Lecture 1

Riemann Surfaces and Quadratic Differentials

Motivations

Plan for the course

1. Basic objects/spaces and motivations
2. Teichmüller theory + mapping class groups
3. $SL(2,\mathbb{R})$ action and orbit closures
4. Dynamics of the Teichmüller flow.

Smooth Surface

Orientable finite type surface (S, Z)

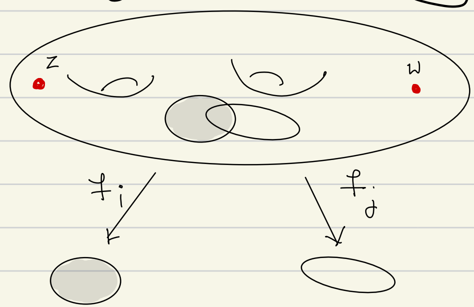
finite genus g

finitely many marked points.

$|Z| = n$

transition functions

$f_j \circ f_i^{-1}$



smooth surface : charts to $\mathbb{R}^2$ + transition functions smooth.

Riemann Surface

Charts to $\mathbb{C} = \mathbb{R}^2$ and requre transition functions to be holomorphic.

Example : $\mathbb{C}P^1$ Riemann sphere

$\mathbb{C}P^1 = \mathbb{C}^2 - \{0\} / \sim \qquad z \sim w$ iff $z = \lambda w$, $\lambda \in \mathbb{C}^*$

Uniformisation : Simply connected Riemann surfaces are biholomorphic to

$\mathbb{C}P^1$, $\mathbb{C}$, $\mathbb{D}$ — unit disc $= \{ z : |z| < 1 \}$

Canonical metrics

$\mathbb{CP}^1$	spherical metric	Gaussian curv	1
$\mathbb{C}$	flat metric	" "	0
$\mathbb{D}$	hyperbolic metric	" "	-1

Gauss – Bonnet $\quad$ $$\underbrace{\int_S K \, dA}_{\text{integral of Gaussian curv.}} = 2\pi \chi(S)$$

S: finite area

Euler charact.

Uniformisation and universal covers

(S, Z) finite type

Universal cover

$\mathbb{CP}^1$	$\mathbb{C}$	$\mathbb{D}$
$2-2g-n>0$	$2-2g-n=0$	$2-2g-n<0$
$\Rightarrow g=0 \quad n \leq 1$	$\Rightarrow g=0\,,\ n=2$	most surfaces
	$g=1\,,\ n=0$

Moduli spaces of Riemann surfaces

To obtain a Riemann surface from its universal cover

$$\pi_1(S) \longrightarrow \text{Isom}(\text{Universal Cover})$$

as a discrete subgrp with finite co-volume, ie lattice.

$\text{Isom}\, \mathbb{CP}^1 \quad = \quad PSL(2, \mathbb{C}) \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts by

Mobius transformations.

$\text{Isom}\, \mathbb{C} \quad =$ subgrp of $PSL(2, \mathbb{C})$ that fixes ∞

$\quad = \mathbb{C} \rtimes \mathbb{C}^*$

$\text{Isom}\, \mathbb{D} \quad =$ subgrp of $PSL(2, \mathbb{C})$ that fixes $\mathbb{H}$

$\quad = PSL(2, \mathbb{R})$

Moduli spaces of Riemann surfaces

$\mathcal{M}_{g,n}$ = moduli space of Riemann surfaces homeomorphic to $S_{g,n}$

= parameter space of equipping $S_{g,n}$ by a Riemann surface str.

= parameter space of canonical metrics on $S_{g,n}$

Moduli space of tori

modular surface

$$\mathcal{M}_1 = \mathbb{H} / SL(2,\mathbb{Z})$$

A conformal torus is given by $\mathbb{C}/\Lambda$ where $\Lambda \cong \mathbb{Z} \oplus \mathbb{Z}$

Marking Λ by a basis we may assume (up to rotation and dilation of $\mathbb{C}$) that

$$\Lambda = \mathbb{Z} 1 \oplus \mathbb{Z} t \quad \text{where} \quad \operatorname{Im} t > 0$$

Change of marking for $\Lambda \cong SL(2,\mathbb{Z})$

Holomorphic 1-forms

Suppose that X is a finite type Riemann surface.

Consider its cotangent bundle T^*X.

A holomorphic 1-form is a holomorphic section

$$\omega : X \longrightarrow T^*X$$

Locally $\omega = f(z)\,dz$

f holomorphic.

Moduli spaces of holomorphic 1-forms

Defined in a similar way to moduli spaces of Riemann surfaces

Example: for tori

$$H = SL(2,\mathbb{R}) / SL(2,\mathbb{Z}).$$

Translation Surfaces

Suppose that ω is a holomorphic 1-form on S.
Suppose that $p \in S$ is a base-point.

Consider $x \in S$ and suppose that $\gamma(p,x)$ is any contour from p to x. Set $\displaystyle \phi(x) = \int_{\gamma(p,x)} \omega$

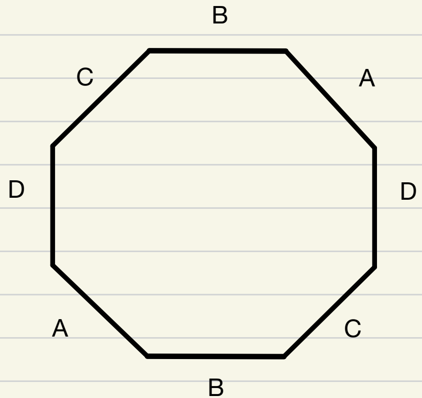
This defines charts $\phi : S - \underbrace{Z(\omega)}_{\text{zeroes of } \omega} \longrightarrow \mathbb{C}$

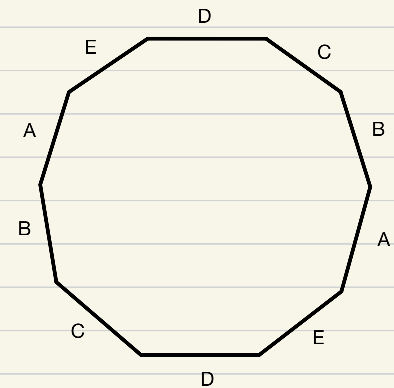
Transition functions are translations.

Examples of translation surfaces

1 Tori

2 Genus 2





Quadratic Differentials

A quadratic differential is a meromorphic section of the symmetric square of T^*X.

simple poles at all marked points and only at marked points.

Locally $q = f(z)\, dz^2$

meromorphic with simple poles.

Important: Moduli space Q of q. diff $= T^*\mathcal{M}_{g,n}$

Half-translation surfaces

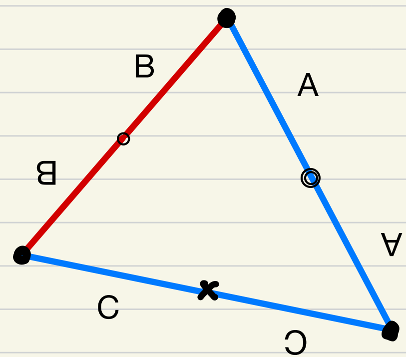
Choice of $\sqrt{q}$ and contour integration defines charts from $S - Z(q) \longrightarrow \mathbb{C}$

zeroes and poles of q

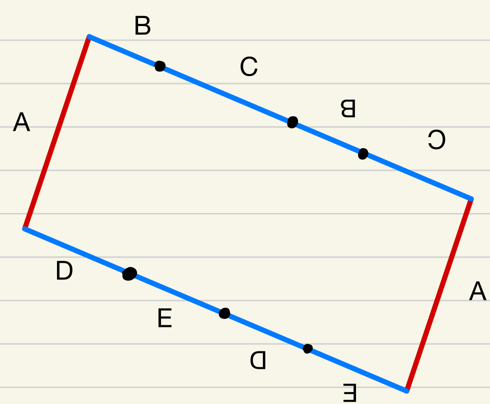
Transition functions are translations and half-translations.

Examples of half-translation surfaces

1 $S_{0,4}$



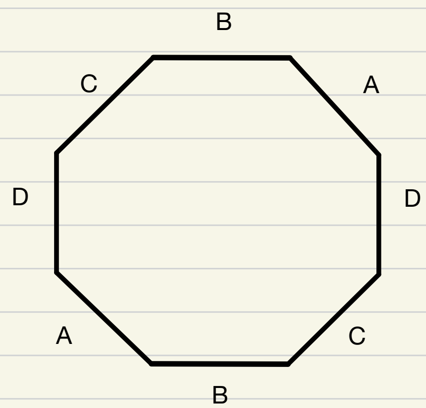
2 Genus 2

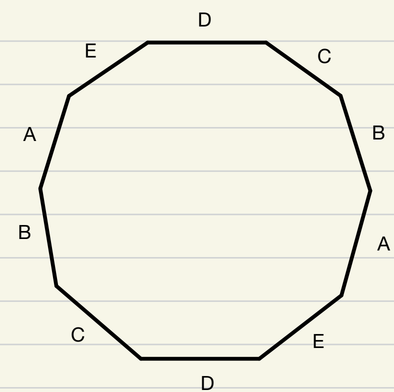


Singular flat metrics

Charts to $\mathbb{C}$ with transitions $z \to \pm z + c$ define a flat metric on $S - Z(q)$.

A zero of order k ($k = -1$ at a pole) $\longrightarrow$ cone angle $\pi(k+2)$





$SL(2,\mathbb{R})$ action

The group $SL(2,\mathbb{R})$ acts on the charts to $\mathbb{C}$.

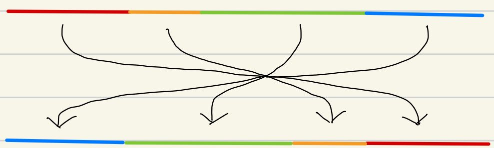
A translation / half-translation is taken to another translation / half-translation by $SL(2,\mathbb{R})$.

So it descends to the moduli spaces of quadratic differentials.

Diagonal part of the action by $\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ is called the Teichmüller flow.

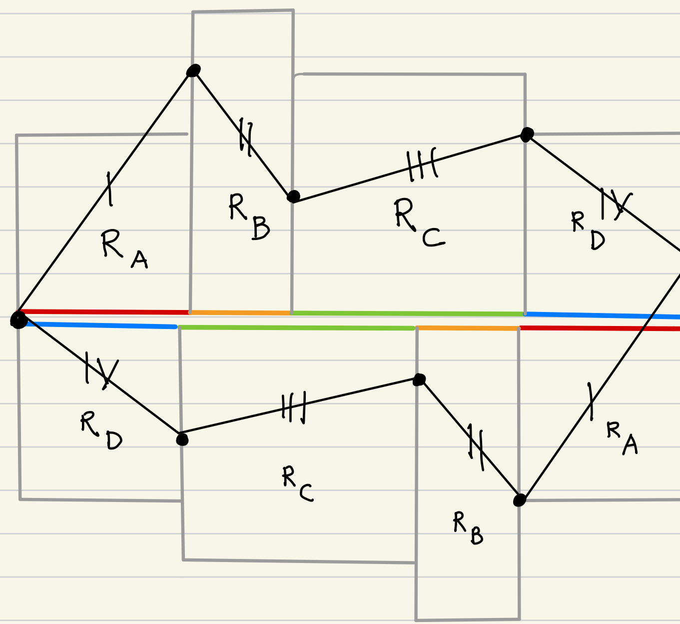
Motivations

interval exchange transformations (IET)



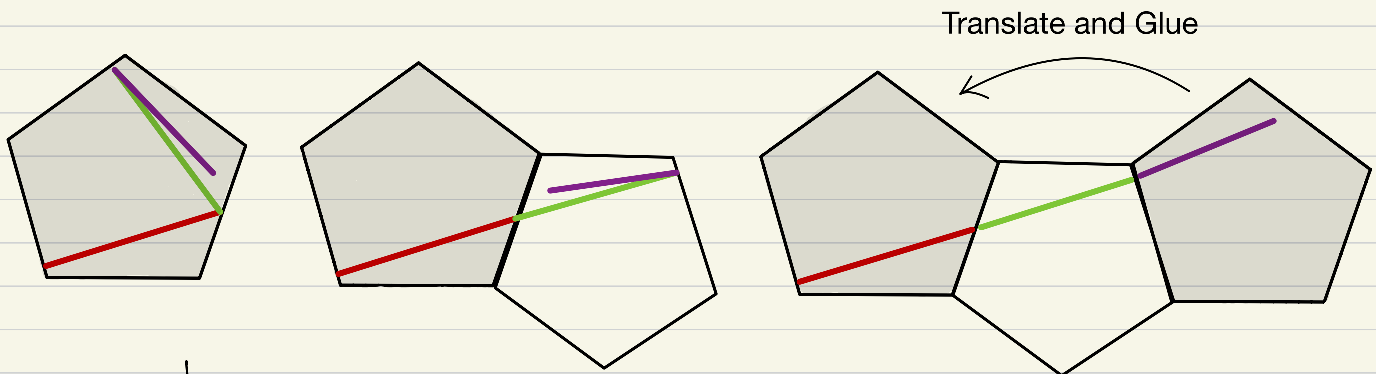
Suspensions of IET~~s~~ give flat surfaces.

A zippered rectangles construction

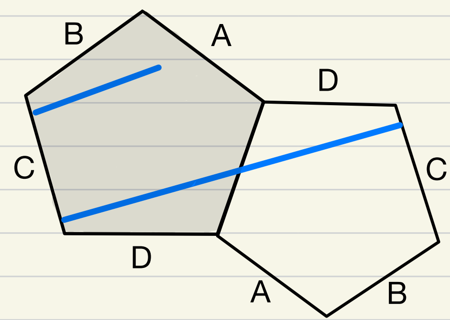


Motivations

Rational billiards

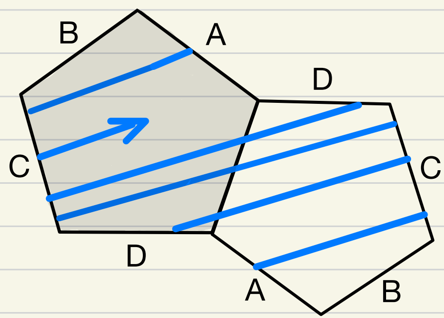


Polygonal angles
rational multiples
of π



Renormalisation

Long trajectories of a straight line flow on a fixed flat surface can be renormalised by time-1 trajectories of straight line flows on a varying family.



rotation

$$r_{-\theta}\, g_t\, r_\theta$$

to shorten

$$\longrightarrow$$

Teich. flow

$$\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

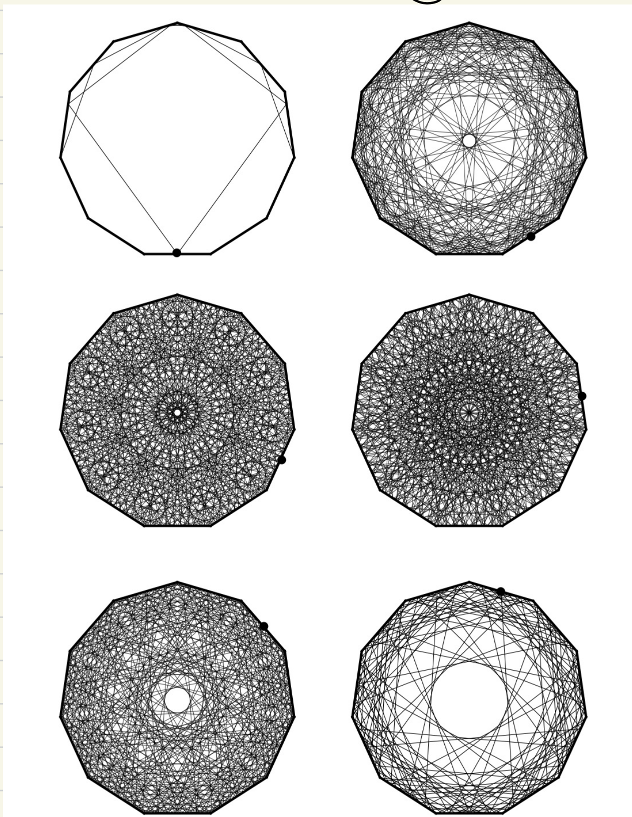
Consequences of renormalisation

1 Straight line flow on a flat surface is not "hyperbolic"

2 Teich flow on Q is semi-hyperbolic

Zorich (1997): Deviations of ergodic averages for typical interval exchange maps are related to the Lyapunov spectrum of the Kontsevich-Zorich cocycle for Teichmüller flow.

Billiards in regular polygons



References

1) Library of billiard trajectories
D. Davis – S. Lelievre

2) Billiards in regular polygons
C. McMullen

Topology of moduli spaces and dynamics

Our recent work: flow detects π_1

So suffices to understand π_1 and its monodromies.

A lot of open questions here including

Kontsevich $K(\pi,1)$ conjecture.