2A Multivariable Calculus 2013

Ex Sheet 5

Tutorial Exercises

T1 Evaluate the following using beta functions:

$$\text{(a)} \int_0^{\pi/2} \sin^3 x \cos^2 x \, dx, \quad \text{(b)} \int_0^{\pi/2} \sin^7 x \cos^3 x \, dx, \quad \text{(c)} \int_0^{\pi} \cos^5 x \, dx,$$

$$\text{(d)} \int_0^{\pi/2} \sin^4 x \cos^2 x \, dx, \quad \text{(e)} \int_0^{\pi} \sin^5 x \, dx, \quad \text{(f)} \int_0^{\pi} \sin^2 x \cos^3 x \, dx,$$

$$\text{(g)} \int_0^{2\pi} \sin^3 x \cos^3 x \, dx, \quad \text{(h)} \int_0^{2\pi} \sin^4 x \cos^3 x \, dx,$$

T2 Given the change of variables

$$u = \frac{1}{3}(x+y) \qquad v = \frac{1}{3}(x-2y)$$

express x and y in terms of u and v.

T3 By making a suitable change of variables, evaluate

$$\iint xy \, dx \, dy$$

over the region enclosed by the two hyperbolas $xy = 1$ and $xy = 7$ and the two parabolas $y = 2x^2$ and $y = 4x^2$.

T4 Use double integration and an appropriate change of variables to find the area of the parallelogram enclosed by the lines $y = x + 1$, $y = x + 2$, $y = 5 - 3x$, $y = 9 - 3x$

T5 Evaluate

$$\iiint yz^2 \, dxdydz$$

throughout the cube given by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

Further Exercises

F1 Evaluate the following using beta functions:

$$\text{(a)} \int_0^{\pi/2} \cos^6 x \, dx, \quad \text{(b)} \int_0^{\pi/2} \sin^2 x \, dx, \quad \text{(c)} \int_0^{\pi/2} \sin x \cos x \, dx,$$

$$\text{(d)} \int_0^{\pi/2} \sin^7 x \, dx, \quad \text{(e)} \int_0^{\pi} \sin^6 x \, dx, \quad \text{(f)} \int_0^{\pi} \sin^2 x \cos^3 x \, dx,$$

$$\text{(g)} \int_0^{\pi} \cos^6 x \, dx \quad \text{(h)} \int_0^{\pi} \sin^3 x \cos^2 x \, dx, \quad \text{(i)} \int_0^{\pi} \sin^4 x \cos^4 x \, dx,$$

$$\text{(j)} \int_0^{2\pi} \cos^4 x \, dx, \text{(k)} \int_0^{2\pi} \sin^3 x \cos^2 x \, dx, \quad \text{(l)} \int_0^{2\pi} \sin^2 x \cos^6 x \, dx,$$

Lecture 9

- **Key Points:**
  - use Beta functions to solve integrals involving sine and cosine
  - use symmetry properties of sine and cosine to simplify an integral
  - definition of a Jacobian
  - theorem for changing variables in double integrals
- **Read:**
  - Stewart Section 15.10 (p1064)
- **Textbook Exercises:**
  - Exercises 15.10 (p1071) Qs 1–14

Lecture 10

- **Key Points:**
  - use change of variables to evaluate double integrals
  - find the area of domains using change of variables
  - definition of a triple integral
- **Read:**
  - Stewart Section 15.10 (p1069)
  - Stewart Section 15.7 (p1041)
- **Textbook Exercises:**
  - Exercises 15.10 (p1071) Qs 15–26
  - Exercises 15.7 (p1049) Q 2

F2 Evaluate the integral

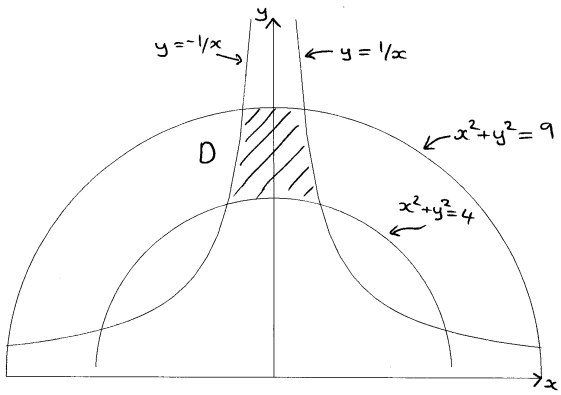
$$\int_0^3 dx \int_{x/4}^{x/4+2} \left(\frac{x+y}{4}\right)\, dy,$$

using the change of variables $u = \frac{x}{4}, \quad v = \frac{x+y}{2}$.

F3 Evaluate

$$\iint_D x^4 - y^4 \, dxdy$$

where D is the region illustrated below.



F4 By making a suitable change of variables, evaluate

$$\iint x^2 \, dx\, dy$$

over the square enclosed by the lines $x+y=-1$, $x+y=1$, $y-x=-1$, $y-x=1$.

F5 By making a suitable change of variables, evaluate

$$\iint \frac{y^2}{x} \, dxdy$$

over the region in the first quadrant enclosed by the four parabolas $y^2 = x$, $y^2 = 2x$, $y = x^2$, $y = 4x^2$

F6 Evaluate $\iint_R (x^2+y^2) dA$, where R is the parallelogram with vertices $(0,0)$, $(2,0)$, $(3,1)$ and $(1,1)$.

F7 Show that the area of the region in the first quadrant enclosed by the two hyperbolas $xy = a$, $xy = b$ and the two lines $y = cx$, $y = dx$, where $b > a > 0$ and $d > c > 0$ is

$$\frac{1}{2}(b-a)\log\left(\frac{d}{c}\right).$$

F8 Use change of variables to evaluate

$$\iint x^4 + y^4 \, dxdy$$

over the interior of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

F9 Evaluate

$$\int_3^5 dx \int_1^4 dy \int_1^2 xy \, dz.$$