2A Multivariable Calculus 2013

Ex Sheet **7**

Tutorial Exercises

T1 Find the directional derivative of xyz^2 at the point $(1,5,1)$ in the direction of the vector $(1,-1,2)$.

Solution

The unit vector in the direction of $(1,-1,2)$ is $\mathbf{n} = (1,-1,2)/\sqrt{6}$ and $\nabla\phi = (yz^2, xz^2, 2xyz)(1,5,1) = (5,1,10)$ at $P(1,5,1)$. Therefore the directional derivative is

$$\frac{\partial\phi}{\partial n} = \nabla\phi(1,5,1)\cdot\mathbf{n} = \frac{1}{\sqrt{6}}(1,-1,2)\cdot(5,1,10) = \frac{5-1+20}{\sqrt{6}} = \frac{24}{\sqrt{6}} = 4\sqrt{6}.$$

T2 Let f be a scalar field, $\mathbf{u}$ a unit vector and let θ be the angle between $\mathbf{u}$ and ∇f evaluated at some point P.

a) Show that the directional derivative of f at P in the direction of vector $\mathbf{u}$ is $|\nabla f|\cos\theta$.

b) Deduce that the directional derivative of f at P in the direction of $\mathbf{u}$ is a maximum when $\mathbf{u}$ has the same direction as ∇f. When is this directional derivative a minimum?

c) In what directions from the point $P(1,3,2)$ is the directional derivative of $f = xyz - y^2z$ a maximum and a minimum respectively? Find these directional derivatives.

d) The temperature at a point $P(x,y,z)$ in space is given by $T = x^2 + y^2 - z$. In what direction should an insect at $P(1,1,2)$ move so that it warms up as rapidly as possible?

Solution

(a) The directional derivative is

$$\nabla F\cdot\mathbf{u} = |\nabla F||\mathbf{u}|\cos\theta = |\nabla F|\cos\theta,$$

since $\mathbf{u}$ is a unit vector.

(b) This directional derivative is a maximum when $\cos\theta$ is a maximum, that is when $\cos\theta = 1$. Hence the maximum occurs when the angle between $\mathbf{u}$ and ∇f (at P)is zero, i.e., they have the same direction. The minimum occurs when $\cos\theta = -1$ which means that $\mathbf{u}$ and ∇f have opposite directions.

(c) We have $\operatorname{grad} f = (yz, xz - 2yz, xy - y^2) = (6,-10,-6)$ at $P(1,3,2)$ and so the maximum directional derivative occurs in the direction of $(3,-5,-3)$ and the minimum in the direction of

$(-3, 5, 3)$. The unit vector having the same directions are

$$\pm\frac{(3, -5, -3)}{\sqrt{43}}.$$

Hence the maximum/minimum directional derivative are

$$\pm\frac{1}{\sqrt{43}}(6, -10, -6) \cdot (3, -5, -3) = \pm\frac{86}{\sqrt{43}} = \pm 2\sqrt{43}.$$

(d) We have $\nabla T = (2x, 2y, -1) = (2, 2, -1)$ at $P(1, 1, 2)$. Hence the temperature in the surroundings increases most rapidly in the direction $(2, 2, -1)$. This is the direction the insect should move in.

T3 A scalar field f is called harmonic if the Laplacian of the scalar field is zero. Show the following scalar fields are harmonic.

$$\text{(a) } u(x, y, z) = e^{(x+y)}\cos(\sqrt{2}z), \quad \text{(b)} \quad v(x, y) = x^2 - y^2.$$

Solution

(a)

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = e^{(x+y)}\cos(\sqrt{2}z) + e^{(x+y)}\cos(\sqrt{2}z) + (-(\sqrt{2})^2 e^{(x+y)}\cos(\sqrt{2}z)) = 0.$$

(b)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + (-2) = 0$$

T4 Find the divergence and curl of the vector fields

$$\text{(a) } \mathbf{F} = (3xyz^2, 2xy^3, -x^2yz), \quad \text{(b) } \mathbf{G} = (e^{xz}, x^2 + y^2, yz),$$

at an arbitrary point and at $P(1, 1, 1)$.

Solution

(a) $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = 3yz^2 + 6xy^2 - x^2y = 8$ at $(1, 1, 1)$ and

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3xyz^2 & 2xy^3 & -x^2yz \end{vmatrix} = (-x^2z - 0)\mathbf{i} + (6xyz + 2xyz)\mathbf{j} + (2y^3 - 3xz^2)\mathbf{k}$$

$$= (-x^2z, 8xyz, 2y^3 - 3xz^2) = (-1, 8, -1) \text{ at } (1, 1, 1).$$

(b) $\operatorname{div}\mathbf{G} = \nabla\cdot\mathbf{G} = ze^{xz} + 2y + y = ze^{xz} + 3y = e + 3$ at $(1,1,1)$ and

$$\operatorname{curl}\mathbf{G} = \nabla\times\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{xz} & x^2+y^2 & yz \end{vmatrix} = (z-0)\mathbf{i} + (xe^{xz}-0)\mathbf{j} + (2x-0)\mathbf{k}$$
$$= (z, xe^{xz}, 2x) = (1, e, 2) \text{ at } (1,1,1).$$

T5 Which of the following vector fields are irrotational?

a) $\mathbf{F} = (yz, xz, xy)$,

b) $\mathbf{G} = \sin xy\,\mathbf{i} + \cos yz\,\mathbf{j} + \sin xz\,\mathbf{k}$,

c) $\mathbf{H} = y^2z\,\mathbf{i} + 2xyz\,\mathbf{j} + xy^2\,\mathbf{k}$.

Solution

(a) $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x-x)\mathbf{i} + (y-y)\mathbf{j} + (z-z)\mathbf{k} = \mathbf{0}$. Therefore $\mathbf{F}$ is irrotational.

(b) $\operatorname{curl}\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin xy & \cos yz & \sin xz \end{vmatrix} = (0 + y\sin(yz))\mathbf{i} - (z\cos(xz) - 0)\mathbf{j} + (0 - x\cos(xy))\mathbf{k} \neq \mathbf{0}$.

Therefore $\mathbf{G}$ is not irrotational.

(c) $\operatorname{curl}\mathbf{H} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z & 2xyz & xy^2 \end{vmatrix} = (2xy - 2xy)\mathbf{i} + (y^2 - y^2)\mathbf{j} + (2yz - 2yz)\mathbf{k} = \mathbf{0}$. Therefore $\mathbf{H}$ is irrotational.

Further Exercises

F1 Give two examples from the natural world of (i) a scalar field, (ii) a vector field.

Solution

Scalar fields include height above sea-level or temperature or air pressure as a function of location on the surface of the earth. Vector fields include velocity as a function of position in fluid flow, gravitational force or magnetic force as a function of location in space.

F2 Find the directional derivative of

a) $f = e^{2x-y+z}$ at $P(1,1,-1)$ in the direction $\mathbf{d} = (-1,-3,-5)$;

b) $f = x^3 + 3xy - 3yz + z^3$ at $P(1,2,1)$ in the direction $\mathbf{d} = (1,4,3)$;

c) $f = \sin xy + \log yz$ at $P(\pi,1,2)$ in the direction $\mathbf{d} = (0,1,2)$.

Solution

(a) The unit vector in the direction of $\mathbf{d}$ is $\hat{\mathbf{d}} = \mathbf{d}/\sqrt{35}$ and $\nabla f = e^{2x-y+z}(2,-1,1) = (2,-1,1)$ at $P(1,1,-1)$. Therefore the directional derivative is

$$\nabla f(1,1,-1)\cdot\hat{\mathbf{d}} = \frac{1}{\sqrt{35}}(2,-1,1)\cdot(-1,-3,-5) = \frac{-2+3-5}{\sqrt{35}} = -\frac{4}{\sqrt{35}}.$$

(b) $\hat{\mathbf{d}} = \mathbf{d}/\sqrt{26}$ and $\nabla f = (3x^2+3y, 3x-3z, -3y+3z^2) = (9,0,-3)$ at $P(1,2,1)$. Therefore

$$\nabla f(1,2,1)\cdot\hat{\mathbf{d}} = \frac{1}{\sqrt{26}}(9,0,-3)\cdot(1,4,3) = 0.$$

(c) $\hat{\mathbf{d}} = \mathbf{d}/\sqrt{5}$ and $\nabla f = (y\cos xy, x\cos xy + 1/y, 1/z) = (-1,-\pi+1,\frac{1}{2})$ at $P(\pi,1,2)$. Therefore

$$\nabla f(\pi,1,2)\cdot\hat{\mathbf{d}} = \frac{1}{\sqrt{5}}(-1,-\pi+1,\tfrac{1}{2})\cdot(0,1,2) = \frac{2-\pi}{\sqrt{5}}.$$

F3 Find the directional derivative of $xy + 3yz$ at the point (0, 3, - 2) in the direction of each of the vectors

$$\text{(a) } (2,2,-1), \quad \text{(b) } (1,0,1), \quad \text{(c) } (4,-7,-4).$$

What are the maximum and minimum values of the directional derivative at $(0,3,-2)$ and in which directions do they occur?

Solution

$\text{grad}(xy+3yz) = (y, x+3z, 3y) = (3,-6,9)$ at $(0,3,-2)$.
(a) The unit vector is $\mathbf{n} = (2,2,-1)/3$. Therefore the directional derivative is

$$\frac{\partial\phi}{\partial n} = \nabla\phi(0,3,-2)\cdot\mathbf{n} = (3,-6,9)\cdot\frac{1}{3}(2,2,-1) = \frac{6-12-9}{3} = -5.$$

(b) The unit vector is $\mathbf{n} = (1,0,1)/\sqrt{2}$. Therefore the directional derivative is

$$\frac{\partial\phi}{\partial n} = \nabla\phi(0,3,-2)\cdot\mathbf{n} = (3,-6,9)\cdot\frac{1}{\sqrt{2}}(1,0,1) = \frac{3+9}{\sqrt{2}} = 6\sqrt{2}.$$

(c) The unit vector is $\mathbf{n} = (4,-7,-4)/9$. Therefore the directional derivative is

$$\frac{\partial\phi}{\partial n} = \nabla\phi(0,3,-2)\cdot\mathbf{n} = (3,-6,9)\cdot\frac{1}{9}(4,-7,-4) = \frac{12+42-36}{9} = 2.$$

We want the unit vectors $\mathbf{n}$ such that at $(0,3,-2)$ $\frac{\partial\phi}{\partial n}$ has its maximum and minimum values. Using the definition of dot product, $\frac{\partial\phi}{\partial n}(0,3,-2) = \nabla\phi(0,3,-2)\cdot\mathbf{n} = |\nabla\phi(0,3,-2)||\mathbf{n}|\cos\theta = 1.\sqrt{9+36+81}\cos\theta = \sqrt{126}\cos\theta$. The angle θ is between the two vectors $(3,-6,9)$ and $\mathbf{n}$ So the maximum occurs when $\cos\theta = 1$, i.e. $\theta = 0$, which is when $\mathbf{n}$ is parallel to grad ϕ. The maximum value of the directional derivative is therefore $\sqrt{126}$.

Similarly, the minimum occurs when $\cos\theta = -1$, i.e. $\theta = \pi$, which is when $\mathbf{n}$ lies in the direction $(-3, 6, -9)$, which is the direction opposite to grad ϕ. The minimum value of the directional derivative is therefore $-\sqrt{126}$.

F4 The temperature at the point (x, y, z) is given by

$$T(x, y, z) = (x + 3y)z^2.$$

Find the direction in which you should move from the point $(2, 2, 1)$ in order to achieve (a) the most rapid increase in temperature, (b) the most rapid decrease in temperature.

Solution

grad $T = (z^2, 3z^2, 2(x + 3y)z) = (1, 3, 16)$ at $(2, 2, 1)$.
$\frac{\partial\phi}{\partial n}(1, 3, 16) = \nabla\phi(1, 3, 16) \cdot \mathbf{n} = |\nabla\phi(1, 3, 16)||\mathbf{n}| \cos\theta = 1.\sqrt{266}\cos\theta = \sqrt{266}\cos\theta$. The angle θ is between the two vectors $(1, 3, 16)$ and $\mathbf{n}$ So the maximum (the fastest increase of ϕ) occurs when $\cos\theta = 1$, i.e. $\theta = 0$, which is when $\mathbf{n}$ is parallel to grad ϕ, so is in the direction $(1, 3, 16)$.

Similarly, the minimum (the fastest decrease of ϕ) occurs when $\cos\theta = -1$, i.e. $\theta = \pi$, which is when $\mathbf{n}$ lies in the direction $(-1, -3, -16)$, which is the direction opposite to grad ϕ.

F5 Calculate the divergence of the vector field $\mathbf{F}$ and state whether the vector field is incompressible.

a) $\mathbf{F} = (z\ln(x), yz/x, z^2/x)$,

b) $\mathbf{F} = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$,

c) $\mathbf{F} = x^2\sin(y)(\mathbf{i} - \mathbf{j} + \mathbf{k})$.

Solution

(a) div $\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{z}{x} + \frac{z}{x} + \frac{2z}{x} = \frac{4z}{x}$. $\mathbf{F}$ is NOT incompressible.

(b) div $\mathbf{F} = \nabla\cdot\mathbf{F} = 0 + 0 + 0 = 0$. $\mathbf{F}$ is incompressible.

(c) div $\mathbf{F} = \nabla\cdot\mathbf{F} = 2x - x^2\cos y + 0 = 2x - x^2\cos y$. $\mathbf{F}$ is NOT incompressible.

F6 Calculate the curl of the vector field $\mathbf{F}$ and state whether the vector field is irrotational.

a) $\mathbf{F} = xz\mathbf{i} - y^3\mathbf{j} + xyz\mathbf{k}$,

b) $\mathbf{F} = \cos^2(x)\mathbf{i} - \sin(y)\mathbf{j} + z^4\mathbf{k}$,

c) $\mathbf{F} = \ln(x + z)\mathbf{i} - e^{y^2}\mathbf{j} + xy\mathbf{k}$).

Solution

(a) $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & -y^3 & xyz \end{vmatrix} = (xz-0)\mathbf{i} + (x-yz)\mathbf{j} + (0-0)\mathbf{k} = (xz, -yz, 0)$. Therefore $\mathbf{F}$ is NOT irrotational.

(b) $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \cos^2 x & -\sin y & z^4 \end{vmatrix} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (0-0) = \mathbf{0}$. Therefore $\mathbf{F}$ is irrotational.

(c) $\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \ln(x+z) & -e^{yz} & xy \end{vmatrix} = (x+ye^{yz})\mathbf{i} + (\dfrac{1}{x+z} - y)\mathbf{j} + (0-0)\mathbf{k} = (x+ye^{yz}, \dfrac{1-xy-zy}{x+z}, 0)$.

Therefore $\mathbf{F}$ is NOT irrotational.

F7 Let $\mathbf{F} = (x^2y, yz, x+z)$. Find

(i) $\operatorname{curl}\operatorname{curl}\mathbf{F}$, (ii) $\operatorname{grad}\operatorname{div}\mathbf{F}$.

Solution

We have

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & yz & x+z \end{vmatrix} = (0-y, 0-1, 0-x^2) = -(y, 1, x^2),$$

and

$$\operatorname{div}\mathbf{F} = \frac{\partial(x^2y)}{\partial x} + \frac{\partial(yz)}{\partial y} + \frac{\partial(x+z)}{\partial z} = 2xy + z + 1.$$

Hence (i),

$$\operatorname{curl}\operatorname{curl}\mathbf{F} = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & 1 & x^2 \end{vmatrix} = -(0-0, 0-2x, 0-1) = (0, 2x, 1),$$

and (ii),

$$\operatorname{grad}\operatorname{div}\mathbf{F} = \left(\frac{\partial(2xy+z+1)}{\partial x}, \frac{\partial(2xy+z+1)}{\partial y}, \frac{\partial(2xy+z+1)}{\partial z}\right) = (2y, 2x, 1).$$