2A Multivariable Calculus 2013

Ex Sheet **9**

Tutorial Exercises

T1 Evaluate

$$\int_P xy^2\,dx + x^4y\,dy,$$

where P is the arc of the parabola $y = 2x^2$ from $A(0,0)$ to $B(1,2)$. (a) by parametrising the curve, (b) using x, y coordinates.

T2 The curve C consists of the part of the circle $x^2 + y^2 = 1$ in the first quadrant starting at $(1,0)$ and ending at $(0,1)$. Evaluate

$$\int_C 3xy^2\,dx + x^2y\,dy,$$

(a) by parametrising the curve, (b) using x, y coordinates.

T3 T is the perimeter of the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$ taken in the anticlockwise direction. Evaluate

$$\int_T xy\,dx + 6(1+x)\,dy,$$

(a) by Green's Theorem, (b) directly (if you want to test yourself).

T4 Verify that the vector function

$$\mathbf{F} = (2x + 3yz^2, 3xz^2, 6xyz)$$

is conservative and find a potential function for it, i.e. find a scalar function ϕ for which $\mathbf{F} = \text{grad}\,\phi$. Evaluate

$$\int_C \mathbf{F}\cdot\mathbf{dr}$$

where C is the straight line segment joining $(1,2,5)$ to $(0,6,6)$.

T5 Show that the vector function

$$\mathbf{F} = (3x^2 + 2y^2, 4xy + z^2 - 2z, 2yz - 2y)$$

is conservative and find a potential function for it. Find the work done when $\mathbf{F}$ moves along any curve from the point $(1,0,9)$ and $(2,2,0)$.

Further Exercises

F1 Evaluate

$$\int_L 4y\,dx + 3xy\,dy,$$

where L is the straight line segment from $A(0,2)$ to $B(2,0)$. (a) by parametrising the curve, (b) using x, y coordinates.

Lecture 17

- **Key Points:**
  - computing line integrals directly without using parameterisation
  - statement and proof of Green's Theorem in two dimensions
  - calculating line integrals over closed curves using Green's Theorem
- **Read:**
  - Stewart Section 16.4 (p1108)
- **Textbook Exercises:**
  - Exercises 16.4 (p1113) Qs 1–14

Lecture 18

- **Key Points:**
  - definition of path independence of line integrals
  - definition of a conservative vector field
  - find a potential function associated to a given conservative vector field
  - using the fundamental theorem of calculus to evaluate line integrals of conservative vector fields
- **Read:**
  - Stewart Section 16.3 (p1099)
- **Textbook Exercises:**
  - Exercises 16.3 (p1106) Qs 3–20

F2 Use Green's Theorem to evaluate

$$\int_K 2xy^3\,dx \;+\; 3x^2\,dy,$$

where K is the perimeter of the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ in the anticlockwise direction.

F3 Evaluate

$$\int_C y^3\,dx \;+\; 4xy^2\,dy,$$

where C is the circle $x^2+y^2=a^2$, where $a>0$, in the anticlockwise direction (a) by Green's Theorem, (b) by parameterising the curve.

F4 Use Green's Theorem to evaluate

$$\int_E (5x-4y)\,dx \;+\; (x+2y)\,dy,$$

where E is the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ in the anticlockwise direction. (Recall the area of the standard ellipse is πab, this can be calculated by evaluating a double integral using the change of variables $u=x/a$ and $v=y/b$.) Also evaluate this integral by parameterising the curve keeping in mind that the standard ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has the parametric equations $x=a\cos t$, $y=b\sin t$ $(0\leq t\leq 2\pi)$.

F5 Determine which of the following vector fields are conservative. For those which are conservative, find a potential.

a) $\mathbf{F}=(yz^2,xz^2,2xyz)$,

b) $\mathbf{G}=(x^3y+z,yz,x+y+z^2)$,

c) $\mathbf{H}=\left(\dfrac{2xz}{1+x^2+y^2},\dfrac{2yz}{1+x^2+y^2},\log(1+x^2+y^2)\right)$,

d) $\mathbf{K}(x,y,z)=(2x+6y,6x+6y+5z,5y-8z-3)$

e) $\mathbf{G}(x,y,z)=(2x+yz^2+3z,8y+xz^2,2xyz+3x+6z)$.

Harder challenge problems[1]

[1] Only attempt these if you have been able to do all the other problems successfully.

F6 Evaluate

$$\int_C x^2y\,dx+(y+xy^2)\,dy,$$

where C is the boundary of the region enclosed between $y=x^2$ and $x=y^2$