2A Multivariable Calculus 2013

Ex Sheet **10**

Tutorial Exercises

T1 In $\mathbb{R}^3$ let S be the part of the plane $4x + 2y - z = 37$ enclosed within the infinite cylinder with rectangular section defined by $0 \leq x \leq 5$, $0 \leq y \leq 2$. Evaluate

$$\iint_S 2y \, dS.$$

Solution

We need to calculate $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$. Differentiating the equation for the plane gives:

$$4 - \frac{\partial z}{\partial x} = 0, \quad \text{hence, } \frac{\partial z}{\partial x} = 4, \; 2 - \frac{\partial z}{\partial y} = 0, \quad \text{hence, } \frac{\partial z}{\partial y} = 2.$$

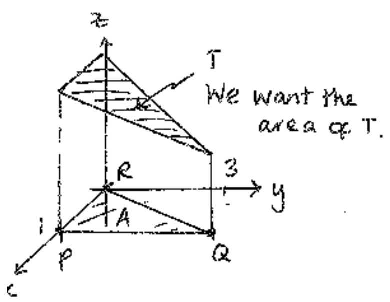
$$\text{Therefore, } \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + 16 + 4} = \sqrt{21}.$$

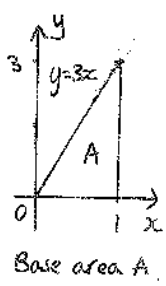
So,

$$I = \iint_A 2y\sqrt{21}\, dxdy = \sqrt{21} \int_0^5 dx \int_0^2 2y \, dy = \sqrt{21} \, [x]_0^5 \left[y^2\right]_0^2 = 20\sqrt{21}.$$

T2 In $\mathbb{R}^3$ let S be the part of the plane $2x + y + 6z = 55$ that is enclosed within the infinite cylinder with triangular cross section determined by the planes $y = 0$, $x = 1$ and $y = 3x$. Using a surface integral find the area of the triangle in which the plane $2x + y + 6z = 55$ meets this cylinder.

Solution





We need to calculate $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$. Differentiating the equation $2x + y + 6z = 55$ gives:

$$2 + 6\frac{\partial z}{\partial x} = 0, \quad \text{hence, } \frac{\partial z}{\partial x} = -1/3, \quad 1 + 6\frac{\partial z}{\partial y} = 0, \quad \text{hence, } \frac{\partial z}{\partial y} = -1/6.$$

$$\text{Therefore, } \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+1/9+1/36}=\sqrt{41}/6,$$

So,

$$\begin{aligned}\text{Area of Triangle T } &= \iint_T 1\,dS = \iint_A \frac{\sqrt{41}}{6}\,dxdy == \frac{\sqrt{41}}{6}\times\text{Area of triangle PQR}\\ &= \frac{\sqrt{41}}{6}\times\frac{1}{2}\text{base}\times\text{height} = \frac{\sqrt{41}}{4}.\end{aligned}$$

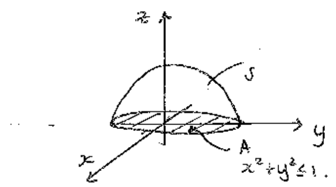
Note A is base defined by the triangle PQR.

T3 Evaluate

$$\iint_S z\,dS,$$

where S is the hemispherical surface given by $x^2+y^2+z^2=1$, $z\geq 0$.

Solution



We need to calculate $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$. Differentiating the equation $x^2+y^2+z^2=1$ gives:

$$2x+2z\frac{\partial z}{\partial x}=0, \quad \text{hence, } \frac{\partial z}{\partial x}=-x/z,\ 2y+2z\frac{\partial z}{\partial y}=0, \quad \text{hence, } \frac{\partial z}{\partial y}=-y/z.$$

$$\text{Therefore, } \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}=1/|z|=1/z,$$

since $z\geq 0$. So,

$$I=\iint_S z\,dS=\iint_A z\frac{1}{z}\,dxdy = \text{ Area of a circle radius 1} = \pi.$$

Note A is obtained by putting $z=0$ into S, thus we have A is a disc of radius 1.

T4 Use Gauss's Divergence Theorem to evaluate

$$\iint_S x^4+y^4+z^4\,dS,$$

where S is the entire surface of the sphere $x^2+y^2+z^2=1$. (You will have to write the integrand as $\mathbf{F}\cdot\mathbf{n}$ for a suitable $\mathbf{F}$ and for the unit normal $\mathbf{n}$.)

Solution

The outward pointing unit normal is $\mathbf{n} = (x, y, z)$. Thus

$$x^4 + y^4 + z^4 = \mathbf{F} \cdot n = (x^3, y^3, z^3) \cdot (x, y, z).$$

Applying Gauss's Divergence Theorem we have,

$$\begin{aligned} I &= \iiint_V \operatorname{div}(x^3, y^3, z^3)\, dxdydz = 3 \iiint_V x^2 + y^2 + z^2\, dxdydz \\ &= 3 \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^1 \rho^4 \sin\phi\, d\rho = 6\pi \int_0^{\pi} \sin\phi\, d\phi \int_0^1 \rho^4\, d\rho \\ &= 6\pi.2.\frac{1}{1}.\left[\frac{\rho^5}{5}\right]_0^1 = \frac{12\pi}{15}. \end{aligned}$$

T5 A closed surface is made up of the cylinder $(x-1)^2 + y^2 = 1$ with $z \geq 0$ and $z \leq 3$. Use the Divergence Theorem to evaluate

$$\iint_S \mathbf{v} \cdot \mathbf{n}\, dS,$$

where $\mathbf{v} = (xy, y^2 + e^{xz^2}, \sin(xy))$ and $\mathbf{n}$ is the outward pointing unit normal.

Solution

Let V be the volume contained by S. Then applying the Divergence Theorem,

$$\begin{aligned} I &= \iiint_V \operatorname{div}(xy, y^2 + e^{xz^2}, \sin(xy))\, dxdydz = \iint_{\{(x,y):(x-1)^2+y^2<1\}} dxdy \int_0^3 3y\, dz \\ &= \iint_{\{(x,y):(x-1)^2+y^2<1\}} 9y\, dxdy = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} 9r^2 \sin\theta\, dr = 0. \end{aligned}$$

T6 Use the Divergence Theorem to evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS,$$

where $\mathbf{F} = \frac{x}{z}\mathbf{i} - \frac{y}{x}\mathbf{j} + \frac{z}{y}\mathbf{k}$, $\mathbf{n}$ is the outward pointing unit normal and S is given by $S = \{(x, y, z) : 1 < x < 4,\ 2 < y < 3,\ 3 < z < 4\}$

Solution

Let V be the volume contained by cuboid S. Then applying the Divergence Theorem,

$$\begin{aligned} I &= \iiint_V \operatorname{div}(x/z, -y/x, z/y)\, dxdydz = \int_1^4 dx \int_2^3 dy \int_3^4 \frac{1}{z} - \frac{1}{x} + \frac{1}{y}\, dz \\ &= \int_1^4 dx \int_2^3 \ln(4/3) - \frac{1}{x} + \frac{1}{y}\, dy = \int_1^4 \ln(2) - \frac{1}{x} dx = \ln(2). \end{aligned}$$

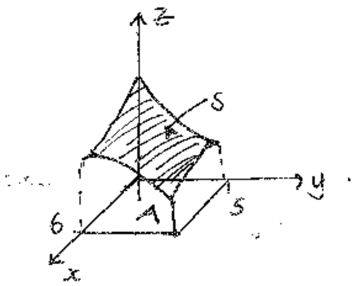
Further Exercises

F1 A vineyard lies on a plane hillside. The base of the vineyard on a map of the area (i.e. the horizontal base section) is determined by by the rectangle $0 \le x \le 6$, $0 \le y \le 5$ and the plane of the vineyard (in the same coordinates) is $x + 3y + z = 21$. The distribution of the grape harvest (in mass per unit area) across the vineyard is given by the function xy at the point (x, y, z). Use surface integrals to find

a) the mass of the total crop of grapes from the vineyard,

b) the actual area on the hillside covered by the vineyard.

Solution



(a) We need to calculate $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$. Differentiating the equation $x + 3y + z = 21$ gives:

$$1 + \frac{\partial z}{\partial x} = 0, \quad \text{hence, } \frac{\partial z}{\partial x} = -1, \; 3 + \frac{\partial z}{\partial y} = 0, \quad \text{hence, } \frac{\partial z}{\partial y} = -3.$$

$$\text{Therefore, } \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+1+9} = \sqrt{11},$$

So,

$$\text{Total crop } = \iint_S xy \, dS = \int_0^6 dx \int_0^5 xy\sqrt{11}\, dy == \sqrt{11}\left[\frac{x^2}{2}\right]_0^6 \left[\frac{y^2}{2}\right]_0^5 = 225\sqrt{11}.$$

So $225\sqrt{11}$ is the mass of grapes.

(b)

$$\text{Area } = \iint_S 1\, dS = \int_0^6 dx \int_0^5 \sqrt{11}\, dy == \sqrt{11}\, [x]_0^6\, [y]_0^5 = 30\sqrt{11}.$$

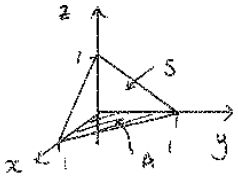
So $30\sqrt{11}$ is the area covered by grapes.

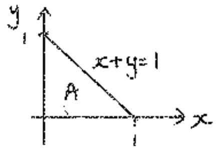
F2 Evaluate

$$\iint_S y \, dS,$$

where S is the plane surface given by the equations $x > 0$, $y > 0$, $z > 0$, and $x + y + z = 1$.

Solution





We need to calculate $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$. Differentiating the equation $x+y+z=1$ gives:

$$1+\frac{\partial z}{\partial x}=0, \quad \text{hence, } \frac{\partial z}{\partial x}=-1, \; 1+\frac{\partial z}{\partial y}=0, \quad \text{hence, } \frac{\partial z}{\partial y}=-1. \text{ Therefore, } \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{3}.$$

So,

$$\iint_S y\,dS=\iint_A y\sqrt{3}\,dxdy=\sqrt{3}\int_0^1 dx\int_0^{1-x} y\,dy=\sqrt{3}\int_0^1 \frac{(1-x)^2}{2}\,dx=\frac{\sqrt{3}}{2}\left[-\frac{(1-x)^3}{3}\right]_0^1=\frac{\sqrt{3}}{6}.$$

Note A is obtained by putting $z=0$ into $x+y+z=1$.

F3 Show that the surface area of the hemisphere given by $x^2+y^2+z^2=a^2$ and $z\geq 0$, where $a>0$, is $2\pi a^2$.

Solution

We need to calculate $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$. Differentiating the equation $x^2+y^2+z^2=a^2$ gives:

$$2x+2z\frac{\partial z}{\partial x}=0, \quad \text{hence, } \frac{\partial z}{\partial x}=-x/z, \quad 2y+2z\frac{\partial z}{\partial y}=0, \quad \text{hence, } \frac{\partial z}{\partial y}=-y/z.$$

$$\text{Therefore, } \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}=a/|z|.$$

On the hemisphere $1/|z|\geq 0$, So,

$$\begin{aligned} I=\iint_S 1\,dS&=\iint_A 1.\frac{a}{z}\,dxdy=\iint_A \frac{a}{\sqrt{a^2-x^2-y^2}}\,dxdy \\ &=\int_0^{2\pi} d\theta\int_0^a \frac{ar}{\sqrt{a^2-r^2}}\,dr=2a\pi\left[-\sqrt{a^2-r^2}\right]_0^a=2a^2\pi. \end{aligned}$$

Note A is obtained by putting $z=0$ into S, thus we A is a disc of radius a.

F4 A tent is in the form of the paraboloid $z=6-x^2-y^2$ for $z>0$. Find its surface area.

Solution

The surface cuts the xy-plane at $z = 0$, which is the circle $x^2 + y^2 = 6$. To find the surface area of the tent we need to calculate $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$. Differentiating the equation $6 - x^2 - y^2 = z$ gives:

$$\frac{\partial z}{\partial x} = -2x,\ \frac{\partial z}{\partial y} = -2y.\ \text{Therefore,}\ \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + 4x^2 + 4y^2}.$$

So the surface area of the tent is given by,

$$\begin{aligned} I &= \iint_S 1\, dS = \iint_A 1.\sqrt{1 + 4x^2 + 4y^2}\, dxdy = \int_0^{2\pi} d\theta \int_0^{\sqrt{6}} r\sqrt{1 + 4r^2}\, dr \\ &= 2\pi \left[\frac{(1 + 4r^2)^{3/2}}{24/2}\right]_0^{\sqrt{6}} = \frac{124\pi}{6} = \frac{62\pi}{3}. \end{aligned}$$

Note A is obtained by putting $z = 0$ into S, thus A is a disc of radius $\sqrt{6}$. The r integral was calculated using the change of variables $u = 1 + 4r^2$.

F5 Using the symmetry of the sine and cosine functions explain in one sentence why

$$\iiint_V x\ dxdydz = 0,$$

where V is the interior of the sphere $x^2 + y^2 + z^2 = a^2$. Use Gauss's Divergence Theorem to evaluate

$$\iint_S x^2z^2 + y^2z^2 + 3xz^2\ dS,$$

where S is the entire surface of the same sphere.

Solution

Let $\mathbf{n} = \frac{1}{a}(x, y, z)$ and $\mathbf{F} = a(xz^2, yz^2, 3xz)$ to give $\mathbf{F} \cdot \mathbf{n} = x^2z^2 + y^2 + z^2 + 3xz^2$.

We apply Gauss's Divergence Theorem with V denoting the interior of the sphere.

$$\begin{aligned} I &= \iiint_V a\, \mathrm{div}(xz^2, yz^2, 3xz)\, dxdydz = a \iiint_V z^2 + z^2 + 3x\, dxdydz \\ &= 2a \iiint_V z^2\, dxdydz + 3a \iiint_V x\, dxdydz \\ &= 2a \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^{a} \rho^4 \cos^2\phi \sin\phi\, d\rho + 0 = 4\pi a \int_0^{\pi} \cos^2\phi \sin\phi\, d\phi \int_0^{a} \rho^4\, d\rho \\ &= 4a\pi.2.\frac{1.1}{3.1}.\left[\frac{\rho^5}{5}\right]_0^a = \frac{8a^6\pi}{15}. \end{aligned}$$

Note, $\iiint_V x\, dxdydz = 0$ because the integrand is an odd function and symmetrical about zero.

F6 Show that for a well behaved closed surface S enclosing a three dimensional region R

$$\frac{1}{3}\iint_S \mathbf{r} \cdot \mathbf{n}\ dS$$

measures the volume of R. (As usual $\mathbf{r} = (x, y, z)$ and $\mathbf{n}$ denotes the outward drawn normal.)

Solution

By Gauss's Divergence Theorem,

$$\frac{1}{3}\iint_S \mathbf{r}\cdot\mathbf{n} = \frac{1}{3}\iiint_V \operatorname{div}\mathbf{r}\,dxdydz = \frac{1}{3}\iiint_V 3\,dxdydz = \iiint_V 1\,dxdydz = \text{ Volume of V.}$$

F7 Use the Divergence Theorem to evaluate

$$\iint_S \mathbf{v}\cdot\mathbf{n}\,dS,$$

where $\mathbf{v} = 7x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$, $\mathbf{n}$ is the outward pointing unit normal and $S = \{x + y + z = 1, x = 0, y = 0, z = 0\}$

Solution

Let V be the volume contained by S. Then applying the Divergence Theorem,

$$\begin{aligned} I &= \iiint_V \operatorname{div}(7x, y, -2z)\,dxdydz = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} 6\,dz \\ &= \int_0^1 dx \int_0^{1-x} 6(1-x-y)dy = \int_0^1 3(1-x)^2\,dx = 1. \end{aligned}$$

F8 Let the surface is given by two spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Use the Divergence Theorem to evaluate

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS,$$

where $\mathbf{F} = (x, -y^2, xz)$ and $\mathbf{n}$ is the outward pointing unit normal.

Solution

Let V be the volume contained by the two spheres. Then applying the Divergence Theorem,

$$\begin{aligned} I &= \iiint_V \operatorname{div}(x, -y^2, xz)\,dxdydz = \iiint_V 1 - 2y + x\,dxdydz \\ &= \int_0^\pi d\phi \int_0^{2\pi} d\theta \int_1^2 \rho^2 \sin\phi - 2\rho^3 \sin\theta \sin^2\phi - \rho^3 \cos\theta \sin^2\phi\,d\rho \\ &= \int_0^\pi \frac{7}{3} 2\pi \sin\phi\,d\phi = \frac{28\pi}{3} \end{aligned}$$