2A Multivariable Calculus 2013

Ex Sheet **8**

Tutorial Exercises

T1 Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$ and $\mathbf{a}$ be a constant vector. Prove the following results

(a) div $\mathbf{r} = 3$, (b) div $(\mathbf{a} \times \mathbf{r}) = 0$, (c) div $(r^n \mathbf{a}) = nr^{n-2}(\mathbf{a} \cdot \mathbf{r})$.

T2 Let $\mathbf{r} = (x, y, z)$ and $r = \sqrt{x^2 + y^2 + z^2}$.

a) Find $\mathrm{div}(r^2\mathbf{r})$.

b) Show that for any smooth function f,

$$\mathrm{grad}\, f(r) = \frac{f'(r)}{r}\mathbf{r}.$$

c) Show that $\mathrm{curl}\,\mathbf{r} = \mathbf{0}$. Deduce that $\mathrm{curl}\, f(r)\mathbf{r} = \mathbf{0}$ for any smooth function f.

d) Determine $\mathrm{grad}(\log r)$ and deduce that $\nabla^2(\log r) = 1/r^2$.

T3 Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$, and $\mathbf{a}$ be a constant vector, show that

$$\mathrm{grad}\left(\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right) = \frac{1}{r^3}\mathbf{a} - \frac{3(\mathbf{a}\cdot\mathbf{r})}{r^5}\mathbf{r}.$$

Using this result, show

$$\mathrm{grad}\left(\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right) + \mathrm{curl}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = \mathbf{0}.$$

T4 Find the parametric form of

a) the line segment joining $P(-1, 2, 1)$ and $Q(4, 2, 0)$,

b) the line passing through a point $(0, 2, 2)$ with direction vector $(-1, 0, 1)$,

c) the curve $x = e^y$ from $(1, 0)$ to $(e, 1)$,

d) the curve $y = \sqrt{x}$ from $(1, 1)$ to $(4, 2)$.

For the line in (a), write the answer in component form, simplifying as far as possible.

T5 Calculate the work done by a force $\mathbf{F}$ in moving a particle along the parametric curve $\mathbf{r}$ where,

a) $\mathbf{F}(x, y) = xy\mathbf{i} + x^3\mathbf{j}$, $\mathbf{r}(t) = t^{1/2}\mathbf{i} + t^{1/4}\mathbf{j}$, $\quad 1 \le t \le 16$.

b) $\mathbf{F}(x, y) = x^2\mathbf{i} + y^2\mathbf{j}$, $\mathbf{r}(t) = (1 + t^2)\mathbf{i} + (2 + \sin(\pi t))\mathbf{j}$, $\quad 0 \le t \le 1$.

Lecture 15

- **Key Points:**
  - statement and proof of nabla identities
  - using nabla identities to prove results
- **Read:**
  - Stewart Section 16.5 (p1115)
- **Textbook Exercises:**
  - Exercises 16.5 (p1122) Qs 23–32

Lecture 16

- **Key Points:**
  - parameteric equations of plane curves
  - work done by a force on a particle expressed as line integrals of a vector fields in $\mathbb{R}^2$
  - calculating a line integral by parameterising the line
- **Read:**
  - Stewart Section 13.1 (p864)
  - Stewart Section 16.2 (p1092)
- **Textbook Exercises:**
  - Exercises 13.1 (p870) Qs 17–20, 40–44
  - Exercises 16.2 (p) Qs 19–22, 39–42

Further Exercises

F1 Let f be a smooth scalar field and $\mathbf{F}$ a smooth vector field. Prove the identity

$$\operatorname{div}(f\mathbf{F}) = f \operatorname{div}\mathbf{F} + \operatorname{grad} f \cdot \mathbf{F}.$$

F2 Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$, and $\mathbf{a}$ be a constant vector. Prove that

$$\text{(a) } \operatorname{div}((\mathbf{a}\cdot\mathbf{r})\mathbf{a}) = \mathbf{a}^2, \quad \text{(b) } \operatorname{curl}((\mathbf{a}\cdot\mathbf{r})\mathbf{a}) = \mathbf{0},$$

$$\text{(c) } \operatorname{div}((\mathbf{a}\cdot\mathbf{r})\mathbf{r}) = 4(\mathbf{a}\cdot\mathbf{r}), \quad \text{(d) } \operatorname{curl}((\mathbf{a}\cdot\mathbf{r})\mathbf{r}) = \mathbf{a}\times\mathbf{r},$$

$$\text{(e) } \operatorname{div}((\mathbf{a}\cdot\mathbf{r})(\mathbf{a}\times\mathbf{r})) = 0, \quad \text{(f) } \operatorname{curl}((\mathbf{a}\cdot\mathbf{r})(\mathbf{a}\times\mathbf{r})) = 3(\mathbf{a}\cdot\mathbf{r})\mathbf{a} - \mathbf{a}^2\mathbf{r},$$

$$\text{(g) } \operatorname{curl}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^2}\right) = \frac{2(\mathbf{r}\cdot\mathbf{a})}{r^4}\mathbf{r}, \quad \text{(h) } \operatorname{curl}(\mathbf{r}\times(\mathbf{a}\times\mathbf{r})) = 3(\mathbf{r}\times\mathbf{a}).$$

F3 Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$, $\lambda \in \mathbb{R}$ and $\mathbf{F} = \lambda r^{-3}\mathbf{r}$. Verify that $\nabla \times \mathbf{F} = \mathbf{0}$ at all points in $\mathbb{R}^3$ except for the origin.

F4 Write down the parametric equations of

a) the circle $x^2 + y^2 = 4$,

b) the circle in the xy-plane with centre $(1, 0, 0)$ and radius 1,

c) the parabola $x = y^2$

d) the ellipse $\dfrac{x^2}{4} + 9y^2 = 1,\ z = 1.$

F5 Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for the given vector field $\mathbf{F}$ and parametric curve $\mathbf{r}$,

a) $\mathbf{F}(x, y) = xy\mathbf{i} + y^2\mathbf{j}$, $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $\quad 0 \le t \le \pi/3$.

b) $\mathbf{F}(x, y) = ln(y)\mathbf{i} - e^x\mathbf{j}$, $\mathbf{r}(t) = \ln(t)\mathbf{i} + t^3\mathbf{j}$, $\quad 0 \le t \le e$.

c) $\mathbf{F}(x, y) = -xy\mathbf{i} + \left(x^2 + 1\right)^{-1}\mathbf{j}$, $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, $\quad -4 \le t \le -1$.