

Mathematics 2A—Multivariate Calculus (2013/14)

D. Bourne

October 24, 2013

Teaching arrangements

- ▶ This section, $2A_1$, meets at 9 a.m. on Tuesday in room 513 Boyd Orr and 9am on Thursday in room 203 Mathematics Building

Teaching arrangements

- ▶ This section, 2A₁, meets at 9 a.m. on Tuesday in room 513 Boyd Orr and 9am on Thursday in room 203 Mathematics Building
- ▶ Other sections meets at 10am and 11am.

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- ▶ **This week:** Lectures on Tuesday and Thursday
- no tutorial this week

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- ▶ Other sections meets at 10am and 11am.
- ▶ **This week:** Lectures on Tuesday and Thursday
- no tutorial this week
- ▶ **Other weeks:** Lectures on Tuesday and Thursday and a tutorial on Monday
- students come to tutorials *every other week*. Go to MyCampus for information on which tutorial group you are in and which weeks you have a tutorial.

Teaching arrangements

- ▶ This section, 2A₃, always meets at 11 a.m. on Tuesday in Lecture Theatre 208 Sir Alexander Stone Building and 11am on Thursday in room 224 (Main Lecture Theatre) Graham Kerr Building.

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- ▶ Seek help with these (and other problems) from tutors
- ▶ Bring lecture notes to tutorials!
- ▶ Tutorials are an important resource and opportunity for getting feedback. Be proactive and ask tutors to look at your work and ask them questions.

Other arrangements

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Office hours: Monday 12-1, Tuesday 12-1, Wednesday 1-2 (or by arrangement)

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- ▶ **Recommended course book:** James Stewart, Multivariable Calculus International Edition, (Seventh Edition), Brooks Cole /Cengage .

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- ▶ Only the Chapter 1 lecture notes will be given out in class. You need to download the notes for Chapters 2,3 and 4 from Moodle yourself in advance.

Syllabus of 2A - *Multivariate Calculus*

General theme: differentiation and integration of functions of several variables and its applications.

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- ▶ introduce partial derivatives,
- ▶ chain rule for partial derivatives.

Functions of one variable

For example, volume V of a sphere is a function of one variable, its radius r ,

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We write $V = f(r)$, where the *rule* is $f(r) = \frac{4}{3}\pi r^3$.

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- ▶ the maximal domain of f is \mathbb{R} .

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The set of all ordered pairs $(a, f(a))$ where $a \in D$. Usually shown as a curve in the plane.

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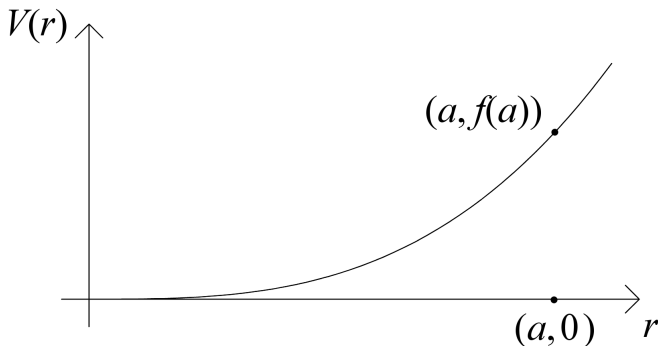
Present example - the set of point $(r, \frac{4}{3}\pi r^3)$ for all $r \geq 0$.

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Functions of two variables

Example

Volume V of a cylinder depends on *two* dimensions, the radius r and the height h - $V = f(r, h)$, where $f(r, h) = \pi r^2 h$ defines a *function of two variables*.

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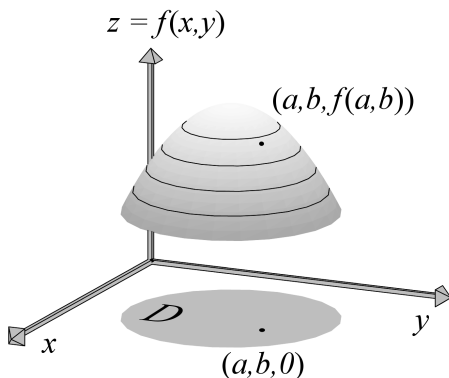
Subset D of \mathbb{R}^2 , i.e., a region in a plane.

If not specified, the maximal domain is assumed.

Functions of two variables

Graph

The set of points $(a, b, c) \in \mathbb{R}^3$ where $(a, b) \in D$ and $c = f(a, b)$ - a *surface*.



Visualisation of surfaces - Spheres

- ▶ Radius r , centre (a, b, c) - points (x, y, z) a distance r from (a, b, c) . Pythagoras's theorem \implies

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2,$$

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- ▶ $+$ means “northern” hemisphere
– means “southern” hemisphere.

Visualisation of surfaces - Spheres

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$$(x + \tfrac{1}{2}\alpha)^2 + (y + \tfrac{1}{2}\beta)^2 + (z + \tfrac{1}{2}\gamma)^2 = \tfrac{1}{4}(\alpha^2 + \beta^2 + \gamma^2) - \delta,$$

- ▶ sphere if and only if $\tfrac{1}{4}(\alpha^2 + \beta^2 + \gamma^2) - \delta > 0$.

Visualisation of surfaces - Spheres

Example 1

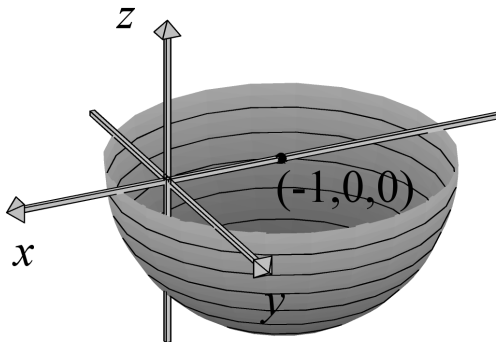
Sketch the graph of $f(x, y) = -\sqrt{1 - 2x - x^2 - y^2}$.

Visualisation of surfaces - Spheres

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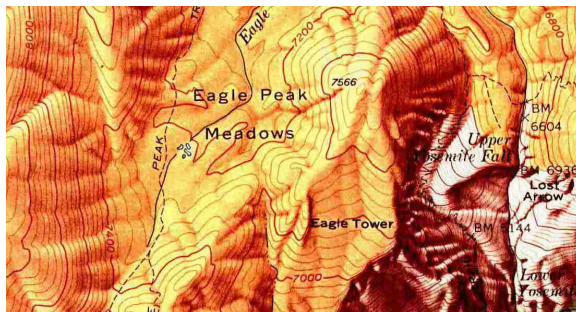
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Answer



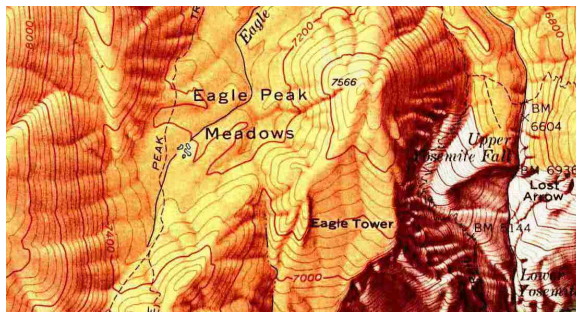
Visualisation of surfaces - Cross-sections

- For a surface $z = f(x, y)$ the set of points satisfying, $f(x, y) = c$, is a *level curve* or *contour*,



Visualisation of surfaces - Cross-sections

- ▶ For a surface $z = f(x, y)$ the set of points satisfying, $f(x, y) = c$, is a *level curve* or *contour*,
- ▶ think of $z = f(x, y)$ as part of the surface of the earth - each level curve represents a particular contour line on its map.



Visualisation of surfaces - Cross-sections

- ▶ More generally, the intersection of plane $x = \text{constant}$ or $y = \text{constant}$ or $z = \text{constant}$ and surface $F(x, y, z) = 0$ is called a *cross-section*,

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- ▶ for different c , L_c may be a curve, a point or points, or the empty set,
- ▶ each point in D lies on one level curve.

Visualisation of surfaces - Cross-sections

Example 2

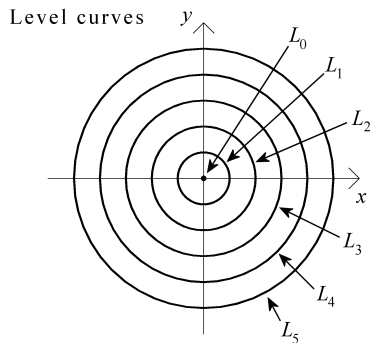
By considering the level curves and the cross-sections $x = 0$ and $y = 0$, obtain a sketch of $z = \sqrt{x^2 + y^2}$.

Visualisation of surfaces - Cross-sections

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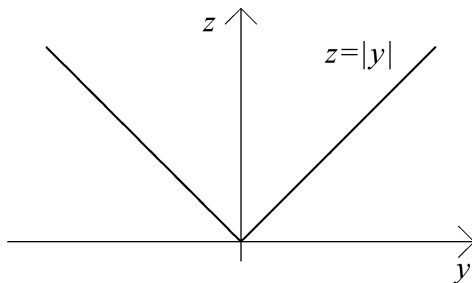
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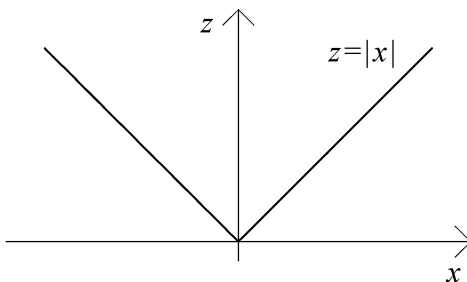
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$y=0$ cross-section

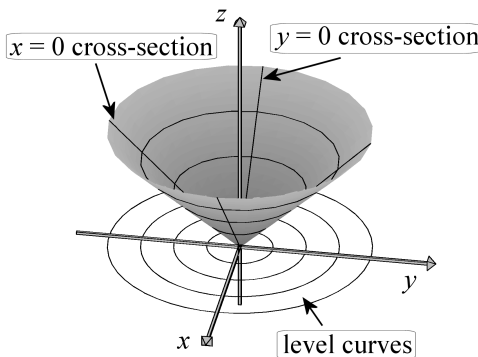


Visualisation of surfaces - Cross-sections

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Visulatisation of surfaces - Ellipsoid

- ▶ An **ellipsoid** of radius r_1 in the x -direction, r_2 in the y -direction and r_3 in the z -direction, with centre (a, b, c) is defined by

$$\frac{(x - a)^2}{(r_1)^2} + \frac{(y - b)^2}{(r_2)^2} + \frac{(z - c)^2}{(r_3)^2} = 1.$$

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- ▶ When $r_1 = r_2 = r_3$ we recover the equation for the sphere.

Visualisation of surfaces - Planes

- ▶ Recall: a plane with normal vector $\mathbf{n} = (\alpha, \beta, \gamma)$ has equation $\alpha x + \beta y + \gamma z = \delta$,

Visualisation of surfaces - Planes

- ▶ Recall: a plane with normal vector $\mathbf{n} = (\alpha, \beta, \gamma)$ has equation $\alpha x + \beta y + \gamma z = \delta$,
- ▶ the graph of $f(x, y) = ax + by + c$ is the plane $z = ax + by + c$ with normal $(a, b, -1)$ passing through the point $(0, 0, c)$.

Visualisation of surfaces - Planes

Example 3

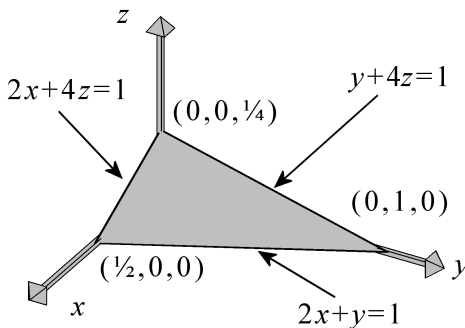
Sketch the part of the surface $2x + y + 4z = 1$ where $x, y, z \geq 0$.

Visualisation of surfaces - Planes

Example 3

Sketch the part of the surface $2x + y + 4z = 1$ where $x, y, z \geq 0$.

Answer



Visualisation of surfaces - Circular cylinder

- ▶ A circular **cylinder** in \mathbb{R}^3 of radius r centred at the origin lying parallel to z -axis, is defined by

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- ▶ This is **NOT** the equation for a circle, because the surface lies in \mathbb{R}^3 .
- ▶ Generalisable to cylinders centred at (a, b, c) , cylinders lying parallel to the x or y axes and cylinders with ellipses as cross sections.

Visualisation of surfaces - Paraboloid

Example 4

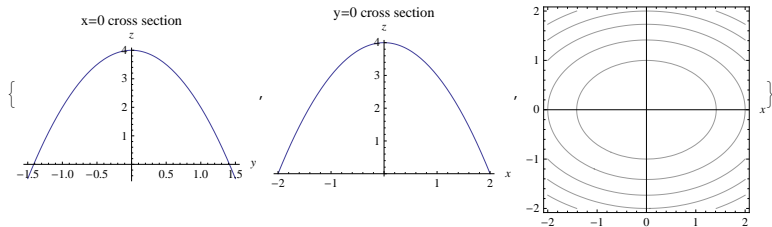
Sketch the region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the plane $z = 2$.

Visualisation of surfaces - Paraboloid

Example 4

Sketch the region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the plane $z = 2$.

Answer

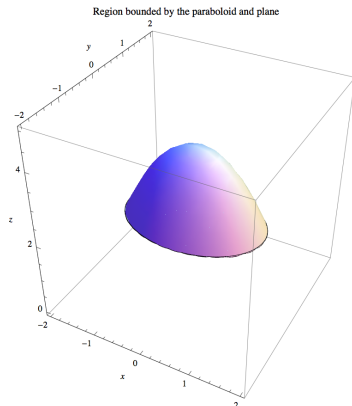


Visualisation of surfaces - Paraboloid

Example 4

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Answer



Ordinary derivative

► Recall:

$$\lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h}$$

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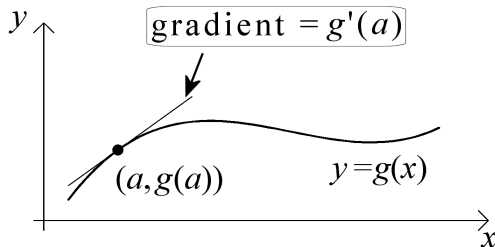
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Partial derivatives

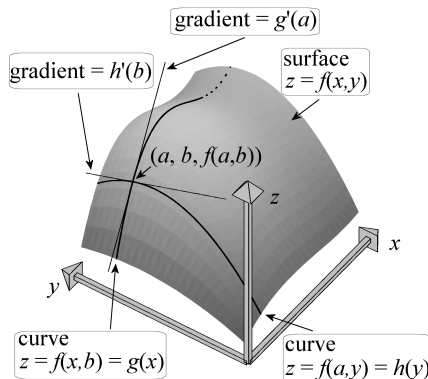
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Partial derivatives

- ▶ On surface $z = f(x, y)$, there is no single meaning of gradient,
- ▶ straight down a mountain side gradient may be very large and traversing the mountain the gradient is much less,
- ▶ necessary to define *two* gradients on cross-section of the surface in the x and y directions.

Partial derivatives

Taking cross-sections $x = a$ and $y = b$ we get the graphs of two functions of *one* variable - $z = f(x, b) = g(x)$ and $z = f(a, y) = h(y)$



Partial derivatives

- ▶ The gradients to $z = g(x)$ and $z = h(y)$ are called the *partial x and y derivatives of f at (a, b)*

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- ▶ for a function of x_1, x_2, \dots, x_n

$$\frac{\partial f}{\partial x_i} = \text{derivative w.r.t. } x_i \text{ with all other variables constant.}$$

Partial derivatives

- Important to distinguish notation used for ordinary and partial derivatives.

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Partial derivatives

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Ordinary derivative : $\frac{df}{dx}$, partial derivative : $\frac{\partial f}{\partial x}$,

- ▶ subscript notation for partial derivatives

$$\frac{\partial f}{\partial x} \equiv f_x, \text{ and } \frac{\partial f}{\partial y} \equiv f_y,$$

Partial derivatives

Example 5

Find f_x , f_y and z_x where

$$(a) f(x, y) = x^3 y^2 + x, \quad (b) z(x, y) = \sin^{-1} \left(\frac{x}{x+y} \right) \text{ and } x, y > 0.$$

Partial derivatives

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[$\sin^{-1} u$ is the inverse sine function and *not* the reciprocal $1/\sin u$.
Domain of \sin^{-1} is $[-1, 1]$ and $x/(x+y)$ lies in this domain.]

Partial derivatives

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Answer

$$(a) f_x = 3x^2 y^2 + 1, \quad f_y = 2x^3 y.$$

$$(b) z_x = \frac{y}{x+y} \frac{1}{\sqrt{2xy + y^2}}.$$

Partial derivatives

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Chain rule

Recall from Level-1:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

We used

$$\frac{\partial}{\partial x} f(g(x, y)) = f'(g(x, y))g_x(x, y).$$

Partial derivatives

Example 6

Find z_x where z is defined implicitly as a function of x and y by the equation

$$x^4 + 2y^2 + z^3 - 2x^2yz = 1$$

Partial derivatives

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Answer

$$z_x = \frac{4x^3 - 4xyz}{2x^2y - 3z^2}$$

Partial derivatives

Example 7

For $r \in \mathbb{R}^+$, let $u = f(r)$ where $r^2 = x^2 + y^2 + z^2$. Show that

$$xu_x + yu_y + zu_z = rf'(r).$$

Higher order derivatives

Let u be a function of x, y, \dots then u_x and u_y are functions of x, y, \dots and so may define

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$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(u_y) = u_{yx}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(u_y) = u_{yy},$$

etc.

Higher order derivatives

There is no automatic guarantee that $u_{xy} = u_{yx}$ but...

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► **Theorem:**

Let u be a function of x, y such u_{xy} and u_{yx} exist and are continuous at a point (a, b) . Then,

$$u_{xy}(a, b) = u_{yx}(a, b).$$

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- Also for functions of more variables and higher order derivatives - e.g. if $u = u(x, y, z)$ then

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- in 2A, we assume the order of taking partial derivatives is unimportant.

Higher order derivatives

Example 8

Determine all second order derivatives of $u = \sin xy$ and verify that $u_{xy} = u_{yx}$.

Higher order derivatives

Example 8

Determine all second order derivatives of $u = \sin xy$ and verify that $u_{xy} = u_{yx}$.

Answers

$$u_{xx} = -y^2 \sin xy,$$

$$u_{xy} = \cos xy - yx \sin xy,$$

$$u_{yx} = \cos xy - xy \sin xy,$$

$$u_{yy} = -x^2 \sin xy.$$

Higher order derivatives

Example 9

Let $u = f(x/y)$, where f is an arbitrary (twice differentiable, with continuous second derivative) function of one variable. Show that

$$xu_x + yu_y = 0,$$

and *deduce* that

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0.$$

Two variable chain rule

- ▶ Chain rule for functions of one variable - used to find derivative of $F(x) = f(u(x))$ -

$$\frac{dF}{dx} = \frac{du}{dx} \frac{df}{du} = u'(x)f'(u(x)).$$

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- ▶ extend this technique to functions of several variables

- ▶ **Theorem**

Let $F(x, y) = f(u(x, y), v(x, y))$. Then

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial f}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial f}{\partial v} \quad \text{and} \quad \frac{\partial F}{\partial y} = \frac{\partial u}{\partial y} \frac{\partial f}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial f}{\partial v}.$$

This is called the *chain rule for functions of two variables*.

Two variable chain rule

- Observe the pattern

$$\frac{\partial F}{\partial x} = \frac{\boxed{\partial u}}{\partial x} \frac{\partial f}{\boxed{\partial u}} + \frac{\boxed{\partial v}}{\partial x} \frac{\partial f}{\boxed{\partial v}},$$

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- extends in an obvious way to functions of any number of variables - if $F(x, y, z) = f(u(x, y, z), v(x, y, z), w(x, y, z))$ then

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial f}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial f}{\partial v} + \frac{\partial w}{\partial x} \frac{\partial f}{\partial w}.$$

Two variable chain rule - Special cases

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Partial derivatives are written as ordinary derivatives when used on functions of one variable.

Two variable chain rule

Example 10

Let $w = u^2 + v^2$ where $u = \sin \theta$ and $v = \cos \phi$. Use the chain rule to calculate w_θ and w_ϕ in terms of θ and ϕ .

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Answer

$$w_\theta = \sin 2\theta, \quad w_\phi = -\sin 2\phi.$$

Examples of ODEs and PDEs

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- ▶ e.g Newton's law of cooling states that
"the rate of change of temperature of an object is proportional to the temperature difference between it and its surroundings"
- ▶ in mathematical terms this is the differential equation

$$\frac{dT}{dt} = k(T - T_0),$$

where $T(t)$ is the temperature, T_0 the temperature of the surroundings and k a constant

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Examples of ODEs and PDEs

- ▶ A *partial differential equation* (PDE) is a relationship between a function of more than one variable and its partial derivatives
- ▶ e.g. the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where $u(x, t)$ is the displacement (from a rest position) of the point x at time t and c is the wave speed.

Definitions

- ▶ The *order* of a differential equation is the order of the highest derivative

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- ▶ the *general solution* includes all possible solutions—includes arbitrary constants (ODE) or arbitrary functions (PDE)
- ▶ a solution without arbitrary constants/functions is called a *particular solution*. This may be found by giving extra conditions in the form of initial or boundary conditions.

First order PDEs

Example 11

Find the general solution of the PDE,

$$\frac{\partial f}{\partial x} = x^2 + y + 9,$$

where f is a function of two independent variables x and y .

First order PDEs

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Answers

Solution is

$$\frac{1}{3}x^3 + xy + 9x + A(y)$$

where A is an arbitrary function.

First order PDEs

Example 12

Find the general solution of the PDE,

$$\frac{\partial^2 f}{\partial x \partial y} = 2x,$$

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First order PDEs

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Find the general solution of the PDE,

$$\frac{\partial^2 f}{\partial x \partial y} = 2x,$$

where f is a function of two independent variables x and y .

Answers

Solution is

$$x^2 y + A(y) + B(x),$$

where A and B are arbitrary functions.

Solving PDEs using change of variable

- ▶ In this section we solve some first and second order PDEs by using a change of independent variables to write it in a simpler form

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- ▶ if we change from x, y to u, v then the chain rule gives

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial z}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial z}{\partial v}.$$

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$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial z}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial z}{\partial v}.$$

- ▶ in fact, for any expression E (e.g. a derivative of z)

$$\frac{\partial}{\partial x}(E) = \frac{\partial u}{\partial x} \frac{\partial}{\partial u}(E) + \frac{\partial v}{\partial x} \frac{\partial}{\partial v}(E) \quad (1)$$

this is used when we consider second order PDEs.

First order PDEs

Example 13

By changing variables from (x, y) to (u, v) , where $u = xy$, $v = x/y$, solve the PDE

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2x^2 \sin(xy).$$

First order PDEs

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$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2x^2 \sin(xy).$$

Answers

Solution is

$$z = -\frac{x}{y} \cos(xy) + A(x/y),$$

where A is an arbitrary function.

First order PDEs

Example 14

By changing variables from (x, y) to (u, v) , where $u = x^3/y$, $v = x$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of partial derivatives with respect to u and v . Hence, solve the PDE

$$x \frac{\partial f}{\partial x} + 3y \frac{\partial f}{\partial y} = \frac{6x^5}{y}.$$

First order PDEs

Example 14

By changing variables from (x, y) to (u, v) , where $u = x^3/y$, $v = x$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in terms of partial derivatives with respect to u and v . Hence, solve the PDE

$$x \frac{\partial f}{\partial x} + 3y \frac{\partial f}{\partial y} = \frac{6x^5}{y}.$$

Answers

Solution is

$$f = \frac{3x^5}{y} + A(x^3/y),$$

where A is an arbitrary function.

Chapter 2: Double and triple integration

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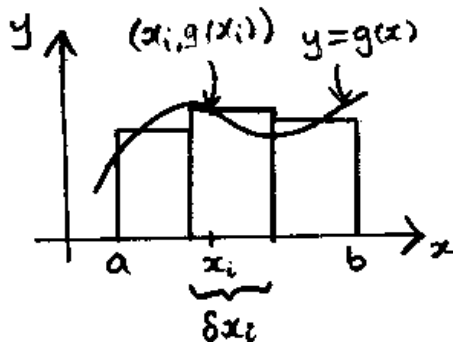
- ▶ Double integration on regular domains,
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- ▶ Double integration and general change of variables,
- ▶ Triple integration on regular domains,
- ▶ Triple integration in spherical coordinates,
- ▶ Using integrals to calculate area, volume and mass.

Area under curves

- ▶ In first year definite integrals arise as “areas under curves”.

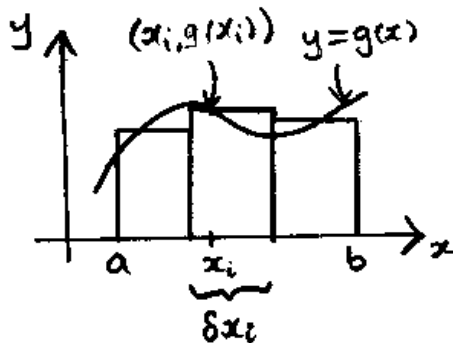
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- ▶ In first year definite integrals arise as “areas under curves”.
- ▶ We approximate the area under the curve the sum of areas of rectangles (called a *Riemann sum*)



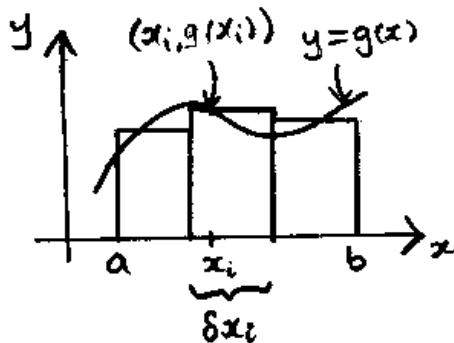
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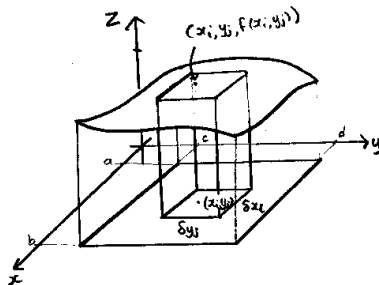
$$\int_a^b g(x) dx = \lim_{N \rightarrow \infty} \sum_{i=1}^N g(x_i) \delta x_i.$$

Double integration on rectangular domains

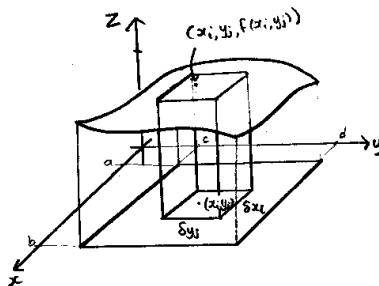
- ▶ Similarly, the “volume under a surface” $z = f(x, y)$ on the set $D \subset \mathbb{R}^2$ is approximated by the sum of the volumes of cuboids.

Double integration on rectangular domains

- ▶ Similarly, the “volume under a surface” $z = f(x, y)$ on the set $D \subset \mathbb{R}^2$ is approximated by the sum of the volumes of cuboids.
- ▶ Divide $R = [a, b] \times [c, d]$ into subrectangles of area $\delta A_{ij} = \delta x_i \delta y_j$ and the cuboid above this has height $f(x_i, y_j)$.



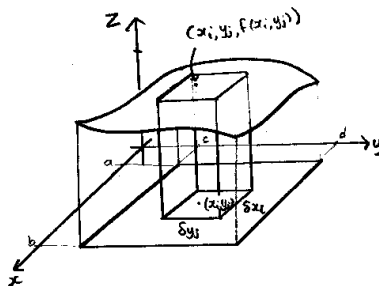
Double integration on rectangular domains



- The whole volume is approximated by

$$\iint_R f(x, y) dA = \iint_R f(x, y) dx dy = \lim_{M, N \rightarrow \infty} \sum_{i=1}^N \sum_{j=1}^M f(x_i, y_j) \delta A_{ij}.$$

Double integration on rectangular domains



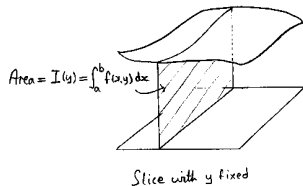
- The whole volume is approximated by

$$\iint_R f(x, y) dA = \iint_R f(x, y) dx dy = \lim_{M, N \rightarrow \infty} \sum_{i=1}^N \sum_{j=1}^M f(x_i, y_j) \delta A_{ij}.$$

- If the limit as $M, N \rightarrow \infty$ exists we say that f is *integrable* over R and $dA = dx dy$ is called the *area element*.

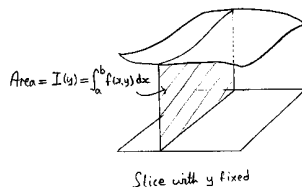
Double integration on rectangular domains

The solid under the curve is made up of slices with y fixed



Double integration on rectangular domains

The solid under the curve is made up of slices with y fixed



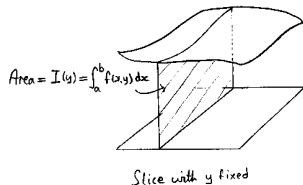
The area under the curve in such a cross section is

$$I(y) = \int_a^b f(x, y) dx,$$

where y is fixed in the integrand.

Double integration on rectangular domains

The solid under the curve is made up of slices with y fixed



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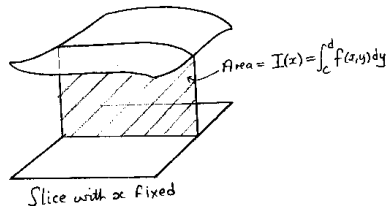
where y is fixed in the integrand. The volume under the surface is then

$$\iint_R f(x, y) dx dy = \int_c^d I(y) dy = \int_c^d \left(\int_a^b f(x, y) dx \right) dy.$$

Double integration on rectangular domains

Instead, summing the areas of cross sections of the solid with x fixed, we have

$$\iint_R f(x, y) \, dx dy = \int_a^b \left(\int_c^d f(x, y) \, dy \right) dx.$$

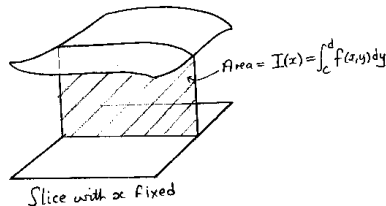


Notation:

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$$\int_a^b dx \int_c^d f(x, y) \, dy \quad \text{for} \quad \int_a^b \left(\int_c^d f(x, y) \, dy \right) dx.$$

Double integration on rectangular domains

Example 1

Evaluate $\iint_R x^2 + y^2 \, dx dy$

where R is $[1, 3] \times [2, 4]$.

Double integration on rectangular domains

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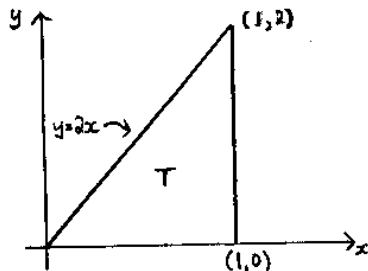
where R is $[1, 3] \times [2, 4]$.

Answer

$$\frac{164}{3}$$

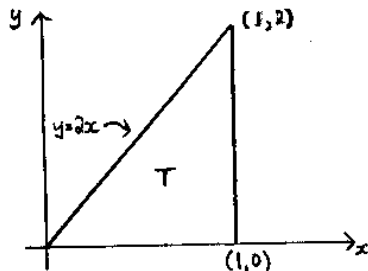
Double integration on regular domains

Consider a more complicated domain T which is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 2)$.



Double integration on regular domains

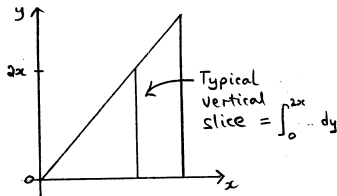
Consider a more complicated domain T which is the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 2)$.



The domain T is bounded by the lines $y = 0$, $x = 1$ and $y = 2x$.

Double integration on regular domains

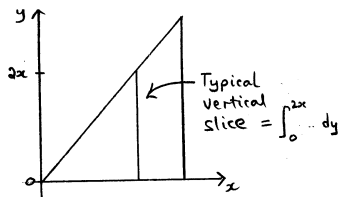
To evaluate a double integral over T we could split T into a collection of **vertical slices**,



integrate with respect to y and then integrate the result with respect to x .

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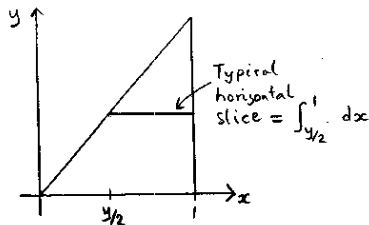
integrate with respect to y and then integrate the result with respect to x .

$$\iint_T f(x, y) \, dx \, dy = \int_0^1 dx \int_0^{2x} f(x, y) \, dy.$$

Notice that the limits in the first integral *depend on x* .

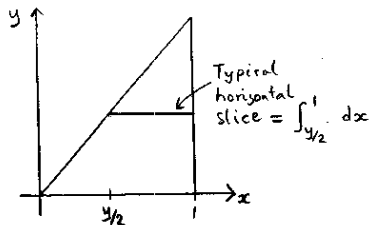
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Alternatively, looking at horizontal slices, with end-points $x = \frac{1}{2}y$, $x = 1$, and summing these from $y = 0$ to $y = 2$.



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Thus the integral is also

$$\iint_T f(x, y) \, dx dy = \int_0^2 dy \int_{\frac{1}{2}y}^1 f(x, y) \, dx.$$

Double integration on regular domains

Definition

Let D be a domain in the x, y -plane. D is said to be

- ▶ Type I (*y-simple*) if it is bounded by lines $x = a$, $x = b$ and curves $y = g(x)$, $y = h(x)$, the intersection of any vertical line $x = c$, where $c \in [a, b]$, is an interval or a single point,

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- ▶ Type II (*x-simple*) if it is bounded by curves $x = g(y)$, $x = h(y)$ and lines $y = a$, $y = b$, the intersection of any horizontal line $y = c$, where $c \in [a, b]$, is an interval or a single point,

Double integration on regular domains

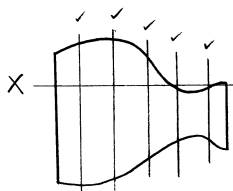
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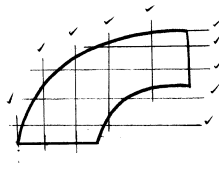
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- ▶ *regular* if it the union of finitely many disjoint type I and type II domains. Every type I and type II domain is regular.

Double integration on regular domains

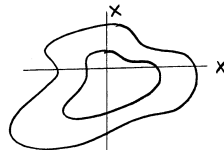
Example



Type I and not type II



Type I and type II



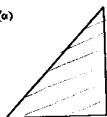
neither type I or type II

Double integration on regular domains

Example 2

State whether each of the domains shown below are type I and/or type II or regular.

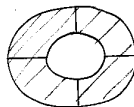
(a)



(b)



(c)

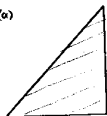


Double integration on regular domains

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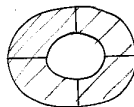
(a)



(b)



(c)



Answers

(a) Both, (b) Type I only, (c) Neither.

Double integration on regular domains

Theorem

If D is the type I domain defined by $g(x) \leq y \leq h(x)$ where $a \leq x \leq b$ then

$$\iint_D f(x, y) \, dx \, dy = \int_a^b dx \int_{g(x)}^{h(x)} f(x, y) \, dy.$$

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The *inner integral* may have a limit depending on the other variable but the *outer integral* has constant limits.

Double integration on regular domains

Example 3

Evaluate

$$\iint_D xy^2 \, dx dy,$$

where D is the region in the first quadrant bounded by the curve $y = 4x^2$, the x axis and the line $x = 1$.

Double integration on regular domains

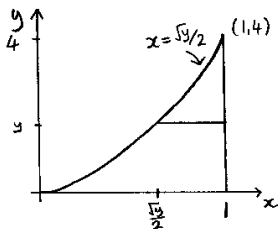
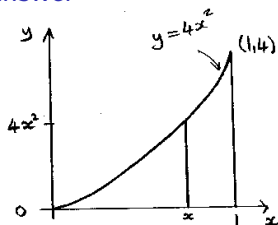
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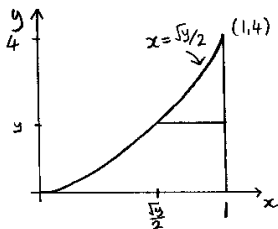
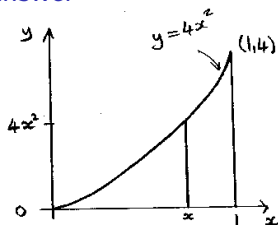
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Answer



$$I = \frac{8}{3}$$

Double integration on regular domains

Example 4

Evaluate

$$I = \iint_D 3x^2 + y^2 \, dx dy,$$

where D is the triangle with vertices $(0, 0)$, $(1, 1)$ and $(2, 1)$.

Double integration on regular domains

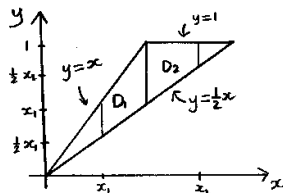
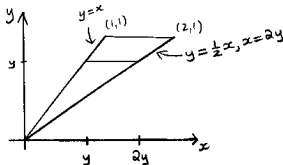
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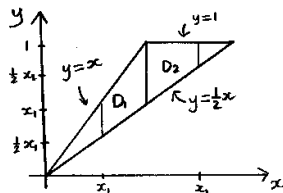
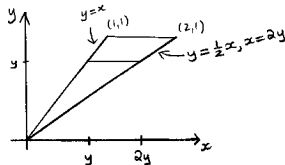
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Answer



$$I = 2.$$

Double integration on regular domains

Example 5

Evaluate

$$I = \int_0^1 dx \int_{\sqrt{x}}^1 \frac{e^{y^2}}{\sqrt{x}} dy.$$

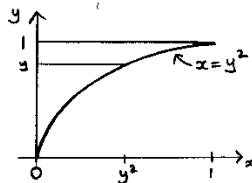
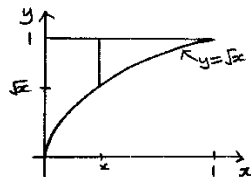
Double integration on regular domains

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Answer



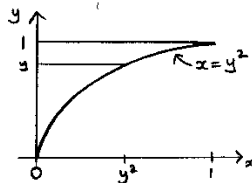
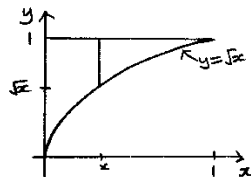
Double integration on regular domains

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Answer



$$I = e - 1.$$

Double integration on regular domains

Example 6

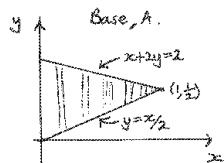
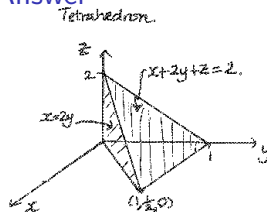
Find the volume of the tetrahedron T , bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

Double integration on regular domains

Example 6

Find the volume of the tetrahedron T , bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

Answer

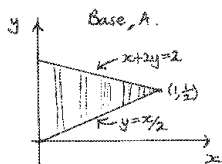
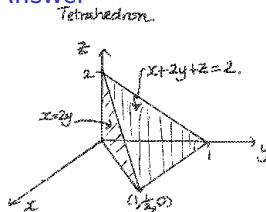


Double integration on regular domains

Example 6

Find the volume of the tetrahedron T , bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

Answer



$$T = \frac{1}{3}.$$

Double integration in polar coordinates

The position of a point (x, y) on the cartesian plane can be specified by r, θ which are

Polar coordinates

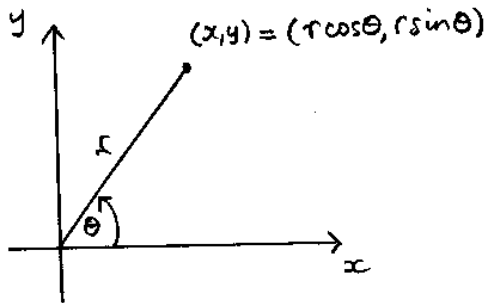
$$x = r \cos \theta, \quad y = r \sin \theta, \quad \theta \in [0, 2\pi), \quad r \geq 0.$$

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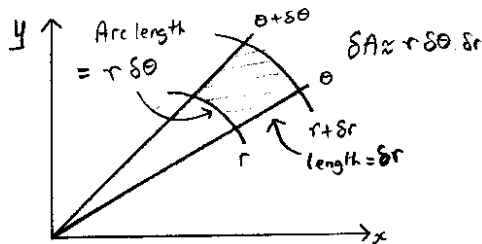


Double integration in polar coordinates

In cartesian coordinates, the area of an elementary rectangle using in the Riemann sum is $\delta A = \delta x \delta y$. In polar coordinates, the area element has area $\delta A \approx r \delta r \delta \theta$.

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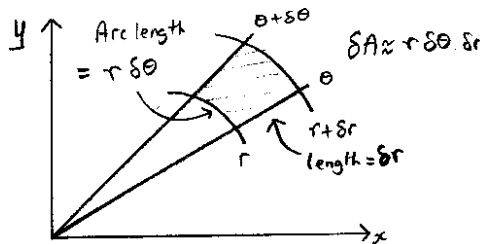


For this reason in polar coordinates, $dA = r dr d\theta$, i.e.,

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When either the domain is circular or the integrand is written in terms of $x^2 + y^2 (= r^2)$, use polar coordinates.

Double integration in polar coordinates

Example 7

Use polar coordinates to evaluate

$$I = \iint_D x + y \, dx dy,$$

where D is part of the annulus between circles of radius 1 and 2, centre $(0, 0)$ lying in upper half plane.

Double integration in polar coordinates

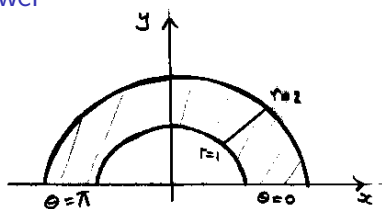
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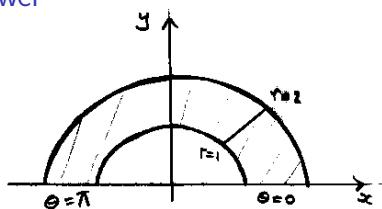
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Answer



$$I = \frac{14}{3}.$$

Double integration in polar coordinates

Example 8

Evaluate

$$I = \iint_D y \, dx dy,$$

where D is the part of the disk of radius $a (> 0)$ and centre $(a, 0)$ lying in the first quadrant.

Double integration in polar coordinates

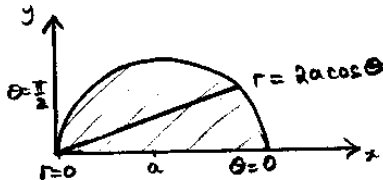
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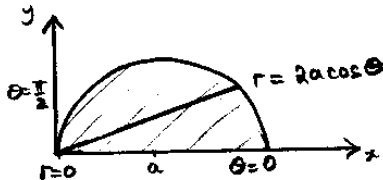
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Answer



$$I = \frac{2a^3}{3}.$$

Beta and Gamma functions

Beta functions can help us easily integrate functions that involve powers of cosine and sine.

► *Beta function:*

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx, \quad p > 0 \text{ and } q > 0.$$

A particularly useful form is,

$$B(p, q) = 2 \int_0^{\pi/2} \sin^{2p-1}(y) \cos^{2q-1}(y) dy .$$

This is found by substituting $x = \sin^2 y$ in the definition of the Beta function.

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$$\Gamma(k) = \int_0^{\infty} x^{k-1} e^{-x} dx, \quad k > 0.$$

Properties of Beta and Gamma functions

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5. $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Properties of Beta and Gamma functions

Result

From the properties of Gamma functions we can derive the following result:

Property of Beta functions

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{(m-1)(m-3)\dots(n-1)(n-3)\dots}{(m+n)(m+n-2)(m+n-4)\dots} K$$

where $K = 1$ unless m and n are both even in which case $K = \pi/2$.

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For example,

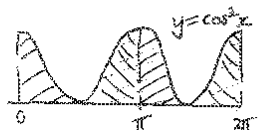
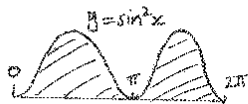
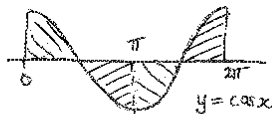
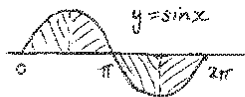
$$\int_0^{\pi/2} \sin^3 x \cos^6 x \, dx = \frac{2.5.3.1}{9.7.5.3.1} = \frac{2}{63}.$$

Simplifying sine and cosine integrals

Properties of the graphs of sine and cosine seen in 1S/X simplify the integral before applying Beta functions.

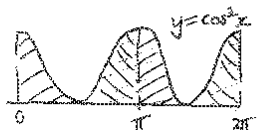
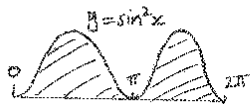
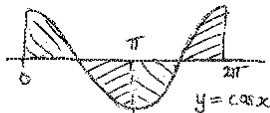
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Simplifying sine and cosine integrals

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We deduce

$$\int_0^\pi \sin x \, dx = 2 \int_0^{\pi/2} \sin x \, dx; \quad \int_0^{2\pi} \sin x \, dx = 0; \quad \int_0^\pi \cos x \, dx = 0;$$

$$\int_0^\pi \sin^2 x \, dx = 2 \int_0^{\pi/2} \sin^2 x \, dx; \quad \int_0^\pi \cos^2 x \, dx = 2 \int_0^{\pi/2} \cos^2 x \, dx \dots \text{et}$$

Beta functions

Example 9

Evaluate:

$$(a) I = \int_0^{\pi} \sin^3 x \cos^4 x \, dx, \quad (b) I = \int_0^{\pi} \sin^3 x \cos^5 x \, dx,$$

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Answers

$$(a) I = \frac{4}{35}, \quad (b) I = 0, \quad (c) I = \frac{\pi}{8}.$$

Change of variables in double integration

Definition

Consider a change of variables x, y to u, v . So $x = x(u, v)$ and $y = y(u, v)$. The *Jacobian* $\frac{\partial(u, v)}{\partial(x, y)}$ is the determinant

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

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If the change of variables is invertible then the Jacobian is nonzero and

$$\frac{\partial(x, y)}{\partial(u, v)} = 1 / \frac{\partial(u, v)}{\partial(x, y)}.$$

Change of variables in double integration

Theorem

Let the change of variables x, y to u, v be invertible on the domain D . Then

$$\iint_D f(x, y) \, dx dy = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du dv,$$

where D is the domain in the xy -plane and S is the corresponding domain in the uv -plane.

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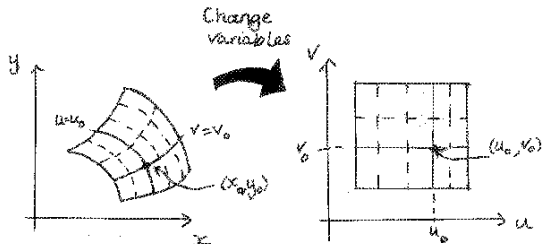
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Often it is convenient to use

$$\frac{\partial(x, y)}{\partial(u, v)} = 1 \bigg/ \frac{\partial(u, v)}{\partial(x, y)}$$

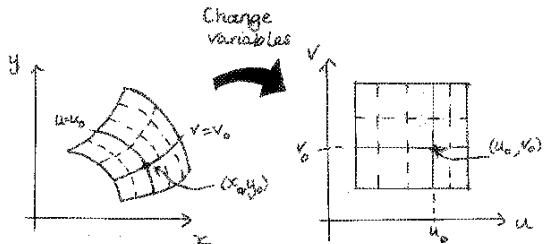
Change of variables in double integration

- The idea here is to choose variables u, v in which the domain is simply described, preferably with constant limits, e.g.



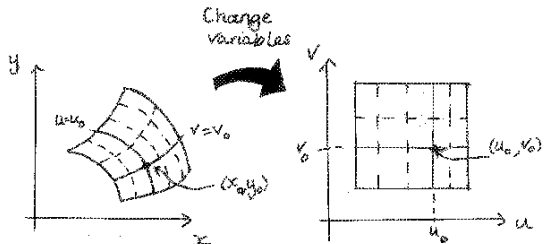
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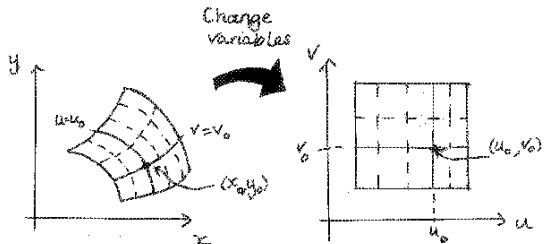


The lines $u = u_0$ and $v = v_0$ in the uv -plane get mapped to curves $x = x(u_0, v)$, $y = y(u_0, v)$ and $x = x(u, v_0)$, $y = y(u, v_0)$ in the xy -plane.

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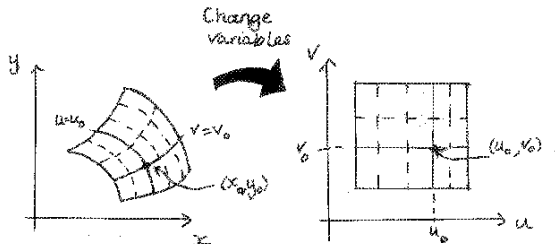


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Change of variables in double integration

- Summing the elements that make up the region D

$$\begin{aligned}\iint_D f(x, y) \, dx dy &= \lim_{N, M \rightarrow \infty} \sum_{i=1}^N \sum_{j=1}^M f(x_i, y_j) \Delta A \\ &= \lim_{N, M \rightarrow \infty} \sum_{i=1}^N \sum_{j=1}^M f(x(u_i, v_j), y(u_i, v_j)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v\end{aligned}$$

which is the Riemann sum for $\iint_S f(x(u, v), y(u, v)) |J| \, du dv$.

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$$J = \begin{vmatrix} (r \cos \theta)_r & (r \cos \theta)_\theta \\ (r \sin \theta)_r & (r \sin \theta)_\theta \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r,$$

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giving the result

$$\iint_D f(x, y) \, dx dy = \iint_S f(r \cos \theta, r \sin \theta) r \, dr d\theta.$$

Change of variables in double integration

Example 10

By making a suitable change of variables, evaluate

$$\iint_D x + 3y \, dx dy,$$

where D is the region bounded by the lines

$$y = x - 1, \quad y = x + 1, \quad y = -x - 1, \quad y = -x + 3.$$

Change of variables in double integration

Example 10

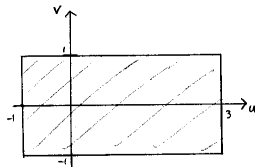
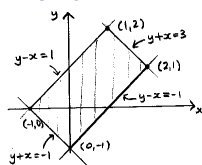
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Change of variables in double integration

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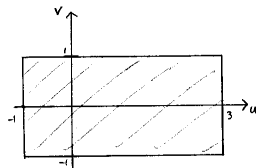
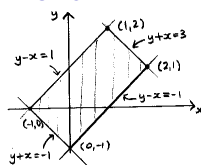
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Answer



$$I = 8.$$

Change of variables in double integration

Example 11

Find the area bounded by the curves $y = e^x$, $y = 2e^x$, $y = e^{-x}$ and $y = 2e^{-x}$.

Change of variables in double integration

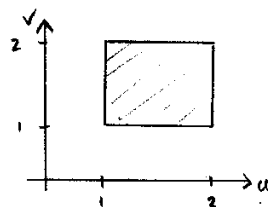
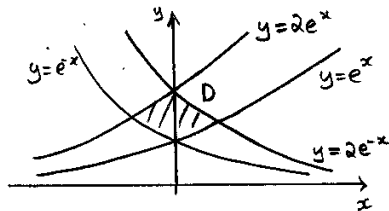
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Answer



Change of variables in double integration

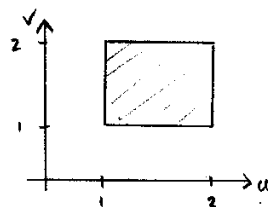
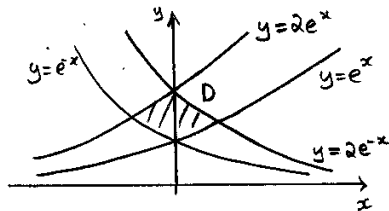
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Answer



$$\text{Area} = 2(3 - 2\sqrt{2}).$$

Triple integration

Define triple integrals for functions of three variables.

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$$\iint_R f(x, y) \, dx dy = \lim_{N, M \rightarrow \infty} \sum_{i=1}^N \sum_{j=1}^M f(x_i, y_j) \delta x_i \delta y_j.$$

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For a triple integral instead of summing over an area $\delta A_{ij} = \delta x_i \delta y_j$, we sum over a volume $\delta V_{ijk} = \delta x_i \delta y_j \delta z_k$ which leads us to

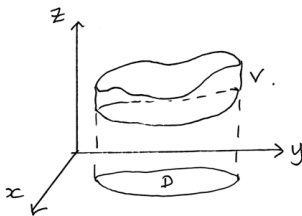
$$\iiint_V f(x, y, z) \, dx dy dz = \lim_{N, M, L \rightarrow \infty} \sum_{i=1}^N \sum_{j=1}^M \sum_{k=1}^L f(x_i, y_j, z_k) \delta x_i \delta y_j \delta z_k.$$

Triple integration

- If V lies between two continuous functions of x and y then

$$\iiint_V f(x, y, z) \, dx dy dz = \iint_D \left(\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \right) dx dy$$

where D is the projection of V onto the xy plane.

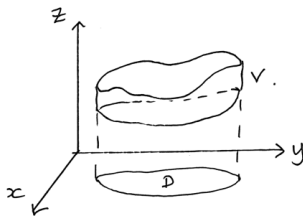


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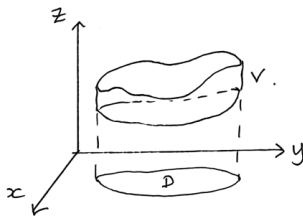


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Triple integration

In general if V lies between two continuous functions of x and y then

Triple integral

$$\iiint_V f(x, y, z) \, dx dy dz = \underbrace{\int_a^b}_{\text{Constants}} dx \underbrace{\int_{h_1(x)}^{h_2(x)}}_{\text{Curves}} dy \underbrace{\int_{g_1(x,y)}^{g_2(x,y)}}_{\text{Surfaces}} f(x, y, z) \, dz.$$

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Triple integration

Example 12

Evaluate

$$I = \iiint_V z \, dx dy dz,$$

where V is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Triple integration

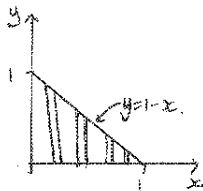
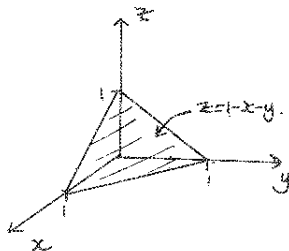
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Answer



Triple integration

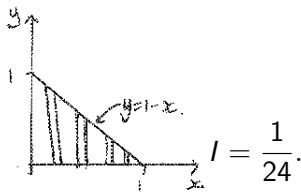
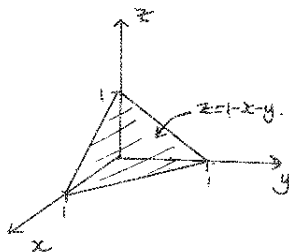
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Example 13

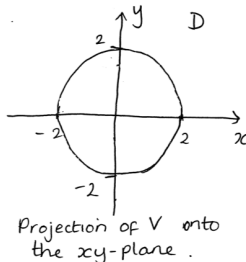
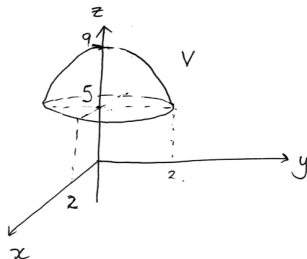
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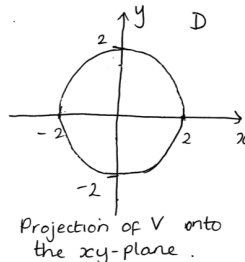
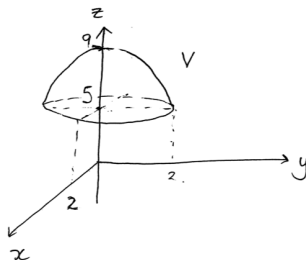


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Answer



Triple integration in spherical coordinates

The position of a point (x, y, z) in cartesian coordinates can be specified by ρ , θ , ϕ which are

Spherical coordinates

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.$$

$$\theta \in [0, 2\pi), \quad \phi \in [0, \pi), \quad \rho \geq 0.$$

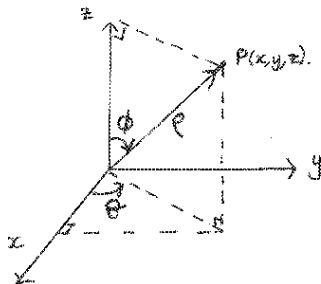
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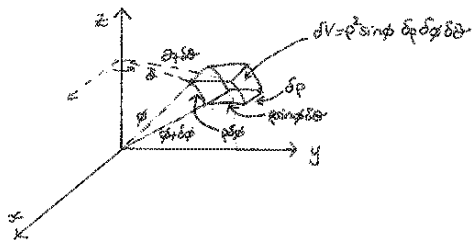


Triple integration in spherical coordinates

In cartesian coordinates, the volume of an elementary cuboid used in the Riemann sum is $\delta V = \delta x \delta y \delta z$. In spherical coordinates, the volume element is $\delta V \approx \rho^2 \sin \phi \delta \theta \delta \phi \delta \rho$.

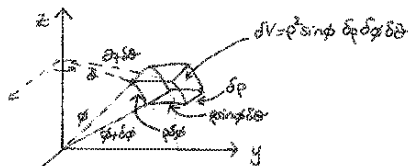
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$$\iiint_V f(x, y, z) dx dy dz = \iiint_V f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\theta d\phi d\rho.$$

When either the domain is spherical or the integrand is written in terms of $x^2 + y^2 + z^2 (= \rho^2)$, use spherical coordinates.

Triple integration in spherical coordinates

Example 14

Use spherical coordinates to evaluate

$$I = \iiint_B \exp((x^2 + y^2 + z^2)^{3/2}) \, dx \, dy \, dz,$$

where B is the unit ball, $B = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$.

Triple integration in spherical coordinates

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Answer

$$I = \frac{4}{3}\pi(e - 1).$$

Triple integration in spherical coordinates

Example 15

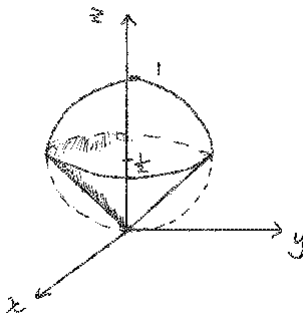
Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Triple integration in spherical coordinates

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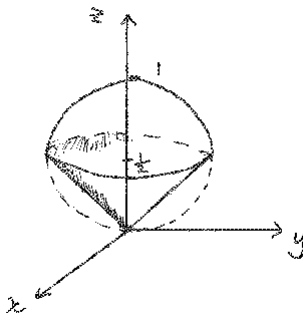


Triple integration in spherical coordinates

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Answer



$$V = \frac{\pi}{8}.$$

Chapter 3: Differentiation of vectors

- ▶ Scalar- and vector-valued functions

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- ▶ vector and scalar fields

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- ▶ Scalar- and vector-valued functions
- ▶ vector and scalar fields
- ▶ types of derivative—grad, div and curl

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- ▶ In Chapters 1 and 2 we considered functions of several variables $f : D \rightarrow \mathbb{R}$ where $D \subset \mathbb{R}^n$

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- ▶ they are *vector-valued* functions—the result is a 2- or 3-vector
- ▶ examples include velocity as a function of time and direction of the Earth's magnetic field.

Parametric equations of curves

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- ▶ position as a function of time is one example. We will revisit parametric equations in Chapter 4.

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- ▶ A function $f(\mathbf{r})$ where $\mathbf{r} = (x, y)$ or (x, y, z) (mapping a vector to a scalar) is called a *scalar field*.

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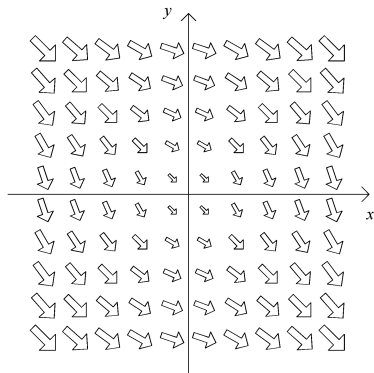
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A typical vector field



e.g. velocity at different points in a fluid.

Different types of derivative

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Name of product	Formula	Type of result	Derivative
Scalar multiplication	$\alpha \mathbf{u}$	Vector	∇f
Scalar or dot product	$\mathbf{u} \cdot \mathbf{v}$	Scalar	$\nabla \cdot \mathbf{F}$
Vector or cross product	$\mathbf{u} \times \mathbf{v}$	Vector	$\nabla \times \mathbf{F}$

Gradient of a scalar field

The *gradient* of a scalar field f is

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Answer

$$\text{grad } f = (2xy + \cosh yz, x^2 + xz \sinh yz, xy \sinh yz).$$

Gradient of a scalar field

Example 2

Let $\mathbf{r} = (x, y, z)$ so that $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. Show that

$$\nabla(r^n) = nr^{n-2}\mathbf{r},$$

for any integer n and deduce the values of $\text{grad}(r)$, $\text{grad}(r^2)$ and $\text{grad}(1/r)$.

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Answers

$$\text{grad}(r) = \frac{\mathbf{r}}{r},$$

$$\text{grad}(r^2) = 2\mathbf{r},$$

$$\text{grad}(1/r) = -\frac{\mathbf{r}}{r^3}.$$

Gradient of a scalar field

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- ▶ Partial derivatives are directional derivatives, e.g.

$$\frac{\partial f}{\partial \mathbf{i}} = \frac{\partial f}{\partial x}.$$

Directional derivative

Example 4

Find the directional derivative of $f = x^2yz^3$ at the point $P(3, -2, -1)$ in the direction of the vector $(1, 2, 2)$.

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Answer

$$\frac{\partial f}{\partial \mathbf{u}}(3, -2, -1) = -38.$$

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Directional derivative

Example 5

Consider $f = \ln(xy + z^3)$ at the point $P(1, 1, 1)$. In what direction does f have the maximal rate of change? What is this maximal rate of change?

Directional derivative

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Consider $f = \ln(xy + z^3)$ at the point $P(1, 1, 1)$. In what direction does f have the maximal rate of change? What is this maximal rate of change?

Answer

Direction is $(1/2, 1/2, 3/2)$. Maximal rate of change is

$$|\nabla f(1, 1, 1)| = \frac{\sqrt{11}}{2}.$$

Divergence of a vector field

- ▶ The *divergence* of a vector field $\mathbf{F} = (F_1, F_2, F_3)$ is the *scalar* obtained as the “scalar product” of ∇ and \mathbf{F} ,

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- ▶ a vector field is said to be *incompressible* (or *solenoidal*) if its divergence is zero.

Divergence of a vector field

- ▶ Consider $\mathbf{F} = (x, y, 0)$, $\mathbf{G} = (x, -y, 0)$ and $\mathbf{H} = (-x, -y, 0)$

Divergence of a vector field

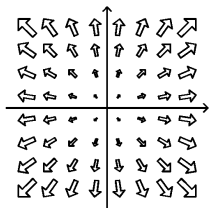
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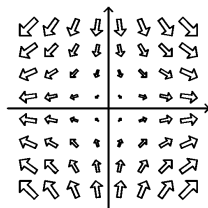
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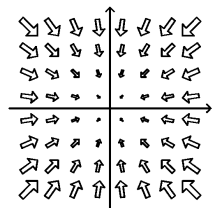
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\mathbf{F} , positive divergence



\mathbf{G} , incompressible



\mathbf{H} , negative divergence

Divergence of a vector field

Example 6

Show that the divergence of $\mathbf{F} = (x - y^2, z, z^3)$ is positive at all points in \mathbb{R}^3 .

Laplacian

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- ▶ Can be extended in a natural way to the Laplacian of a vector field $\mathbf{F} = (F_1, F_2, F_3)$,

$$\nabla^2 \mathbf{F} = (\nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3) .$$

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Answer

$\nabla^2(r^n) = 0$ if and only if $n = 0$ or $n = -1$.

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$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k} .$$

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- ▶ can be calculated using a 3×3 determinant,

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

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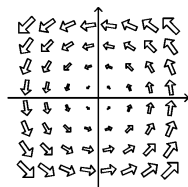
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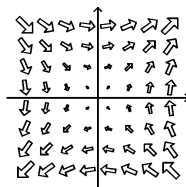
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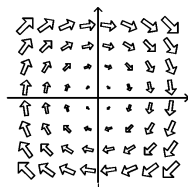
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\mathbf{F} , anticlockwise rotation



\mathbf{G} , irrotational



\mathbf{H} , clockwise rotation

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Determine $\text{curl } \mathbf{F}$ when $\mathbf{F} = (x^2y, xy^2 + z, xy)$.

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Answer

$$\text{curl } \mathbf{F} = (x - 1, -y, y^2 - x^2).$$

Curl of a vector field

Example 9

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Answer

$$\text{curl}(\mathbf{c} \times \mathbf{r}) = 2\mathbf{c}.$$

Nabla identities

Analogues involving div , grad and curl of the elementary rules of differentiation such as linearity $(f + g)'(x) = f'(x) + g'(x)$ the product rule $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$.

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$\text{grad}(f + g) = \text{grad } f + \text{grad } g$	$\text{grad}(fg) = f(\text{grad } g) + (\text{grad } f)g,$
$\text{div}(\mathbf{F} + \mathbf{G}) = \text{div } \mathbf{F} + \text{div } \mathbf{G}$	$\text{div}(f\mathbf{F}) = f \text{div } \mathbf{F} + \text{grad } f \cdot \mathbf{F},$
$\text{curl}(\mathbf{F} + \mathbf{G}) = \text{curl } \mathbf{F} + \text{curl } \mathbf{G}$	$\text{curl}(f\mathbf{F}) = f \text{curl } \mathbf{F} + \text{grad } f \times \mathbf{F},$
$\text{curl grad } f = \mathbf{0},$	$\text{div curl } \mathbf{F} = 0.$

Nabla identities

- Note the special cases

$$\text{grad}(cf) = c \text{ grad } f, \quad \text{div}(c\mathbf{F}) = c \text{ div } \mathbf{F}, \quad \text{curl}(c\mathbf{F}) = c \text{ curl } \mathbf{F},$$

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- e.g.

$$\begin{aligned}\text{curl}(f\mathbf{F}) &= \nabla \times (f\mathbf{F}) \\ &= f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F} \\ &= f \text{ curl } \mathbf{F} + \text{grad } f \times \mathbf{F}.\end{aligned}$$

Nabla identities

Example 10

Prove the identities

$$(i) \operatorname{curl} \operatorname{grad} f = 0, \quad (ii) \operatorname{curl}(f \mathbf{F}) = f \operatorname{curl} \mathbf{F} + \operatorname{grad} f \times \mathbf{F}$$

$$(iii) \operatorname{div}(f \mathbf{F}) = f \operatorname{div} \mathbf{F} + (\operatorname{grad} f) \cdot \mathbf{F}.$$

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Example 11

Let \mathbf{c} be a constant vector and $\mathbf{r} = (x, y, z)$ so that $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. Determine

$$(i) \operatorname{div}(r^n(\mathbf{c} \times \mathbf{r})), \quad (ii) \operatorname{curl}(r^n(\mathbf{c} \times \mathbf{r})).$$

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Answers

$$(i) \operatorname{div}(r^n(\mathbf{c} \times \mathbf{r})) = 0 \quad ,$$

$$(ii) \operatorname{curl}(r^n(\mathbf{c} \times \mathbf{r})) = (n+2)r^n\mathbf{c} - n(\mathbf{r} \cdot \mathbf{c})r^{n-2}\mathbf{r}.$$