# Chapter 4

# Line and surface integrals: Solutions

**Example 4.1** Find the work done by the force $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the curve which runs from $(1, 0)$ to $(0, 1)$ along the unit circle and then from $(0, 1)$ to $(0, 0)$ along the $y$-axis (see Figure 4.1).

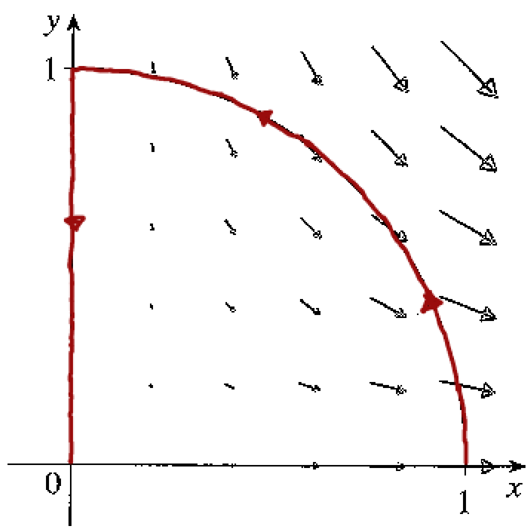


Figure 4.1: Shows the force field $\mathbf{F}$ and the curve $C$. The work done is negative because the field impedes the movement along the curve.

**Solution** : Split the curve $C$ into two sections, the curve $C_1$ and the line that runs along the $y$-axis $C_2$. Then,

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \, .$$

*Curve* $C_1$: Parameterise $C_1$ by $\mathbf{r}(t) = (x(t), y(t)) = (\cos t, \sin t)$, where $0 \leq t \leq \pi/2$ and $\mathbf{F} = (x^2, -xy)$ and $d\mathbf{r} = (dx, dy)$. Hence,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} x^2 \, dx - xy \, dy = \int_0^{\pi/2} \cos^2 t \, \frac{dx}{dt} \, dt - \int_0^{\pi/2} \cos t \sin t \, \frac{dy}{dt} \, dt = -\int_0^{\pi/2} 2\cos^2 t \sin t \, dt = -2/3,$$

by applying Beta functions to solve the integral where $m = 2$, $n = 1$ and $K = 1$.

*Curve* $C_2$: Parameterise $C_2$ by $\mathbf{r}(t) = (x(t), y(t)) = (0, t)$, where $0 \le t \le 1$. Hence,

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_1^0 0 \frac{dx}{dt} dt - \int_1^0 0t \, \frac{dy}{dt} \, dt = 0.$$

So the work done, $W = -2/3 + 0 = -2/3$. Notice the order of limits must reflect the direction along the curve. Work done is negative because the force field impedes the movement along the cure. □

**Example 4.2** Evaluate the line integral $\int_C (y^2)dx + (x)dy$, where $C$ is the is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$

**Solution** :
Parameterise $C$ by $\mathbf{r}(t) = (x(t), y(t)) = (4 - t^2, t)$, where $-3 \le t \le 2$, since $-3 \le y \le 2$. $C$ is illustrated in Figure 4.2. $\mathbf{F} = (y^2, x)$ and $d\mathbf{r} = (dx, dy)$. Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y^2 dx + x dy = \int_{-3}^2 t^2 \frac{dx}{dt} dt - \int_{-3}^2 (4 - t^2) \frac{dy}{dt} dt = \int_{-3}^2 -2t^3 + (4 - t^2) dt = 245/6.$$

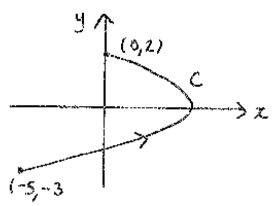


Figure 4.2: Curve $C$, where $C$ is the arc of the parabola $x = 4 - y^2$ from $(-5, 3)$ to $(0, 2)$.

□

**Example 4.3** Evaluate the line integral, $\int_C (x^2 + y^2)dx + (4x + y^2)dy$, where $C$ is the straight line segment from $(6, 3)$ to $(6, 0)$.

**Solution** : We can do this question without parameterising $C$ since $C$ does not change in the $x$-direction. So $dx = 0$ and $x = 6$ with $0 \le y \le 3$ on the curve. Hence

$$I = \int_C (x^2 + y^2)0 + (4x + y^2)dy = \int_3^0 24 + y^2 dy = -81.$$

□

**Example 4.4** Use Green's Theorem to evaluate $\int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4 + 1})dy$, where $C$ is the circle $x^2 + y^2 = 9$.

**Solution** : $P(x,y) = 3y - e^{\sin x}$ and $Q(x,y) = 7x + \sqrt{y^4+1}$. Hence, $\frac{\partial Q}{\partial x} = 7$ and $\frac{\partial P}{\partial y} = 3$. Applying Green's Theorem where $D$ is given by the interior of $C$, i.e. $D$ is the disc such that $x^2+y^2 \le 9$.

$$\int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4+1})dy = \iint_D (7-3)dxdy = \int_0^{2\pi}\int_0^3 4r dr d\theta = \int_0^{2\pi} 18 d\theta = 36\pi$$

The $D$ integral is solved by using polar coordinates to describe $D$. □

**Example 4.5** Evaluate $\int_C (3x-5y)dx + (x-6y)dy$, where $C$ is the ellipse $\frac{x^2}{4} + y^2 = 1$ in the anticlockwise direction. Evaluate the integral by (i) Green's Theorem, (ii) directly.

**Solution** : i) **Green's Theorem:** $P(x,y) = 3x-5y$ and $Q(x,y) = x-6y$. Hence, $\frac{\partial Q}{\partial x} = 1$ and $\frac{\partial P}{\partial y} = -5$. Applying Green's Theorem where $D$ is given by the interior of $C$, i.e. $D$ is the ellipse such that $x^2/4+y^2 \le 1$.

$$\int_C (3x-5y)dx + (x+6y)dy = \iint_D (1-(-5))dxdy = 6\iint_D 1 dxdy = 6 \times (\text{Area of the ellipse}) = 6 \times 2\pi.$$

See chapter 2 for calculating the area of an ellipse by change of variables for a double integral.

(ii) **Directly:** Parameterise $C$ by $x(t) = 2\cos t$, $y(t) = \sin t$, where $0 \le t \le 2\pi$.

$$\begin{aligned}
I &= \textstyle\int_0^{2\pi} (6\cos t - 5\sin t)\frac{dx}{dt}dt + (2\cos t - 6\sin t)\frac{dy}{dt}dt \\
&= \textstyle\int_0^{2\pi} 18\cos t \sin t + 10\sin^2 t + 2\cos^2 t\, dt \\
&= 0 + 40\textstyle\int_0^{\pi/2} \sin^2 t\, dt + 8\int_0^{\pi/2} \cos^2 t\, dt \\
&= 0 + 40\frac{\pi}{2}(1/2) + 8\frac{\pi}{2}(1/2) = 12\pi.
\end{aligned}$$

The integrals are calculated using symmetry properties of $\cos t$ and $\sin t$ and beta functions. Using the table of signs below we see that $\int_0^{2\pi} \sin^2 t = 4\int_0^{\pi/2} \sin t\, dt$ etc.

| Quadrant | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| $\cos t$ | + | − | − | + | |
| $\sin t$ | + | + | − | − | |
| $\cos t \sin t$ | + | − | + | − | 0 |
| $\sin^2 t$ | + | + | + | + | 4 |
| $\cos^2 t$ | + | + | + | + | 4 |

□

**Example 4.6** Vector fields $\mathbf{V}$ and $\mathbf{W}$ are defined by

$$\mathbf{V} = (2x-3y+z, -3x-y+4z, 4y+z)$$
$$\mathbf{W} = (2x-4y-5z, -4x+2y, -5x+6z)\ .$$

One of these is conservative while the other is not. Determine which is conservative and denote it by $\mathbf{F}$. Find a potential function $\phi$ for $\mathbf{F}$ and evaluate

$$\int_C \mathbf{F}\cdot d\mathbf{r}\ ,$$

where $C$ is the curve from A(1,0,0) to B(0,0,1) in which the plane $x+z=1$ cuts the hemisphere given by $x^2+y^2+z^2=1$, $y \ge 0$.

**Solution** : We have

$$\operatorname{curl}\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-3y+z & -3x-y+4z & 4y+z \end{vmatrix}$$
$$= \big(0,1,0\big) \neq \mathbf{0}.$$

Since $\operatorname{curl}\mathbf{V} \neq \mathbf{0}$, $\mathbf{V}$ is **NOT** conservative.

We have

$$\operatorname{curl}\mathbf{W} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-4y-5z & -4x+2y & -5x+6z \end{vmatrix}$$
$$= \big(0,0,0\big) = \mathbf{0}.$$

Since $\operatorname{curl}\mathbf{W} = \mathbf{0}$, $\mathbf{W}$ **is** conservative.

Suppose that $\operatorname{grad}\phi = \mathbf{W}$. Then

$$\frac{\partial \phi}{\partial x} = 2x - 4y - 5z, \tag{1}$$
$$\frac{\partial \phi}{\partial y} = -4x + 2y, \tag{2}$$
$$\frac{\partial \phi}{\partial z} = -5x + 6z. \tag{3}$$

Integrating (1) with respect to $x$, holding the other variables constant, we get

$$\phi = \int_{y,z \text{ fixed}} 2x - 4y - 5z \, dx = x^2 - 4yx - 5zx + A(y,z),$$

where $A$ is an arbitrary function. Substituting this expression into (2) gives,

$$-4x + \frac{\partial A}{\partial y} = -4x + 2y, \quad \text{i.e. } \frac{\partial A}{\partial y} = 2y,$$

and therefore

$$A(y,z) = \int_{z \text{ fixed}} (2y)\, dy = y^2 + B(z),$$

where $B$ is an arbitrary function, giving

$$\phi = x^2 - 4yx - 5zx + y^2 + B(z).$$

Finally, substituting this into (3) gives

$$-5x + \frac{dB}{dz} = -5x + 6z, \quad \text{i.e. } \frac{dB}{dz} = 6z,$$

so that $B = 3z^2 + C$, where $C$ is a constant. Hence, by taking $C = 0$ we obtain a potential

$$\phi = x^2 - 4yx - 5zx + y^2 + 3z^2.$$

Notice that the potential function is not unique; we may always add an arbitrary constant to a potential and it remains a potential.

So the line integral is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \operatorname{grad} \phi \cdot d\mathbf{r} = \phi(0,0,1) - \phi(1,0,0) = 3 - 1 = 2.$$

□

**Example 4.7** Evaluate

$$\iint_S z^2 \, dS$$

where $S$ is the hemisphere given by $x^2 + y^2 + z^2 = 1$ with $z \geq 0$.

**Solution** : We first find $\frac{\partial z}{\partial x}$ etc. These terms arise because $dS = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\, dxdy$. Since this

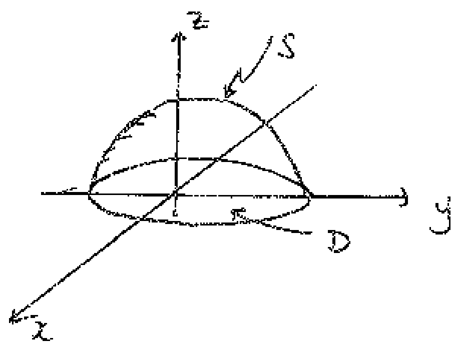


Figure 4.3: Shows the hemisphere $S$ and the projection $D$ onto the $xy$-plane.

change of variables relates to the surface $S$ we find these derivatives by differentiating both sides of the surface $x^2 + y^2 + z^2 = 1$ with respect to $x$, giving $2x + 2z\frac{\partial z}{\partial x} = 0$. Hence, $\frac{\partial z}{\partial x} = -x/z$. Similarly, $\frac{\partial z}{\partial y} = -y/z$. Hence,

$$\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} = 1/z.$$

Then the integral becomes the following, where $D$ is the projection of the surface, $S$, onto the $xy$-plane. i.e. $D = \{(x, y) : x^2 + y^2 \leq 1\}$. (See Figure 4.3)

$$\begin{aligned}
\iint_S z^2 dS &= \iint_D z^2 \frac{1}{z}\, dxdy \\
&= \iint_D \sqrt{1 - x^2 - y^2}\, dxdy \\
&= \int_0^{2\pi} d\theta \int_0^1 \sqrt{1 - r^2}\, r\, dr \\
&= -\int_0^{2\pi} d\theta \int_1^0 \frac{1}{2}\sqrt{u}\, du \\
&= \int_0^{2\pi} \frac{1}{3}\, d\theta \\
&= 2\pi/3.
\end{aligned}$$

$\square$

**Example 4.8** Find the area of the ellipse cut on the plane $2x + 3y + 6z = 60$ by the circular cylinder $x^2 + y^2 = 2x$.

**Solution** : The surface $S$ lies in the plane $2x+3y+6z = 60$ so we use this to calculate $dS = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\, dxdy$.

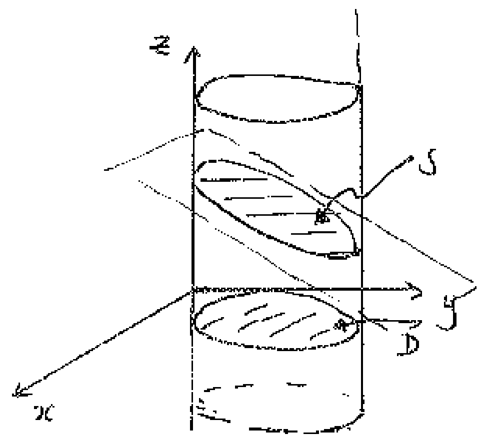


Figure 4.4: A sketch of the surface $S$ and the projection onto the $xy$-plane.

Differentiating the equation for the plane with respect to $x$ gives,

$$2 + 6\frac{\partial z}{\partial x} = 0 \quad \text{thus, } \frac{\partial z}{\partial x} = -1/3.$$

Differentiating the equation for the plane with respect to $y$ gives,

$$3 + 6\frac{\partial z}{\partial y} = 0 \quad \text{thus, } \frac{\partial z}{\partial y} = -1/2.$$

Hence,

$$\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} = \sqrt{1 + \frac{1}{9} + \frac{1}{4}} = 7/6.$$

Then the area of $S$ is found be calculating the surface integral over $S$ for the function $f(x, y, z) = 1$. The the projection of the surface, $S$, onto the $x-y$-plane is given by $D = \{(x, y) : x^2 - 2x + y^2 = (x-1)^2 + y^2 \leq 1\}$ (See Figure 4.4). Hence the area of $S$ is given by

$$\begin{aligned}\iint_S 1\, dS &= \iint_D 1\frac{7}{6}\, dxdy \\ &= \frac{7}{6}\iint_D 1\, dxdy \\ &= \frac{7}{6} \times \text{ Area of D} = \frac{7}{6}\pi.\end{aligned}$$

Note, since $D$ is a circle or radius 1 centred at $(1, 0)$ the area of $D$ is the area of a unit circle which is $\pi$. $\square$

**Example 4.9** Use Gauss' Divergence Theorem to evaluate

$$I = \iint_S x^4y + y^2z^2 + xz^2\, dS,$$

where $S$ is the entire surface of the sphere $x^2+y^2+z^2=1$.

**Solution** : In order to apply Gauss' Divergence Theorem we first need to determine $\mathbf{F}$ and the unit normal $\mathbf{n}$ to the surface $S$. The normal is $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 2y, 2z)$, where $f(z,y,z) = x^2+y^2+z^2-1 = 0$ describes the surface $S$. We require the unit normal, so $\mathbf{n} = (2x,2y,2z)/|(2x,2y,2z)| = (2x,2y,2z)/2 = (x,y,z)$. To find $\mathbf{F} = (F_1, F_2, F_3)$ we note that

$$\begin{aligned}\mathbf{F}\cdot\mathbf{n} &= x^4y + y^2z^2 + xz^2 \\ &= F_1x + F_2y + F_3z\end{aligned}$$

Hence, comparing terms we have $F_1 = x^3y$, $F_2 = yz^2$ and $F_3 = xz$. Applying the Divergence Theorem noting that $V$ is the volume enclosed by the sphere $S$ gives

$$\begin{aligned}I = \iint_S \mathbf{F}\cdot\mathbf{n}dS &= \iiint_V \operatorname{div}\mathbf{F}dxdydz \\ &= \iiint_V (3x^2y + z^2 + x)\, dxdydz \\ &= 0 + \iiint_V z^2\, dxdydz + 0 \\ &= \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^1 \rho^2\cos^2\phi\, \rho^2 \sin\phi\, d\rho \\ &= 2\pi \int_0^{\pi} \cos^2\phi \sin\phi\, d\phi \int_0^1 \rho^4 d\rho \\ &= 2\pi \times 2 \times \frac{1\cdot 1}{3\cdot 1} \times 1 = \frac{4\pi}{15}.\end{aligned}$$

**Remarks**

1. As $V$ is a sphere it is natural to use spherical polar coordinates to solve the integral. Thus, $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$ and $dxdydz = \rho^2\sin\phi\, d\rho d\theta d\phi$.
2. $\iiint_V 3x^2y\, dxdydz = 0$ and $\iiint_V x\, dxdydz = 0$ from the symmetry of the cosine and sine functions. We look at the signs in each quadrant as $\theta$ changes. Think about a fixed $\phi$. $\cos\theta$ and $\sin\theta$ terms in $x^2y$ and $x$ then have the following signs

| Quadrant | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| $\cos\theta$ | + | − | − | + | |
| $\sin\theta$ | + | + | − | − | |
| $x^2y$ | + | + | − | − | 0 |
| $x$ | + | + | − | − | 0 |

   The positive and negative contribution from the integral cancel out in these two cases so the integrals are zero.

□

**Example 4.10** Find $I = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS$ where $\mathbf{F} = (2x, 2y, 1)$ and where $S$ is the entire surface consisting of $S_2$=the part of the paraboloid $z = 1 - x^2 - y^2$ with $z = 0$ together with $S_1$=disc $\{(x,y) : x^2+y^2 \le 1\}$. Here $\mathbf{n}$ is the outward pointing unit normal.

**Solution** : Applying the Divergence Theorem noting that $V$ is the volume enclosed by $S_1$ and $S_2$ (see

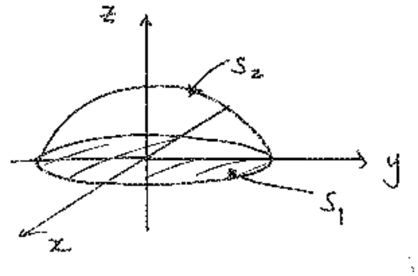


Figure 4.5: Illustration of surfaces $S_1$ and $S_2$.

Figure 4.5) and $\operatorname{div}\mathbf{F} = 2+2+0$ gives

$$\begin{aligned}
I = \iint_S \mathbf{F}\cdot\mathbf{n}dS &= \iiint_V \operatorname{div}\mathbf{F}dxdydz \\
&= \iiint_V 4dxdydz \\
&= 4\iint_{\{(x,y):x^2+y^2\le 1\}} dxdy \int_0^{1-x^2-y^2} 1dz \\
&= 4\iint_{\{(x,y):x^2+y^2\le 1\}} 1-x^2-y^2dxdy \\
&= 4\int_0^{2\pi} d\theta \int_0^1 (1-r^2)r\; dr \\
&= 4\times 2\pi(1/2-1/4) = 2\pi.
\end{aligned}$$

□