2A Multivariable Calculus 2013

Ex Sheet 8

Tutorial Exercises

T1 Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$ and $\mathbf{a}$ be a constant vector. Prove the following results

(a) $\text{div } \mathbf{r} = 3$, (b) $\text{div } (\mathbf{a} \times \mathbf{r}) = 0$, (c) $\text{div } (r^n \mathbf{a}) = nr^{n-2}(\mathbf{a} \cdot \mathbf{r})$.

Solution

$$\text{(a) div } \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\text{(b) } \mathbf{a} \times \mathbf{r} = (c_2 z - c_3 y, c_3 x - c_1 z, c_1 y - c_2 x), \quad \text{so} \quad \text{div } (\mathbf{a} \times \mathbf{r}) = \frac{\partial c_2 z - c_3 y}{\partial x} + \frac{\partial c_3 x - c_1 z}{\partial y} + \frac{\partial c_1 y - c_2 x}{\partial z} = 0.$$

$$\text{(c) div } (r^n \mathbf{a}) = r^n \text{ div } \mathbf{a} + \text{grad}(r^n) \cdot \mathbf{a} = 0 + nr^{n-2}(\mathbf{a} \cdot \mathbf{r})$$

(using the result from Ex 3.2 and Ex 3.10 from the notes).

T2 Let $\mathbf{r} = (x, y, z)$ and $r = \sqrt{x^2 + y^2 + z^2}$.

a) Find $\text{div}(r^2 \mathbf{r})$.

b) Show that for any smooth function f,

$$\text{grad } f(r) = \frac{f'(r)}{r} \mathbf{r}.$$

c) Show that $\text{curl } \mathbf{r} = \mathbf{0}$. Deduce that $\text{curl } f(r)\mathbf{r} = \mathbf{0}$ for any smooth function f.

d) Determine $\text{grad}(\log r)$ and deduce that $\nabla^2(\log r) = 1/r^2$.

Solution

a) Using the identity

$$\text{div}(f\mathbf{F}) = f \text{ div } \mathbf{F} + \text{grad } f \cdot \mathbf{F},$$

we get

$$\text{div}(r^2 \mathbf{r}) = r^2 \text{ div } \mathbf{r} + \text{grad}(r^2) \cdot \mathbf{r}.$$

Since $\mathbf{r} = (x, y, z)$ and $r = \sqrt{x^2 + y^2 + z^2}$, we also have $\text{div } \mathbf{r} = 3$ and $\text{grad}(r^2) = 2\mathbf{r}$ (see example in the lecture notes: $\text{grad}(r^n) = nr^{n-2}\mathbf{r}$). Therefore,

$$\text{div}(r^2 \mathbf{r}) = 3r^2 + 2\mathbf{r} \cdot \mathbf{r} = 5r^2.$$

b) $\text{grad } f(r) = f'(r) \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z} \right) = f'(r) \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) = \frac{f'(r)}{r} \mathbf{r}$ as required.

c) We have

$$\operatorname{curl}\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (0-0)\mathbf{k} = \mathbf{0}.$$

Using the identity

$$\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + \operatorname{grad} f \times \mathbf{F},$$

we get

$$\operatorname{curl}(f(r)\mathbf{r}) = f(r)\operatorname{curl}\mathbf{r} + \operatorname{grad} f(r) \times \mathbf{r} = \frac{f'(r)}{r}\mathbf{r} \times \mathbf{r} = \mathbf{0}.$$

since $\mathbf{r} \times \mathbf{r} = \mathbf{0}$.

d) Using (b), $\operatorname{grad}(\log r) = \mathbf{r}/r^2$. Therefore

$$\begin{aligned} \nabla^2(\log r) &= \operatorname{div}\operatorname{grad}(\log r) = \nabla \cdot (r^{-2}\mathbf{r}) \\ &= r^{-2}\nabla \cdot \mathbf{r} + \operatorname{grad}(r^{-2}) \cdot \mathbf{r} = 3r^{-2} - 2r^{-4}\mathbf{r}\cdot\mathbf{r} \qquad = 1/r^2, \end{aligned}$$

as required.

T3 Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$, and $\mathbf{a}$ be a constant vector, show that

$$\operatorname{grad}\left(\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right) = \frac{1}{r^3}\mathbf{a} - \frac{3(\mathbf{a}\cdot\mathbf{r})}{r^5}\mathbf{r}.$$

Using this result, show

$$\operatorname{grad}\left(\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right) + \operatorname{curl}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = \mathbf{0}.$$

Solution

Using the identities

$$\operatorname{grad}(fg) = f\operatorname{grad}(g) + \operatorname{grad}(f)g, \quad \operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + \operatorname{grad} f \times \mathbf{F},$$

we have

$$\begin{aligned} \operatorname{grad}\left(r^{-3}(\mathbf{a}\cdot\mathbf{r})\right) &= r^{-3}\operatorname{grad}(\mathbf{a}\cdot\mathbf{r}) + \operatorname{grad}(r^{-3})(\mathbf{a}\cdot\mathbf{r}) \\ &= r^{-3}\mathbf{a} - 3r^{-5}(\mathbf{a}\cdot\mathbf{r})\mathbf{r}, \end{aligned}$$

and

$$\begin{aligned} \operatorname{curl}\left(r^{-3}(\mathbf{a}\times\mathbf{r})\right) &= r^{-3}\operatorname{curl}(\mathbf{a}\times\mathbf{r}) + \operatorname{grad}(r^{-3}) \times (\mathbf{a}\times\mathbf{r}) \\ &= 2r^{-3}\mathbf{a} - 3r^{-5}\mathbf{r}\times(\mathbf{a}\times\mathbf{r}) \\ &= 2r^{-3}\mathbf{a} - 3r^{-5}\left((\mathbf{r}\cdot\mathbf{r})\mathbf{a} - (\mathbf{r}\cdot\mathbf{a})\mathbf{r}\right) \\ &= -r^{-3}\mathbf{a} + 3r^{-5}(\mathbf{a}\cdot\mathbf{r})\mathbf{r}. \end{aligned}$$

Hence the sum is the zero vector as required.

T4 Find the parametric form of

a) the line segment joining $P(-1,2,1)$ and $Q(4,2,0)$,

b) the line passing through a point $(0,2,2)$ with direction vector $(-1,0,1)$,

c) the curve $x = e^y$ from $(1,0)$ to $(e,1)$,

d) the curve $y = \sqrt{x}$ from $(1,1)$ to $(4,2)$.

For the line in (a), write the answer in component form, simplifying as far as possible.

Solution

a) The parametric form is $\mathbf{r} = (1-t)(-1,2,1) + t(4,2,0),\ t \in [0,1]$. In component form this is

$$x = 5t - 1,\ y = 2,\ z = 1 - t, \quad t \in [0,1].$$

b) The parametric form is $\mathbf{r} = (0,2,2) + t(-1,0,2),\ t \in [0,1]$.

c) The parametric form is $\mathbf{r} = (e^t, t),\ t \in [0,1]$.

d) The parametric form is $\mathbf{r} = (t, \sqrt{t}),\ t \in [1,4]$.

T5 Calculate the work done by a force $\mathbf{F}$ in moving a particle along the parametric curve $\mathbf{r}$ where,

a) $\mathbf{F}(x,y) = xy\mathbf{i} + x^3\mathbf{j}$, $\mathbf{r}(t) = t^{1/2}\mathbf{i} + t^{1/4}\mathbf{j}, \quad 1 \le t \le 16.$

b) $\mathbf{F}(x,y) = x^2\mathbf{i} + y^2\mathbf{j}$, $\mathbf{r}(t) = (1+t^2)\mathbf{i} + (2+\sin(\pi t))\mathbf{j}, \quad 0 \le t \le 1.$

Solution

a) $\frac{d\mathbf{r}}{dt} = ((1/2)t^{-1/2}, (1/4)t^{-3/4}), \quad \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = (1/2)t^{1/4)} + (1/4)t^{3/4}$, hence,

$$I = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^{16} (1/2)t^{1/4)} + (1/4)t^{3/4} dt = 1069/35.$$

b) $\frac{d\mathbf{r}}{dt} = (2t, \pi\cos(\pi t)), \quad \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 2t(1+t^2)^2 + \pi\cos(\pi t)(2+\sin(\pi t))^2$, hence,

$$I = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 2t(1+t^2)^2 + \pi\cos(\pi t)(2+\sin(\pi t))^2 dt = 7/3.$$

Note the integral of second term is 0.

Further Exercises

F1 Let f be a smooth scalar field and $\mathbf{F}$ a smooth vector field. Prove the identity

$$\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \operatorname{grad} f \cdot \mathbf{F}.$$

Solution

Let $\mathbf{F} = (F_1, F_2, F_3)$. Then

$$\begin{aligned}\operatorname{div}(f\mathbf{F}) = \nabla \cdot (f\mathbf{F}) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (fF_1, fF_2, fF_3) \\ &= \frac{\partial}{\partial x}(fF_1) + \frac{\partial}{\partial y}(fF_2) + \frac{\partial}{\partial z}(fF_3) \\ &= f\frac{\partial F_1}{\partial x} + f\frac{\partial F_2}{\partial y} + f\frac{\partial F_3}{\partial z} + \frac{\partial f}{\partial x}F_1 + \frac{\partial f}{\partial y}F_2 + \frac{\partial f}{\partial z}F_3 \\ &= f(\nabla \cdot \mathbf{F}) + (\nabla f) \cdot \mathbf{F} = f \operatorname{div} \mathbf{F} + \operatorname{grad} f \cdot \mathbf{F},\end{aligned}$$

as required.

F2 Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}|$, and $\mathbf{a}$ be a constant vector. Prove that

$$\text{(a) } \operatorname{div}\left((\mathbf{a}\cdot\mathbf{r})\mathbf{a}\right) = \mathbf{a}^2, \quad \text{(b) } \operatorname{curl}((\mathbf{a}\cdot\mathbf{r})\mathbf{a}) = \mathbf{0},$$

$$\text{(c) } \operatorname{div}\left((\mathbf{a}\cdot\mathbf{r})\mathbf{r}\right) = 4(\mathbf{a}\cdot\mathbf{r}), \quad \text{(d) } \operatorname{curl}\left((\mathbf{a}\cdot\mathbf{r})\mathbf{r}\right) = \mathbf{a}\times\mathbf{r},$$

$$\text{(e) } \operatorname{div}\left((\mathbf{a}\cdot\mathbf{r})(\mathbf{a}\times\mathbf{r})\right) = 0, \quad \text{(f) } \operatorname{curl}\left((\mathbf{a}\cdot\mathbf{r})(\mathbf{a}\times\mathbf{r})\right) = 3(\mathbf{a}\cdot\mathbf{r})\mathbf{a} - \mathbf{a}^2\mathbf{r},$$

$$\text{(g) } \operatorname{curl}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^2}\right) = \frac{2(\mathbf{r}\cdot\mathbf{a})}{r^4}\mathbf{r}, \quad \text{(h) } \operatorname{curl}\left(\mathbf{r}\times(\mathbf{a}\times\mathbf{r})\right) = 3(\mathbf{r}\times\mathbf{a}).$$

Solution

(a) Using the identity

$$\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \operatorname{grad} f \cdot \mathbf{F},$$

we get

$$\operatorname{div}((\mathbf{a}\cdot\mathbf{r})\mathbf{a}) = (\mathbf{a}\cdot\mathbf{r}) \operatorname{div} \mathbf{a} + \operatorname{grad}((\mathbf{a}\cdot\mathbf{r})) \cdot \mathbf{a} = (\mathbf{a}\cdot\mathbf{r})0 + \mathbf{a}\cdot\mathbf{a} = \mathbf{a}^2.$$

(b) Using the identity

$$\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl} \mathbf{F} - \mathbf{F} \times \operatorname{grad} f,$$

we get

$$\operatorname{curl}((\mathbf{a}\cdot\mathbf{r})\mathbf{a}) = (\mathbf{a}\cdot\mathbf{r}) \operatorname{curl} \mathbf{a} - \mathbf{a} \times \operatorname{grad}((\mathbf{a}\cdot\mathbf{r})) = \mathbf{0} - \mathbf{a}\times\mathbf{a} = \mathbf{0}.$$

(c)

$$\operatorname{div}((\mathbf{a}\cdot\mathbf{r})\mathbf{r}) = (\mathbf{a}\cdot\mathbf{r}) \operatorname{div} \mathbf{r} + \operatorname{grad}((\mathbf{a}\cdot\mathbf{r})) \cdot \mathbf{r} = (\mathbf{a}\cdot\mathbf{r})3 + \mathbf{a}\cdot\mathbf{r} = 4(\mathbf{a}\cdot\mathbf{r}).$$

(d)

$$\operatorname{curl}((\mathbf{a}\cdot\mathbf{r})\mathbf{r}) = (\mathbf{a}\cdot\mathbf{r}) \operatorname{curl} \mathbf{r} - \mathbf{r} \times \operatorname{grad}((\mathbf{a}\cdot\mathbf{r})) = \mathbf{0} - \mathbf{r}\times\mathbf{a} = \mathbf{a}\times\mathbf{r}.$$

(e)

$$\operatorname{div}((\mathbf{a}\cdot\mathbf{r})(\mathbf{a}\times\mathbf{r})) = (\mathbf{a}\cdot\mathbf{r}) \operatorname{div}(\mathbf{a}\times\mathbf{r}) + (\mathbf{a}\times\mathbf{r}) \cdot \operatorname{grad}((\mathbf{a}\cdot\mathbf{r})) = (\mathbf{a}\cdot\mathbf{r})0 - (\mathbf{a}\times\mathbf{r})\mathbf{a} = 0 - [\mathbf{a},\mathbf{r},\mathbf{a}] = 0.$$

(f)

$$\begin{aligned}\operatorname{curl}((\mathbf{a}\cdot\mathbf{r})(\mathbf{a}\times\mathbf{r})) &= (\mathbf{a}\cdot\mathbf{r}) \operatorname{curl}(\mathbf{a}\times\mathbf{r}) - (\mathbf{a}\times\mathbf{r}) \times \operatorname{grad}((\mathbf{a}\cdot\mathbf{r})) = (\mathbf{a}\cdot\mathbf{r})2\mathbf{a} - (\mathbf{a}\times\mathbf{r})\times\mathbf{a} \\ &= 2(\mathbf{a}\cdot\mathbf{r})\mathbf{a} + \mathbf{a}\times(\mathbf{a}\times\mathbf{r}) = 2(\mathbf{a}\cdot\mathbf{r})\mathbf{a} + (\mathbf{a}\cdot\mathbf{r})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{r} = 3(\mathbf{a}\cdot\mathbf{r})\mathbf{a} - \mathbf{a}^2\mathbf{r}.\end{aligned}$$

(g)

$$\operatorname{curl}(\frac{1}{r^2}(\mathbf{a}\times\mathbf{r})) = \frac{1}{r^2}\operatorname{curl}(\mathbf{a}\times\mathbf{r}) - (\mathbf{a}\times\mathbf{r})\times\operatorname{grad}(\frac{1}{r^2}) = \frac{1}{r^2}2\mathbf{a} - (\mathbf{a}\times\mathbf{r})\times(\frac{-2}{r^4}\mathbf{r})$$
$$= \frac{2}{r^2}\mathbf{a} - \frac{2}{r^4}\mathbf{r}\times(\mathbf{a}\times\mathbf{r}) = \frac{2}{r^2}\mathbf{a} - \frac{2}{r^4}(r^2\mathbf{a} - (\mathbf{r}\cdot\mathbf{a})\mathbf{r}) = \frac{2}{r^2}\mathbf{a} - \frac{2}{r^2}\mathbf{a} + \frac{2(\mathbf{r}\cdot\mathbf{a})}{r^4}\mathbf{r} = \frac{2(\mathbf{r}\cdot\mathbf{a})}{r^4}\mathbf{r}.$$

(h)

$$\operatorname{curl}(\mathbf{r}\times(\mathbf{a}\times\mathbf{r})) = \operatorname{curl}(r^2\mathbf{a} - (\mathbf{r}\cdot\mathbf{a})r) = \operatorname{curl}(r^2\mathbf{a}) - \operatorname{curl}(\mathbf{r}\cdot\mathbf{a})\mathbf{r}$$
$$= (r^2\operatorname{curl}\mathbf{a} - \mathbf{a}\times\operatorname{grad}(r^2)) - (\mathbf{a}\times\mathbf{r}) = \mathbf{0} - (\mathbf{a}\times 2\mathbf{r}) - (\mathbf{a}\times\mathbf{r}) = -3(\mathbf{a}\times\mathbf{r}) = 3(\mathbf{r}\times\mathbf{a}).$$

F3 Let $\mathbf{r} = (x,y,z)$, $r = |\mathbf{r}|$, $\lambda \in \mathbb{R}$ and $\mathbf{F} = \lambda r^{-3}\mathbf{r}$. Verify that $\nabla\times\mathbf{F} = \mathbf{0}$ at all points in $\mathbb{R}^3$ except for the origin.

Solution

Using the identity

$$\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + \operatorname{grad} f\times\mathbf{F},$$

we take $f = \lambda r^{-3}$ and $\mathbf{F} = \mathbf{r}$ to give

$$\nabla\times(\lambda r^{-3}\mathbf{r} = \lambda r^{-3}\operatorname{curl}\mathbf{r} + \operatorname{grad}\left(\lambda r^{-3}\right) = \quad \mathbf{0} + \lambda(-3)r^{-5}\mathbf{r} = \mathbf{0}.$$

Using $\operatorname{curl}\mathbf{r} = \mathbf{0}$ from T2(c) and $\operatorname{grad}(r^n) = nr^{n-2}\mathbf{r}$ from Ex 3.2 of the notes. Lastly since $\mathbf{r}$ is parallel to itself we have $\mathbf{r}\times\mathbf{r} = \mathbf{0}$.

F4 Write down the parametric equations of

a) the circle $x^2 + y^2 = 4$,

b) the circle in the xy-plane with centre $(1,0,0)$ and radius 1,

c) the parabola $x = y^2$

d) the ellipse $\frac{x^2}{4} + 9y^2 = 1,\ z = 1$.

Solution

(a) $x = 2\cos\theta,\ y = 2\sin\theta,\ \theta\in[0,2\pi)$.
(b) $x = 1+\cos\theta,\ y = \sin\theta,\ z = 0,\ \theta\in[0,2\pi)$.
(c) $x = t^2,\ y = t,\ t\in\mathbb{R}$.
(d) $x = 2\cos\theta,\ y = \frac{1}{3}\sin\theta,\ z = 1,\ \theta\in[0,2\pi)$.

F5 Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for the given vector field $\mathbf{F}$ and parametric curve $\mathbf{r}$,

a) $\mathbf{F}(x,y) = xy\mathbf{i} + y^2\mathbf{j}$, $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $\quad 0\le t\le\pi/3$.

b) $\mathbf{F}(x,y) = ln(y)\mathbf{i} - e^x\mathbf{j}$, $\mathbf{r}(t) = \ln(t)\mathbf{i} + t^3\mathbf{j}$, $\quad 0 \leq t \leq e$.

c) $\mathbf{F}(x,y) = -xy\mathbf{i} + \left(x^2+1\right)^{-1}\mathbf{j}$, $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, $\quad -4 \leq t \leq -1$.

Solution

a) $\frac{d\mathbf{r}}{dt} = (-\sin t, \cos t)$, $\quad \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 0$, hence,

$$I = \int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

b) $\frac{d\mathbf{r}}{dt} = (1/t, 3t^2)$, $\quad \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{3}{t}\ln(t) - 3t^3$, hence,

$$I = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^e \frac{3}{t}\ln(t) - 3t^3 dt = \left[-(3/4)t^4 + (1/2)(\ln t)^2\right]_1^e = 3/2 - (3/4)e^4 + 3/4 = 9/3 - (3/4)e^4.$$

c) $\frac{d\mathbf{r}}{dt} = (1, 2t)$, $\quad \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{2t}{t^2+1} - t^3$, hence,

$$I = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-4}^{-1} \frac{2t}{t^2+1} - t^3 dt = \left[-\frac{t^4}{4} + \ln|t^2+1|\right]_{-4}^{-1} = \frac{255}{4} + \ln\left(\frac{2}{17}\right).$$