2A Multivariable Calculus 2013

Ex Sheet 5

Tutorial Exercises

T1 Evaluate the following using beta functions:

(a) $\displaystyle\int_0^{\pi/2} \sin^3 x \cos^2 x \, dx$, (b) $\displaystyle\int_0^{\pi/2} \sin^7 x \cos^3 x \, dx$, (c) $\displaystyle\int_0^{\pi} \cos^5 x \, dx$,

(d) $\displaystyle\int_0^{\pi/2} \sin^4 x \cos^2 x \, dx$, (e) $\displaystyle\int_0^{\pi} \sin^5 x \, dx$, (f) $\displaystyle\int_0^{\pi} \sin^2 x \cos^3 x \, dx$,

(g) $\displaystyle\int_0^{2\pi} \sin^3 x \cos^3 x \, dx$, (h) $\displaystyle\int_0^{2\pi} \sin^4 x \cos^3 x \, dx$,

Solution

(a) $\frac{2}{15}$, (b) $\frac{1}{40}$, (c) 0, (d) $\frac{\pi}{32}$, (e) $\frac{16}{15}$, (f) 0, (g) 0, (h) 0.

T2 Given the change of variables

$$u = \frac{1}{3}(x+y) \qquad v = \frac{1}{3}(x-2y)$$

express x and y in terms of u and v.

Solution

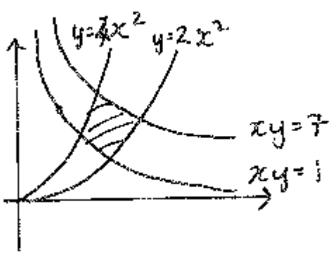
$x = 2u + v$, $y = u - v$.

T3 By making a suitable change of variables, evaluate

$$\iint xy \, dx \, dy$$

over the region enclosed by the two hyperbolas $xy = 1$ and $xy = 7$ and the two parabolas $y = 2x^2$ and $y = 4x^2$.

Solution



Let $u = xy$ and $v = y/x^2$, so $1 \leq u \leq 7$ and $2 \leq v \leq 4$. The Jacobian is

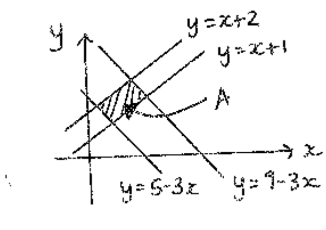
$$\frac{\partial(u,v)}{\partial(x,y)} = (y).(1/x^2) - x.(-2y/x^3) = 3y/x^2,$$

and so $\dfrac{\partial(x,y)}{\partial(u,v)} = x^2/3y$. Hence the integral is

$$I = \int_1^7 du \int_2^4 xy.\left|\frac{x^2}{3y}\right| dv = \frac{1}{3}\int_1^7 du \int_2^4 \frac{u}{v}\, dv$$
$$= \frac{1}{3}\int_1^7 u\, du \int_2^4 \frac{1}{v}\, dv = \frac{1}{3}\left[\frac{u^2}{2}\right]_1^7 [\log v]_2^4$$
$$= \frac{1}{6}.48.(\log 4 - \log 2) = 8\log 2.$$

T4 Use double integration and an appropriate change of variables to find the area of the parallelogram enclosed by the lines $y = x+1$, $y = x+2$, $y = 5-3x$, $y = 9-3x$

Solution



Let $u = y - x$ and $v = y + 3x$, so $1 \le u \le 2$ and $5 \le v \le 9$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1).(1) - 1.(3) = -4,$$

and so $\dfrac{\partial(x,y)}{\partial(u,v)} = -1/4$. Hence this area, A, is

$$\iint_A dA = \int_1^2 du \int_5^9 1.\left|\frac{-1}{4}\right| dv = \frac{1}{4}[u]_1^2[v]_5^9 = 1.$$

T5 Evaluate

$$\iiint yz^2\, dxdydz$$

throughout the cube given by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

Solution

$$I = \int_0^1 dx \int_0^1 dy \int_0^1 yz^2\, dz = [x]_0^1 [\tfrac{1}{2}y^2]_0^1 [\tfrac{1}{3}z^3]_0^1 = \frac{1}{6}.$$

Further Exercises

F1 Evaluate the following using beta functions:

$$\text{(a)} \int_0^{\pi/2} \cos^6 x\, dx, \quad \text{(b)} \int_0^{\pi/2} \sin^2 x\, dx, \text{ (c)} \int_0^{\pi/2} \sin x \cos x\, dx,$$
$$\text{(d)} \int_0^{\pi/2} \sin^7 x\, dx, \quad \text{(e)} \int_0^{\pi} \sin^6 x\, dx, \quad \text{(f)} \int_0^{\pi} \sin^2 x \cos^3 x\, dx,$$
$$\text{(g)} \int_0^{\pi} \cos^6 x\, dx \quad \text{(h)} \int_0^{\pi} \sin^3 x \cos^2 x\, dx, \quad \text{(i)} \int_0^{\pi} \sin^4 x \cos^4 x\, dx,$$
$$\text{(j)} \int_0^{2\pi} \cos^4 x\, dx, \text{(k)} \int_0^{2\pi} \sin^3 x \cos^2 x\, dx, \quad \text{(l)} \int_0^{2\pi} \sin^2 x \cos^6 x\, dx,$$

Solution

(a) $\frac{5\pi}{32}$, (b) $\frac{\pi}{4}$, (c) $\frac{1}{2}$, (d) $\frac{16}{35}$, (e) $\frac{5\pi}{16}$, (f) 0, (g) $\frac{5\pi}{16}$, (h) $\frac{4}{15}$, (i) $\frac{3\pi}{128}$, (j) $\frac{3\pi}{4}$, (k) 0, (l) $\frac{5\pi}{64}$,

F2 Evaluate the integral

$$\int_0^3 dx \int_{x/4}^{x/4+2} \left(\frac{x+y}{4}\right)\, dy,$$

using the change of variables $u = \frac{x}{4}, \quad v = \frac{x+y}{2}$.

Solution

Upon rearranging we have $x = 4u$ and $y = 2v - 4u$. Since $0 \le x \le 3$, we then have $0 \le u \le 3/4$. Similarly, given $x/4 \le y \le x/4 + 2$ we obtain $\frac{5}{2}u \le v \le \frac{5}{2}u + 1$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (\frac{1}{4})(\frac{1}{2}) - \frac{1}{2}(0) = \frac{1}{8},$$

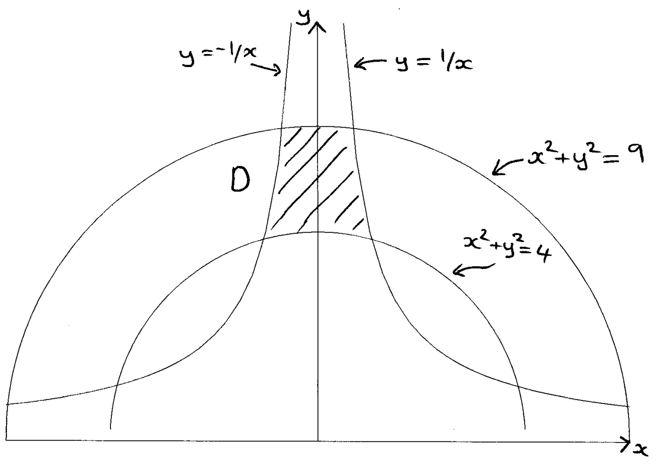
and so $\dfrac{\partial(x,y)}{\partial(u,v)} = 8$. Hence the integral is

$$I = \int_0^{\frac{3}{4}} du \int_{\frac{5}{2}u}^{\frac{5}{2}u+1} \frac{v}{2}.|8|\, dv == 2\int_0^{\frac{3}{4}} 1 + 5u\, du = \frac{69}{16}$$

F3 Evaluate

$$\iint_D x^4 - y^4\, dxdy$$

where D is the region illustrated below.



Solution

Let $u = x^2 + y^2$ and $v = xy$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (2x).x - (2y).y = 2(x^2 - y^2),$$

and so $\dfrac{\partial(x,y)}{\partial(u,v)} = 1/(2(x^2 - y^2))$. Now, since in D $y > |x|$ and hence $x^2 - y^2 < 0$,

$$(x^4 - y^4)\left|\frac{1}{2(x^2 - y^2)}\right| = -\frac{x^2 + y^2}{2} = -\frac{u}{2},$$

hence the integral is

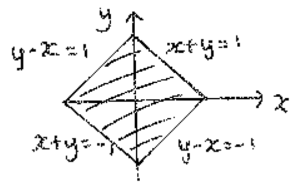
$$-\iint_D \frac{1}{2} u \, du dv = -\int_4^9 \left(\int_{-1}^{1} \frac{1}{2} u \, dv\right) du = -\int_4^9 u \, du = -\frac{65}{2}.$$

F4 By making a suitable change of variables, evaluate

$$\iint x^2 \, dx \, dy$$

over the square enclosed by the lines $x + y = -1$, $x + y = 1$, $y - x = -1$, $y - x = 1$.

Solution



Let $u = x + y$ and $v = y - x$, so $-1 \leq u \leq 1$ and $-1 \leq v \leq 1$. Solving for x and y in terms of u and v we get $x = (u - v)/2$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (1).(1) - 1.(-1) = 2,$$

and so $\dfrac{\partial(x,y)}{\partial(u,v)} = 1/2$. Hence the integral is

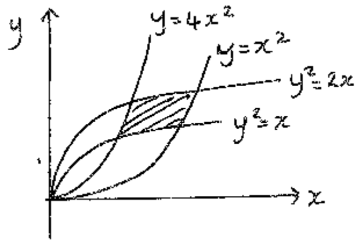
$$I = \int_{-1}^{1} du \int_{-1}^{1} x^2 . \left|\frac{1}{2}\right| dv = \frac{1}{2}\int_{-1}^{1} du \int_{-1}^{1} (\frac{1}{2}(u-v))^2 \, dv$$
$$= \frac{1}{8}\int_{-1}^{1} du \int_{-1}^{1} u^2 - 2uv + v^2 \, dv = \frac{1}{8}\int_{-1}^{1} [u^2 v - uv^2 + \frac{v^3}{3}]_{-1}^{1} du$$
$$= \frac{1}{8}\int_{-1}^{1} 2u^2 - \frac{2}{3} \, du = \frac{1}{8}\left[2\frac{u^3}{3} + \frac{2}{3}u\right]_{-1}^{1} = \frac{1}{3}.$$

F5 By making a suitable change of variables, evaluate

$$\iint \frac{y^2}{x} \, dxdy$$

over the region in the first quadrant enclosed by the four parabolas $y^2 = x$, $y^2 = 2x$, $y = x^2$, $y = 4x^2$

Solution



Let $u = y^2/x$ and $v = y/x^2$, so $1 \le u \le 2$ and $1 \le v \le 4$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (-y^2/x^2).(1/x^2) - (2y/x).(-2y/x^3) = 3y^2/x^4,$$

and so $\dfrac{\partial(x,y)}{\partial(u,v)} = x^4/(3y^2)$. Hence the integral is

$$I = \int_1^2 du \int_1^4 \frac{y^2}{x} . \left|\frac{x^4}{3y^2}\right| dv = \frac{1}{3}\int_1^2 du \int_1^4 x^3 \, dv.$$

Now, $y = vx^2$, substituting this into the expression for u gives, $u = v^2x^4/x = v^2x^3$. Hence, $x^3 = u/v^2$. Thus,

$$I = \frac{1}{3}\int_1^2 du \int_1^4 \frac{u}{v^2} \, dv = \frac{1}{3}[\frac{u^2}{2}]_1^2[-\frac{1}{v}]_1^4 = \frac{3}{8}.$$

F6 Evaluate $\iint_R (x^2 + y^2) dA$, where R is the parallelogram with vertices $(0,0)$, $(2,0)$, $(3,1)$ and $(1,1)$.

Solution

The parallelogram is bounded by the lines $y = 0$, $y = 1$, $y = x$ and $y = x - 2$. Letting $u = x - y$ and $v = y$ the domain can be described by $0 \leq u \leq 2$ and $0 \leq v \leq 1$.

Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (1).(1) - 0.(-1) = 1,$$

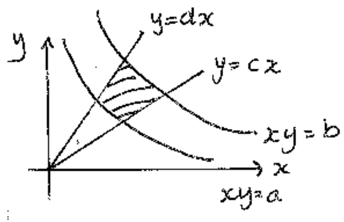
and so $\dfrac{\partial(x,y)}{\partial(u,v)} = 1$. Hence the integral is

$$\iint_R dA = \int_0^1 dv \int_0^2 (u+v)^2 + v^2 \,|1|\, du = \int_0^1 \left[\frac{1}{3}(u+v)^3 + v^2 u\right]_0^2 dv = 6.$$

F7 Show that the area of the region in the first quadrant enclosed by the two hyperbolas $xy = a$, $xy = b$ and the two lines $y = cx$, $y = dx$, where $b > a > 0$ and $d > c > 0$ is

$$\frac{1}{2}(b-a)\log\left(\frac{d}{c}\right).$$

Solution



Let $u = xy$ and $v = y/x$, so $a \leq u \leq b$ and $c \leq v \leq d$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = (y).(1/x) - x.(-y/x^2) = 2y/x,$$

and so $\dfrac{\partial(x,y)}{\partial(u,v)} = x/(2y)$. Hence this area, A, is

$$\iint_A dA = \int_a^b du \int_c^d 1.\left|\frac{x}{2y}\right| dv = \frac{1}{2}\left[u\right]_a^b \left[\log v\right]_c^d = \frac{1}{2}(b-a)\log(d/c).$$

F8 Use change of variables to evaluate

$$\iint x^4 + y^4 \, dxdy$$

over the interior of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution

Let $x = au$ and $y = bv$, so the ellipse becomes $u^2 + v^2 \leq 1$ and so $0 \leq u \leq 1$ and $0 \leq v \leq 1$. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = (a).(b) - 0.(0) = ab,$$

Hence the integral is

$$\begin{aligned}
I &= \int\int_{u^2+v^2 \leq 1} (a^4u^4 + b^4v^4)ab\, dudv \\
&= ab(a^4 + b^4) \int_0^{2\pi} d\theta \int_0^1 r^5 \cos^4\theta\, dr, \quad \text{(By symmetry } \int\int_{\text{circle}} u^4\, dudv = \int\int_{\text{circle}} v^4\, dudv\text{)} \\
&= ab(a^4 + b^4) \int_0^{2\pi} \cos^4\theta\, d\theta \left[\frac{r^6}{6}\right]_0^1 \\
&= ab(a^4 + b^4) 4 \frac{3.1}{4.2} \frac{\pi}{2} \frac{1}{6} = \frac{\pi ab}{8}(a^4 + b^4).
\end{aligned}$$

F9 Evaluate

$$\int_3^5 dx \int_1^4 dy \int_1^2 xy\, dz.$$

Solution

$$I = \int_5^3 dx \int_1^4 [xyz]_1^2\, dy = \int_5^3 [\frac{1}{2}xy^2]_1^4\, dx = [\frac{15}{4}x^2]_3^5 = 60.$$