Proof of the Change of Variables Theorem for a special case

We prove

$$\iint_D f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v))\,\underbrace{|J|}\,du\,dv$$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

for the special case:
- f = constant,
- a linear change of variables

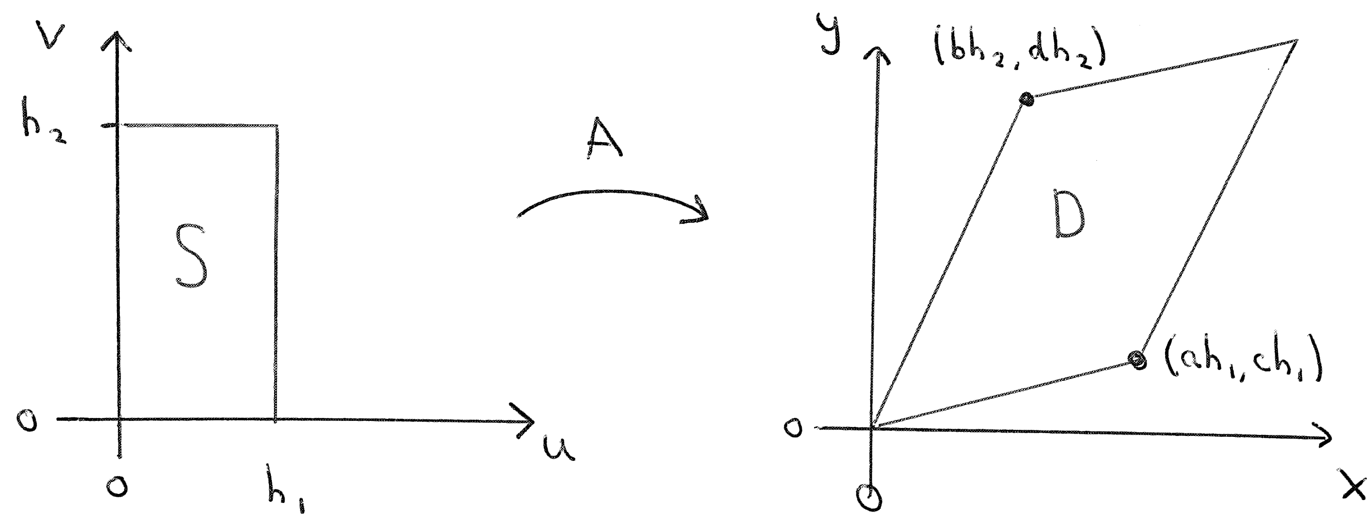
  $$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix},$$

  $\therefore\ x_u = a$, $x_v = b$, $y_u = c$, $y_v = d$

  which implies that $\dfrac{\partial(x,y)}{\partial(u,v)} = \det(A)$

  and $|J| = |\det A| = |ad - bc|$,
- S is the rectangle $[0,h_1] \times [0,h_2]$,

  which implies that D is a parallelogram.



$$\iint_D f \, dx\,dy \overset{\substack{f \text{ is constant} \\ \downarrow}}{=} f \iint_D 1 \, dx\,dy$$

$$= f \cdot \operatorname{area}(D)$$

$$\overset{\text{since } D \text{ is a parallelogram}}{=} f \left| \underline{k} \cdot \left(h_1 \begin{pmatrix} a \\ c \end{pmatrix} \times h_2 \begin{pmatrix} b \\ d \end{pmatrix} \right) \right|$$

$$= f \underbrace{h_1 h_2}_{\operatorname{area}(S)} \underbrace{|ad - bc|}_{|J|}$$

$$= f\,|J| \iint_S 1 \, du\,dv$$

$$\overset{\text{since } f \text{ and } |J| \text{ are constant}}{=} \iint_S f \, |J| \, du\,dv. \qquad \square$$

<u>Remarks</u>

- The same proof works for an affine change of variables

$$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}.$$

- If f is not constant, then you could approximate it by a piecewise constant function.