Lecture 2

Teichmüller theory + Mapping class groups

Teichmüller space of marked tori

Last lecture : universal cover of a torus is $\mathbb{C}$

Conformal automorphism of $\mathbb{C}$: $z \longrightarrow az+b$

For $z \longrightarrow az+b$ to be fixed pt free : $a=1$ and $b \neq 0$

$\Rightarrow$ a conformal torus given by $\mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z} \oplus \mathbb{Z}$

Choosing generators, assume they give translations

$z \longrightarrow z+s$ and $z \longrightarrow z+t$

where the ordered basis $\{s,t\}$ gives orientation of $\mathbb{C}$

Teichmüller space of marked tori

The conformal tori

$$\mathbb{C}/\langle z \to z+s,\ z \to z+t \rangle = \mathbb{C}/\langle z \to z+1,\ z \to z+t/s \rangle$$

Set $\tau = \dfrac{t}{s}$ and note $\mathrm{Im}(\tau) > 0$

Thus $\mathrm{Teich}(S_1) = \mathbb{H} = \{ \tau \in \mathbb{C} \,/\, \mathrm{Im}\,\tau > 0 \}$

Change of marking : change ordered basis for $\mathbb{Z} \oplus \mathbb{Z}$

$\simeq SL(2, \mathbb{Z})$.

Moduli space of tori

Suppose $\{(a,c),(b,d)\}$ is an ordered basis of Λ.

Then it acts by $z \to z + a + c\tau$ and $z \to z + b + d\tau$

Up to a bi-holomorphism

$$\tau' = \frac{b+d\tau}{a+c\tau}$$

right action by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\mathcal{M}_1 = SL(2,\mathbb{Z}) \backslash \mathbb{H}$$

mapping class group.

Elliptic Curves

Suppose that $\Lambda \subseteq \mathbb{C}$ is a lattice — orbit of 0 under two translations

Suppose that $M(\mathbb{C}/\Lambda)$ is the field of meromorphic fncs.

Lifting to $\mathbb{C}$, f in $M(\mathbb{C}/\Lambda)$ gives a doubly periodic fnc on $\mathbb{C}$.

Weierstrass P- function

$$p(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

$$p'(z) = \frac{-2}{z^3} + \sum_{\lambda \in \Lambda} \frac{-2}{(z-\lambda)^3}$$

Suppose that $\Lambda = \mathbb{Z}\alpha + \mathbb{Z}\beta$, $\alpha, \beta \in \mathbb{C}^*$

Elliptic Curves

$$p(z+\alpha) = p(z) + \int_{\gamma(z, z+\alpha)} p'(z) = p(z) + a$$

any contour that avoids poles

$$p(z+\beta) = p(z) + b$$

Note $p(z) = p(-z)$ and so $p(-\alpha/2) = p(\alpha/2) \Rightarrow a = 0$

similarly $b = 0 \implies p$ is doubly periodic

Theorem: Any doubly periodic function is a rational function of p, p'

Elliptic Curves

Theorem : The map $\mathbb{C} \xrightarrow{\pi} \mathbb{CP}^2$ given by $\pi(z) = (p(z), p'(z))$ gives an isomorphism between $\mathbb{C}/\Lambda$ and

$$y^2 = 4x^3 + ax + b$$

where a and b are related to $p(z)$ by Eisenstein series.

Thus $\mathcal{M}_1$ can also be thought of as the moduli space of elliptic curves.

Holomorphic 1 forms on tori

By the maximum principle, any holo. fnc on $\mathbb{C}/\Lambda$ is constant.

The form dz on $\mathbb{C}$ is translation invariant so descends to a holomorphic 1-form on $\mathbb{C}/\Lambda$.

$\omega = f\,dz$

Let ω be any holo 1-form on $\mathbb{C}/\Lambda$, then ω/dz defines a holomorphic fnc $\Rightarrow$ $\omega = \text{const} \cdot dz$

Note that $\operatorname{Re}\omega$ and $\operatorname{Im}\omega$ define vertical and horizontal dir.

$$\Rightarrow \Omega^1 \mathcal{M}_1 = H(0) = SL(2,\mathbb{Z}) \backslash SL(2,\mathbb{R})$$

notation

Teichmüller theory

Suppose that (S, Z) is a finite-type surface
$\llcorner$ finite set of pts

A marked Riemann surface is a Riemann surface along with a homeomorphism $\phi : (S, Z) \longrightarrow (X, Z(X))$.

Two marked Riem. surfaces $\phi : (S, Z) \longrightarrow (X, Z(X))$ and $\phi' : (S, Z) \longrightarrow (X', Z(X'))$ if there exists a holomorphic map $h : (X', Z(X')) \longrightarrow (X, Z(X))$ s.t

$\phi^{-1} \circ h \circ \phi' : (S, Z) \longrightarrow (S, Z)$ is isotopic to identity

Teichmüller theory

$\mathrm{Teich}(S,Z)$ = space of marked Riemann surfaces homeomorphic to $(S,Z)/\sim$

Mapping class grp = orient. pres. diffeos of (S,Z) modulo isotopy

notation

$\mathrm{Mod}(S,Z)$

acts on $\mathrm{Teich}(S,Z)$ by changing the marking by pre-composition : suppose α is an orient. pres. diffeo

new marking by $\phi \circ \alpha$

Moduli space $M(S,Z) = \mathrm{Teich}(S,Z) / \mathrm{Mod}(S,Z)$

Moduli spaces of holomorphic 1-forms

It is the space of un-marked translation surfaces.

Riemann-Roch:
Suppose that ω is holomorphic 1-form on a Riem surface X. Let $p_1 \ldots, p_j$ be zeroes of ω. Recall that cone angle at p_i is $2\pi(k_i+1)$ where k_i is the order of the zero at p_i. Then

$$\sum k_i = 2g-2$$

Moduli spaces of holomorphic 1-forms

The moduli space of holomorphic 1-forms is stratified by the tuple $(\kappa_1, \ldots, \kappa_j)$.

Strata are not connected; each stratum has at most 3- components

Classification of components: Kontsevich-Zorich

Quadratic differentials

Deformation of complex structure by quasi-conformal maps

Conformal maps $\longrightarrow$ Cauchy Riemann equations

$$\partial f / \partial \bar{z} = 0$$

Quasi-conformal maps $\longrightarrow$ Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z}$$

Beltrami coefficient / differential

Quadratic differentials

Dilation / quasi-conformality constant

$$K = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$ ratio of major to minor axis

Measurable Riemann mapping

$\mathbb{D} \xrightarrow{\mu} \mathbb{C}$ $\quad \|\mu\|_\infty < 1$ and μ measurable

then there exists a quasi-conf homeo $\mathbb{D} \xrightarrow{f} \mathbb{C}$

with $\mu(f) = \mu$

Extremal quasi-conformal maps: Given two homeo. Riemann surfaces give an extremal quasi-conf. map between them.

Quadratic differentials

Teichmüller's theorem: Given marked Riem. surfaces $(X, Z(X))$ and $(X', Z(X'))$ there is a unique extremal quasi-conf map $f: (X, Z(X)) \to (X', Z(X'))$.

Moreover, there is a quadratic diff q on X and q' on Y s.t f is given by the diagonal element $\begin{bmatrix} e^{\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix}$ acting on the half-translation surface q and its image is q'.

Moduli spaces of quadratic differentials

This is the space of unmarked half-translation surfaces.

This is stratified by the cone angle data at singularities $\underline{\kappa} = (\kappa_1, \dots, \kappa_j)$ such that $\sum (\kappa_i - 2) = 4g - 4$

Classification of connected components of strata:

Kontsevich - Zorich, Lanneau, Chen-Möller

Invariants: abelian or quadratic, hyper-ellipticity, odd / even spin for abelian with κ_i even, regularity

Examples

Local calculation : consider 1-form $z\,dz$ in the neighbourhood of zero in $\mathbb{C}$.

Define $\phi(w) = \int_0^w z\,dz = \dfrac{w^2}{2}$

$\Rightarrow$ a branched covering of degree 2

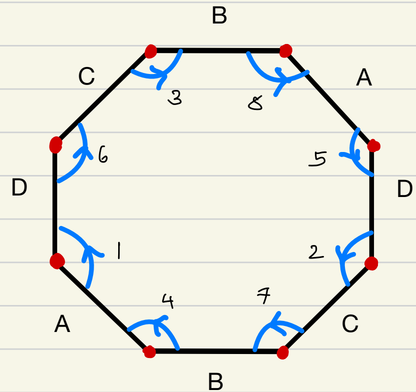
Similarly consider quadratic differential $z\,dz^2$ in nbhd of zero.
Then by introducing a slit we can form $\sqrt{z}\,dz$ on the complement

$\Rightarrow \phi(w) = z^{3/2} \Rightarrow$ each half plane mapped to sector with angle $3\pi/2$

Examples

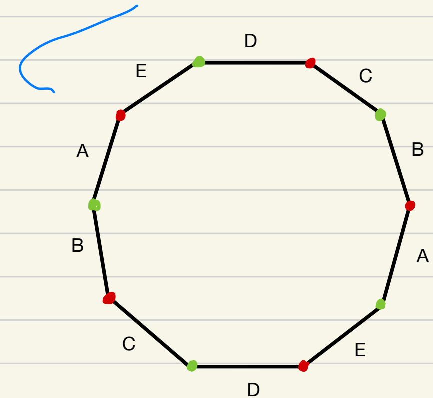
Genus 2

stratum $H(1,1)$

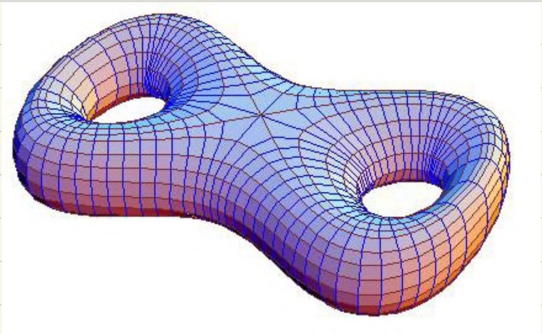


holomorphic 1-form with a single zero of order two

Stratum : $H(2)$

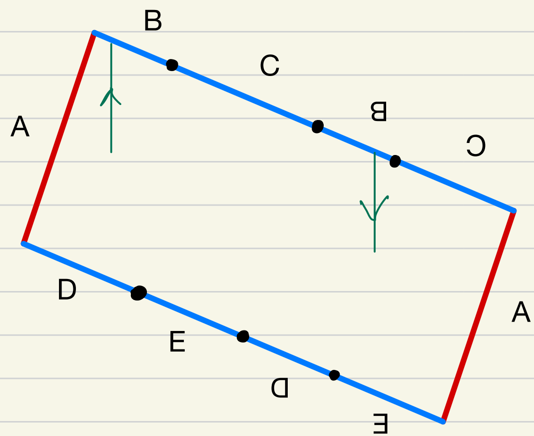


holo 1-form with simple zeroes



Examples

Genus 2

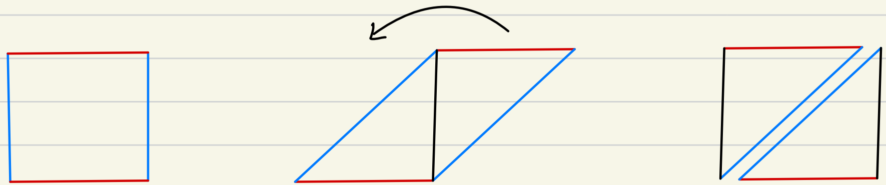


A holomorphic quadratic differential with two zeroes of order 2 (local str. $z^2 dz^2$)

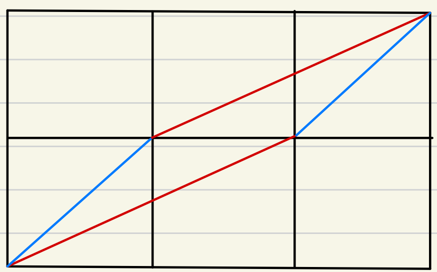
Stratum : $Q(2,2)$

Mapping class group

Scissors congruence



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

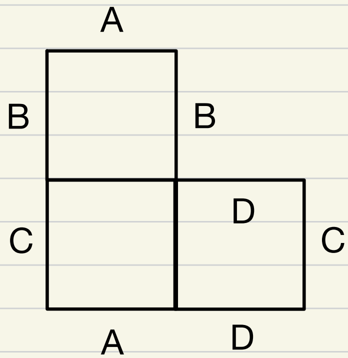


Anosov map

of the torus

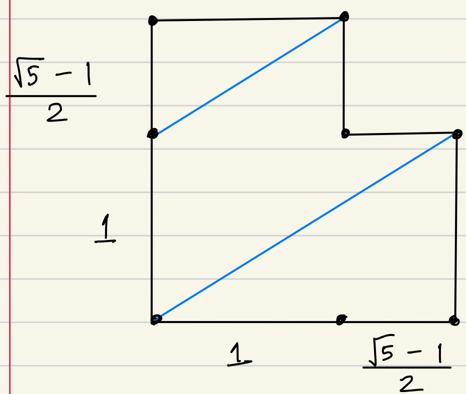
$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Mapping class group



square tiled L-shaped

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ twists twice in B and once in C



golden L-shaped

$\begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 1 \end{bmatrix}$ twists once in both horizontal cylinders

Next Week

Nielsen-Thurston classification for mapping classes.