2A Multivariable Calculus 2013

Ex Sheet **2**

Tutorial Exercises

T1 Find all second order partial derivatives of

$$\text{(a) } z = x\log(1+y), \quad \text{(b) } z = \sin(xy), \quad \text{(c) } z = \left(\frac{x}{y}\right)^2.$$

Check in each case that $z_{xy} = z_{yx}$.

T2 Let $\phi(x,y) = f(u)$, where $u = x^2y^3$ and f is a twice differentiable function of one variable. Show that

$$\frac{\partial \phi}{\partial x} = 2xy^3 f'(u) \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x^2} = 4x^2y^6 f''(u) + 2y^3 f'(u).$$

Find similar expressions for $\frac{\partial \phi}{\partial y}$ and $\frac{\partial^2 \phi}{\partial y^2}$. Hence show that

$$9x^2\frac{\partial^2 \phi}{\partial x^2} - 4y^2\frac{\partial^2 \phi}{\partial y^2} + 3x\frac{\partial \phi}{\partial x} = 0.$$

T3 Let $z(x,y) = e^x g(y-4x)$, where g is an arbitrary twice differentiable function of one variable. Show that

$$\frac{\partial z}{\partial x} + 4\frac{\partial z}{\partial y} = z.$$

By taking suitable partial derivatives of this equation, show that

$$\frac{\partial^2 z}{\partial x^2} + 8\frac{\partial^2 z}{\partial x \partial y} + 16\frac{\partial^2 z}{\partial y^2} = z.$$

T4 Let $z = z(x,y)$. Use the chain rule[1] for functions of two variables to find z_x and z_y when

$$\text{(a) } z = \tan^{-1} r, \text{ where } r = \sqrt{x^2+y^2}, \quad \text{(b) } z = \cos uv, \text{ where } u = xy,\ v = \frac{x}{y}.$$

T5 Find the general solution of the PDE

$$\frac{\partial \phi}{\partial x} = y\cos(2x+y),$$

where ϕ is a function of two independent variables x and y.

Further Exercises

F1 Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, where
(a) $x\ln z + y = 3$, (b) $x^2y + y^2z = z^3$,

Lecture 3

- **Key Points:**
  - implicit partial differentiation
  - finding higher order partial derivatives
  - Clairaut's Theorem
- **Read:**
  - Stewart Section 14.3 (p930)
- **Textbook Exercises:**
  - Exercises 14.3 (p) Qs 53–72

[1] In (b) you verify the answers you get by first expressing z directly in terms x and y.

F2 Let $\phi(x,y) = f(r)$ where $r^2 = x^2 + y^2$ and let f be a twice differentiable function of one variable. Show that

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = f''(r) + \frac{1}{r}f'(r).$$

F3 Find the value of n such that the function $2xy + x^n y^{2n}$ is a solution of the partial differential equation

$$2x^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2} + 18f = 36xy.$$

F4 Let $f(x,y) = 3r^2 + 2\log r$, where $r^2 = x^2 + y^2$. Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 12.$$

By suitable partial differentiation of this equation, deduce that

$$\frac{\partial^4 f}{\partial x^4} + 2\frac{\partial^4 f}{\partial x^2\partial y^2} + \frac{\partial^4 f}{\partial y^4} = 0.$$

F5 Let $z(x,y) = (x+y)^2 + h(xy)$, where h is an arbitrary twice differentiable function of one variable. Show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2(x^2 - y^2).$$

Deduce that

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = 2(x^2 - y^2).$$

F6 Let $z = z(x,y)$. Use the chain rule for functions of two variables to find z_x and z_y when

(a) $z = f(u),$ where $u = \sin(x-y),$ (b) $z = \log(1+uv),$ where $u = x+y,\ v = x-y,$
(c) $z = \phi(u,v),$ where $u = e^x,\ v = e^y,$ (d) $z = u^2 + v^2,$ where $u = a(x,y),\ v = b(x,y).$

F7 Find the general solution of the PDE

$$\frac{\partial f}{\partial y} = xy\exp(y^2) + 4\log x,$$

where f is a function of two independent variables x and y.

Lecture 4

- **Key Points:**
  - chain rule for functions of several variables
  - definition of a PDE
  - finding the general solution to a PDE
- **Read:**
  - Stewart Section 14.5 (p948)
  - Stewart Section 14.3 (p932)
- **Textbook Exercises:**
  - Exercises 14.5 (p954) Qs 1–34