x^2-y^2}$ (≤ 0) is the hemisphere *below* the x, y-plane. See Figure 1.1. $\square$

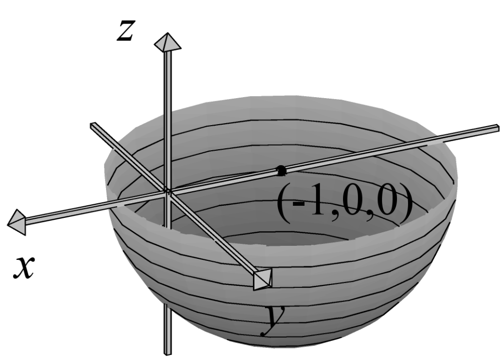

Figure 1.1: Graph of $f(x,y) = -\sqrt{1-2x-x^2-y^2}$

Example 1.2 By considering the level curves and the cross-sections $x = 0$ and $y = 0$, obtain a sketch of $z = \sqrt{x^2+y^2}$.

Solution : The level curves are defined by

$$L_c = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} = c\}.$$

For $c < 0$, $L_c = \emptyset$ (since $\sqrt{\cdots} \geq 0$), $L_0 = \{(0,0)\}$ (since $x^2+y^2 = 0 \implies x = y = 0$) and for $c > 0$, $L_c = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = c^2\}$, the circle of radius c, centre $(0,0)$.

Fixing $x = 0$ we get $z = \sqrt{y^2} = |y|$ and fixing $y = 0$ we get $z = |x|$. These cross-sections are illustrated in Figure 1.2.

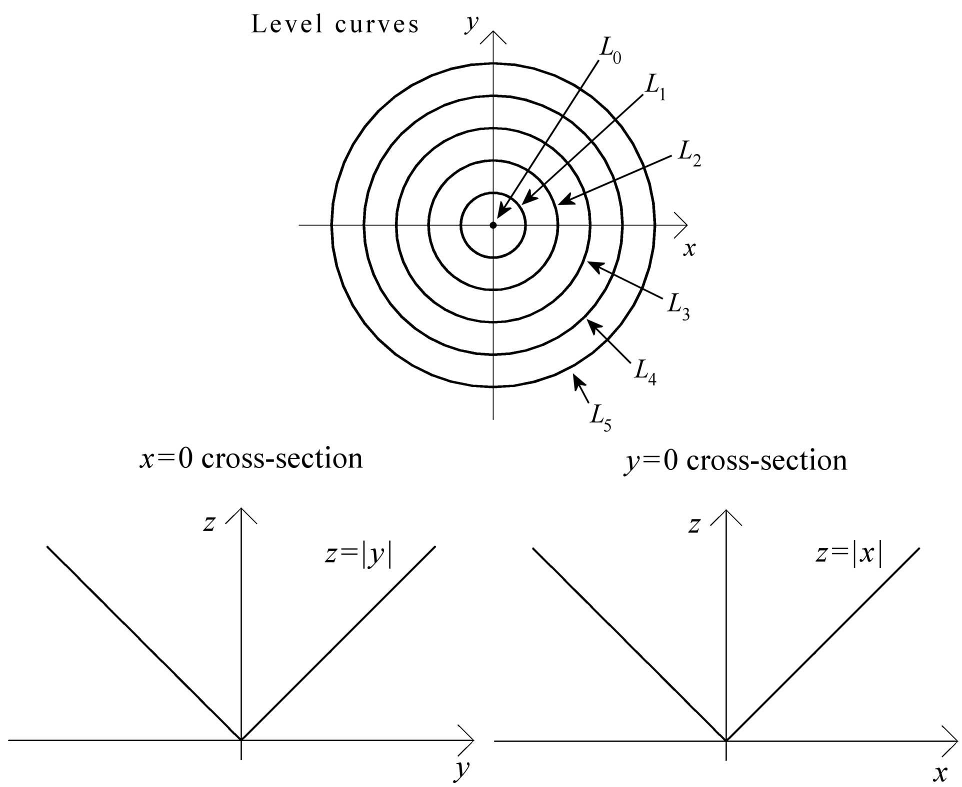

Figure 1.2: Cross sections

Putting this information together, we see that the surface defined by $z = \sqrt{x^2 + y^2}$ is a (circular) *cone* with vertex at $(0, 0)$ (Figure 1.3).

□

Example 1.3 Sketch the part of the surface

$$2x + y + 4z = 1,$$

where $x, y, z \geq 0$.

Solution : We consider the cross-section with the coordinate planes $x = 0$ (y, z-plane), $y = 0$ (x, z-plane) and $z = 0$ (x, y-plane).

The cross-section of $2x + y + 4z = 1$ with $x = 0$ is the line $y + 4z = 1$ (lying in the y, z-plane). This passed through the points $(0, 0, \frac{1}{4})$ and $(0, 1, 0)$. In a similar way we obtain the cross-section with the other coordinate planes; $2x + 4z = 1$ in the x, z-plane, passing through $(0, 0, \frac{1}{4})$ and $(\frac{1}{2}, 0, 0)$ and $2x + y = 1$ in the x, y-plane, passing through $(0, 1, 0)$ and $(\frac{1}{2}, 0, 0)$.

A sketch of the plane is shown in Figure 1.4. □

Example 1.4 Sketch the region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the plane $z = 2$.

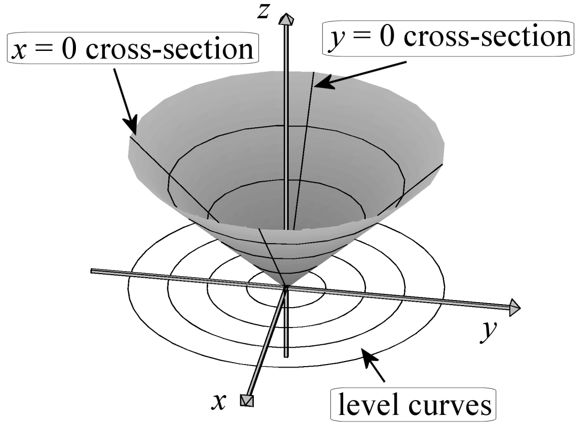

Figure 1.3: The cone $z = \sqrt{x^2 + y^2}$

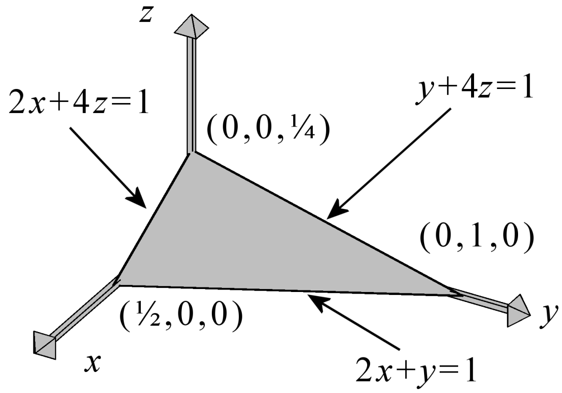

Figure 1.4: The plane $2x + y + 4z = 1$

Solution : The level curves of the paraboloid for $c > 4$ are $L_c = \emptyset$ (since $4 - x^2 - 2y^2 \leq 4$). For $c \leq 4$ are defined by the ellispse

$$L_c = \{(x, y) \in \mathbb{R}^2 : 4 - x^2 - 2y^2 = c\}.$$

In particular, the level curve where the plane insects the paraboloid is given by $L_2 = \{(x, y) \in \mathbb{R}^2 : 2 = x^2 + 2y^2\}$.

The cross section of the paraboloid when fixing $x = 0$ is the curve $z = 4 - 2y^2$ and fixing $y = 0$ gives the cross section $z = 4 - x^2$. These cross-sections are both parabolas and are illustrated in Figure 1.5.

Putting this information together, the region bounded by the parabolid and the plane is illustrated in Figure 1.6.

□

Example 1.5 Find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial z}{\partial x}$ where

$$\text{(a) } f(x, y) = x^3y^2 + x, \quad \text{(b) } z(x, y) = \sin^{-1}\left(\frac{x}{x + y}\right) \text{ and } x, y > 0.$$

[Note that $\sin^{-1} u$ is the inverse sine function (sometimes written as $\arcsin u$), and *not* the reciprocal $1/\sin u$. The domain of $\sin^{-1}$ is $[-1, 1]$ and, since $x, y > 0$, $x/(x + y)$ lies in this domain.]

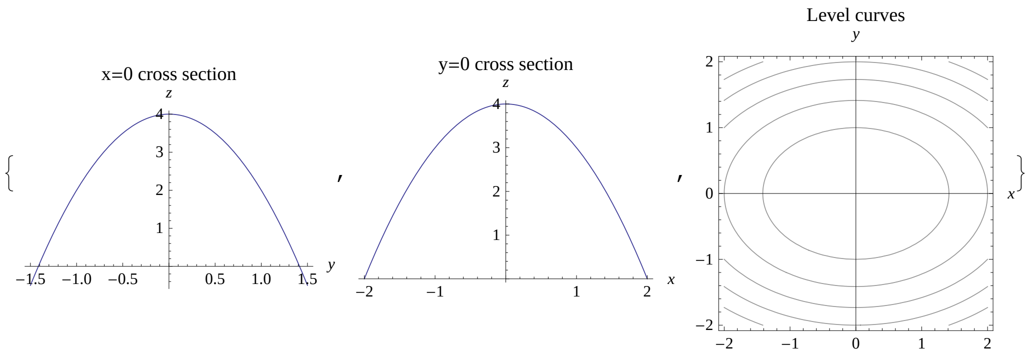

Figure 1.5: Cross sections and level curves of the paraboloid $z = 4 - x^2 - 2y^2$

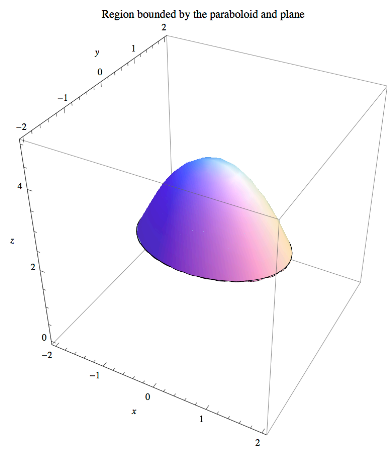

Figure 1.6: The region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the plane $z = 2$.

Solution : (a) To calculate the partial x derivative, we think of y as a constant and differentiate in the usual way with respect to x. Hence, we have

$$\frac{\partial f}{\partial x} = y^2 \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(x) = 3x^2y^2 + 1.$$

For the y derivative, we think of x as a constant and differentiate with respect to y;

$$\frac{\partial f}{\partial y} = x^3 \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(x) = 2x^3y.$$

Answer: $f_x = 3x^2y^2 + 1$ and $f_y = 2x^3y$.

(b) Let $u = x/(x+y)$. So, by the chain rule

$$\frac{\partial z}{\partial x} = \frac{d}{du}(\sin^{-1} u)\frac{\partial u}{\partial x}.$$

We have

$$\frac{d}{du}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} = \frac{1}{\sqrt{1-\left(\dfrac{x}{x+y}\right)^2}} = \frac{|x+y|}{\sqrt{(x+y)^2 - x^2}} = \frac{x+y}{\sqrt{2xy+y^2}},$$

since $x, y > 0$. Also, by the quotient rule,

$$\frac{\partial u}{\partial x} = \frac{\frac{\partial}{\partial x}(x).(x+y) - x.\frac{\partial}{\partial x}(x+y)}{(x+y)^2} = \frac{y}{(x+y)^2}.$$

Hence

$$\frac{\partial z}{\partial x} = \frac{y}{x+y}\frac{1}{\sqrt{2xy+y^2}}.$$

Answer: $z_x = \frac{y}{x+y}\frac{1}{\sqrt{2xy+y^2}}$. □

Example 1.6 Find $\dfrac{\partial z}{\partial x}$ where z is defined implicitly as a function of x and y by the equation

$$x^4 + 2y^2 + z^3 - 2x^2yz = 1.$$

Solution : Differentiating implicitly with respect to x gives

$$4x^3 + 0 + 3z^2\frac{\partial z}{\partial x} - 4xyz - 2x^2y\frac{\partial z}{\partial x} = 0.$$

Rearranging to solve for $\frac{\partial z}{\partial x}$ yields

$$\frac{\partial z}{\partial x} = \frac{4x^3 - 4xyz}{2x^2y - 3z^2}.$$

□

Example 1.7 For $r \in \mathbb{R}^+$, let $u = f(r)$ where $r^2 = x^2 + y^2 + z^2$. Show that

$$xu_x + yu_y + zu_z = rf'(r).$$

Solution : By the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} = f'(r)\frac{\partial r}{\partial x},$$

and similarly,

$$u_y = f'(r)\frac{\partial r}{\partial y}, \quad u_z = f'(r)\frac{\partial r}{\partial z}.$$

Now

$$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2 + y^2 + z^2) \qquad \text{i.e., } 2r\frac{\partial r}{\partial x} = 2x.$$

Therefore

$$\frac{\partial r}{\partial x} = \frac{x}{r},$$

and similarly,

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

Thus we have

$$xu_x + yu_y + zu_z = \frac{x^2}{r}f'(r) + \frac{y^2}{r}f'(r) + \frac{z^2}{r}f'(r) = \frac{(x^2 + y^2 + z^2)}{r}f'(r) = rf'(r),$$

as required. □

Example 1.8 Determine all second order derivatives of $u = \sin xy$ and verify that $u_{xy} = u_{yx}$.

Solution : We have first derivatives

$$u_x = y\cos xy, \quad u_y = x\cos xy.$$

Hence, the second derivatives are

$$\begin{aligned}
u_{xx} &= \frac{\partial}{\partial x}(u_x) = y\frac{\partial}{\partial x}\big(\cos xy\big) = -y^2\sin xy,\\
u_{xy} &= \frac{\partial}{\partial y}(u_x) = \frac{\partial}{\partial y}(y).\cos xy + y\frac{\partial}{\partial y}\big(\cos xy\big) = \cos xy - yx\sin xy,\\
u_{yx} &= \frac{\partial}{\partial x}(u_y) = \frac{\partial}{\partial x}(x).\cos xy + x\frac{\partial}{\partial x}\big(\cos xy\big) = \cos xy - xy\sin xy,\\
u_{yy} &= \frac{\partial}{\partial y}(u_y) = x\frac{\partial}{\partial y}\big(\cos xy\big) = -x^2\sin xy.
\end{aligned}$$

Hence $u_{xy} = u_{yx} = \cos xy - xy\sin xy$ as required. □

Example 1.9 Let $u = f(x/y)$, where f is an arbitrary (twice differentiable, with continuous second derivative) function of one variable. Show that

$$xu_x + yu_y = 0,$$

and *deduce* that

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0.$$

Solution : Using the chain rule, we have,

$$\begin{aligned}
u_x &= f'\left(\frac{x}{y}\right)\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}f'\left(\frac{x}{y}\right),\\
u_y &= f'\left(\frac{x}{y}\right)\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = -\frac{x}{y^2}f'\left(\frac{x}{y}\right).
\end{aligned}$$

So,

$$xu_x + yu_y = x\frac{1}{y}f'\left(\frac{x}{y}\right) - y\frac{x}{y^2}f'\left(\frac{x}{y}\right) = 0.$$

[Although we *could* proceed by calculating u_{xx}, u_{xy} and u_{yy} and taking the appropriate combination, it is much less work to deduce the final part as indicated below.]

Since $xu_x + yu_y = 0$, its x- and y-derivatives must also equal 0. Hence

$$xu_{xx} + u_x + yu_{yx} = 0, \tag{1}$$

and

$$xu_{xy} + yu_{yy} + u_y = 0. \tag{2}$$

Taking $x \times (1) + y \times (2)$ [the need to have the correct coefficient for u_{xx} and u_{yy} dictates the choice of this combination of (1) and (2)] we get

$$x^2u_{xx} + xu_x + xyu_{yx} + yxu_{xy} + y^2u_{yy} + yu_y = 0.$$

Since $u_{xy} = u_{yx}$ and $xu_x + yu_y = 0$, we get

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0.$$

as required.

□

Example 1.10 Let $w = u^2 + v^2$ where $u = \sin\theta$ and $v = \cos\phi$. Use the chain rule to calculate w_θ and w_ϕ in terms of θ and ϕ.

Solution : Using the chain rule, we have

$$w_\theta = \frac{\partial u}{\partial \theta}\frac{\partial w}{\partial u} + \frac{\partial v}{\partial \theta}\frac{\partial w}{\partial v} = \cos\theta.2u + 0.2v = 2\cos\theta\sin\theta = \sin 2\theta,$$

and

$$w_\phi = \frac{\partial u}{\partial \phi}\frac{\partial w}{\partial u} + \frac{\partial v}{\partial \phi}\frac{\partial w}{\partial v} = 0.2u + (-\sin\phi).2v = -2\sin\phi\cos\phi = -2\sin 2\phi.$$

□

Example 1.11 Find the general solution of the PDE,

$$\frac{\partial f}{\partial x} = x^2 + y + 9,$$

where f is a function of two independent variables x and y.

Solution : Integrating with respect to x and treating y as fixed gives

$$f = \int_{y \text{ fixed}} x^2 + y + 9dx = \frac{x^3}{3} + xy + 9x + A(y)\,,$$

where A is an arbitrary function depending on the fixed variable y. □

Example 1.12 Find the general solution of the PDE,

$$\frac{\partial^2 f}{\partial x \partial y} = 2x,$$

where f is a function of two independent variables x and y.

Solution The PDE can be expressed as

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2x,$$

Integrating with respect to x and treating y as fixed gives

$$\frac{\partial f}{\partial y} = \int_{y \text{ fixed}} 2xdx = x^2 + A(y)\,,$$

where A is an arbitrary function depending on the fixed variable y. Integrating with respect to y, holding x fixed then gives

$$f = \int_{x \text{ fixed}} x^2 + A(y)dy = x^2y + \int_{x \text{ fixed}} A(y)dy + B(x) = x^2y + C(y) + B(x).$$

B is an arbitrary function of the fixed variable x. Since A was an arbitrary function of y its intergral is also and arbitrary function of y so let's call this function $C(y) = \int_{x \text{ fixed}} A(y)dy$. □

Example 1.13 By changing variables from (x, y) to (u, v), where $u = xy$, $v = x/y$, solve the PDE

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2x^2\sin(xy).$$

Solution By the chain rule,

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v} = y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v},$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}.$$

Therefore,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right) + y\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right) = 2xy\frac{\partial z}{\partial u}.$$

Inverting the change of variables we have

$$x = \sqrt{uv}, \quad y = \sqrt{\frac{u}{v}},$$

and so, after the change of variable the PDE becomes,

$$2u\frac{\partial z}{\partial u} = 2uv\sin u,$$

i.e.,

$$\frac{\partial z}{\partial u} = v\sin u.$$

Then

$$z = \int_{v \text{ fixed}} v\sin u\, du = -v\cos u + A(v),$$

and in terms of x and y this is $z = -\frac{x}{y}\cos(xy) + A\left(\frac{x}{y}\right)$, where A is an arbitrary function. □

Example 1.14 By changing variables from (x, y) to (u, v), where $u = x^3/y$, $v = x$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of parital derivatives with respect to u and v. Hence, solve the PDE

$$x\frac{\partial f}{\partial x} + 3y\frac{\partial f}{\partial y} = \frac{6x^5}{y}.$$

Solution By the chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial f}{\partial v} = \frac{3x^2}{y}\frac{\partial f}{\partial u} + 1\frac{\partial f}{\partial v},$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial f}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial f}{\partial v} = \frac{-x^3}{y^2}\frac{\partial f}{\partial u} - 0\frac{\partial f}{\partial v}.$$

Therefore,

$$x\frac{\partial f}{\partial x} + 3y\frac{\partial f}{\partial y} = x\left(\frac{3x^2}{y}\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) + y\left(\frac{-x^3}{y}\frac{\partial f}{\partial u}\right) = x\frac{\partial f}{\partial v}.$$

Inverting the change of variables we have

$$x = v, \quad y = \frac{v^3}{u},$$

and so, after the change of variable the PDE becomes,

$$v\frac{\partial f}{\partial v} = 6uv^2,$$

i.e.,

$$\frac{\partial f}{\partial u} = 6uv.$$

Then

$$f = \int_{u \text{ fixed}} 6uv\, dv = 3v^2 u + A(u),$$

and in terms of x and y this is $f = 3\frac{x^5}{y} + A\left(\frac{x^3}{y}\right)$, where A is an arbitrary function. $\square$