2A Multivariable Calculus 2013

Ex Sheet **7**

Tutorial Exercises

T1 Find the directional derivative of xyz^2 at the point $(1, 5, 1)$ in the direction of the vector $(1, -1, 2)$.

T2 Let f be a scalar field, $\mathbf{u}$ a unit vector and let θ be the angle between $\mathbf{u}$ and ∇f evaluated at some point P.

a) Show that the directional derivative of f at P in the direction of vector $\mathbf{u}$ is $|\nabla f| \cos\theta$.

b) Deduce that the directional derivative of f at P in the direction of $\mathbf{u}$ is a maximum when $\mathbf{u}$ has the same direction as ∇f. When is this directional derivative a minimum?

c) In what directions from the point $P(1, 3, 2)$ is the directional derivative of $f = xyz - y^2 z$ a maximum and a minimum respectively? Find these directional derivatives.

d) The temperature at a point $P(x, y, z)$ in space is given by $T = x^2 + y^2 - z$. In what direction should an insect at $P(1, 1, 2)$ move so that it warms up as rapidly as possible?

T3 A scalar field f is called harmonic if the Laplacian of the scalar field is zero. Show the following scalar fields are harmonic.

$$\text{(a) } u(x, y, z) = e^{(x+y)} \cos(\sqrt{2}z), \quad \text{(b)} \quad v(x, y) = x^2 - y^2.$$

T4 Find the divergence and curl of the vector fields

$$\text{(a) } \mathbf{F} = (3xyz^2, 2xy^3, -x^2yz), \quad \text{(b) } \mathbf{G} = (e^{xz}, x^2 + y^2, yz),$$

at an arbitrary point and at $P(1, 1, 1)$.

T5 Which of the following vector fields are irrotational?

a) $\mathbf{F} = (yz, xz, xy)$,

b) $\mathbf{G} = \sin xy\, \mathbf{i} + \cos yz\, \mathbf{j} + \sin xz\, \mathbf{k}$,

c) $\mathbf{H} = y^2 z\, \mathbf{i} + 2xyz\, \mathbf{j} + xy^2\, \mathbf{k}$.

Further Exercises

F1 Give two examples from the natural world of (i) a scalar field, (ii) a vector field.

F2 Find the directional derivative of

Lecture 13

- **Key Points:**
  - definition of directional derivative,
  - maximising the directional derivative,
  - divergence of vector field
- **Read:**
  - Stewart Section 14.6 (p957)
  - Stewart Section 16.5 (p1118)
- **Textbook Exercises:**
  - Exercises 14.6 (p967) Qs 11–34
  - Exercises 16.5 (p1121) Qs 1–8

Lecture 14

- **Key Points:**
  - definition of a incompressible vector field,
  - Laplacian of scalar and vector field,
  - curl of a vector field,
  - definition of an irrotational vector field
- **Read:**
  - Stewart Section 16.5 (p1118)
- **Textbook Exercises:**
  - Exercises 16.5 (p1121) Qs 1–8, 12, 21–22

a) $f = e^{2x-y+z}$ at $P(1,1,-1)$ in the direction $\mathbf{d} = (-1,-3,-5)$;

b) $f = x^3 + 3xy - 3yz + z^3$ at $P(1,2,1)$ in the direction $\mathbf{d} = (1,4,3)$;

c) $f = \sin xy + \log yz$ at $P(\pi,1,2)$ in the direction $\mathbf{d} = (0,1,2)$.

F3 Find the directional derivative of $xy + 3yz$ at the point (0, 3, - 2) in the direction of each of the vectors

$$\text{(a) } (2,2,-1), \quad \text{(b) } (1,0,1), \quad \text{(c) } (4,-7,-4).$$

What are the maximum and minimum values of the directional derivative at $(0,3,-2)$ and in which directions do they occur?

F4 The temperature at the point (x,y,z) is given by

$$T(x,y,z) = (x+3y)z^2.$$

Find the direction in which you should move from the point $(2,2,1)$ in order to achieve (a) the most rapid increase in temperature, (b) the most rapid decrease in temperature.

F5 Calculate the divergence of the vector field **F** and state whether the vector field is incompressible.

a) $\mathbf{F} = (z\ln(x), yz/x, z^2/x)$,

b) $\mathbf{F} = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$,

c) $\mathbf{F} = x^2\sin(y)(\mathbf{i} - \mathbf{j} + \mathbf{k})$.

F6 Calculate the curl of the vector field **F** and state whether the vector field is irrotational.

a) $\mathbf{F} = xz\mathbf{i} - y^3\mathbf{j} + xyz\mathbf{k}$,

b) $\mathbf{F} = \cos^2(x)\mathbf{i} - \sin(y)\mathbf{j} + z^4\mathbf{k}$,

c) $\mathbf{F} = \ln(x+z)\mathbf{i} - e^{y^2}\mathbf{j} + xy\mathbf{k})$.

F7 Let $\mathbf{F} = (x^2y, yz, x+z)$. Find

$$\text{(i) } \operatorname{curl}\operatorname{curl}\mathbf{F}, \quad \text{(ii) } \operatorname{grad}\operatorname{div}\mathbf{F}.$$