Lecture 3

Teichmüller theory + Mapping class groups

continued

Lecture slides posted on my webpage

maths.gla.ac.uk/~vgadre/

Recap

$$\underset{\text{marked}}{(\overset{\text{surface}}{S}, Z)} \longrightarrow \underset{\text{Riemann surface}}{(X, Z(X))}$$

Teich space = space of marked Riemann surfaces

Mapping class grp = orient. pres. diffeos of the surface modulo isotopy

notation
Mod — acts on Teich by changing the marking.

Moduli space of Riemann surfaces $\mathcal{M}$

= Teich / Mod

$$\begin{array}{ccc} (S, Z) & \longrightarrow & (X, Z(X)) \\ & & \uparrow \text{biholo} \\ (S, Z) & \longrightarrow & (X', Z(X')) \end{array}$$

Mapping class groups

$\mathrm{Mod} = \pi_0(\mathrm{Diffeo}^+ \text{ of the surface})$

For instance, when genus $= 0$ and n marked points $\longrightarrow$ mapping classes that preserve a chosen marked point

$\simeq$ the braid group B_{n-1}

Classification in $SL(2,\mathbb{Z}) = \mathrm{Mod}(S_1)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\mathbb{C}P^1$ by $\dfrac{az+b}{cz+d}$

Solve for fixed points $\dfrac{az+b}{cz+d} = z$

$\Longrightarrow \quad cz^2 + (d-a)z - b = 0$ and so solns

$$z = \frac{-(d-a) \pm \sqrt{(d-a)^2 + 4bc}}{2c}$$

$$= \frac{-(d-a) \pm \sqrt{(d+a)^2 - 4}}{2c} \qquad ad - bc = \underline{1}$$

Note $d + a =$ trace

Classification in $SL(2,\mathbb{Z}) = \text{Mod}(S_1)$

$SL(2,\mathbb{R})$ - With real entries, if

1) $|\text{Tr}(A)| < 2$ then roots complex, one in $\mathbb{H}$ and another in $\mathbb{H}^- = \{z \mid \text{Im}\, z < 0\}$ elliptic

2) $|\text{Tr}(A)| = 2$ then a repeated root on $\mathbb{R}$
A is parabolic and can be conjugated in $SL(2,\mathbb{Z})$ to $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$

3) $|\text{Tr}(A)| > 2$ then distinct real roots.
A is hyperbolic

Classification in $SL(2,\mathbb{Z}) = \mathrm{Mod}(S_1)$

Eigenvalues for linear action of A on $\mathbb{R}^2$

charact. poly $\qquad \lambda^2 - \mathrm{Tr}(A)\lambda + 1 = 0$

$$\Rightarrow \quad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{(\mathrm{Tr}(A))^2 - 4}}{2}$$

1) $|\mathrm{Tr}(A)| < 2 \Rightarrow \lambda, \lambda^{-1}$ complex conjugates

$\Rightarrow A$ is a rotation

In $SL(2,\mathbb{Z})$, $\mathrm{Tr}(A) = 0$ or $\mathrm{Tr}(A) = \pm 1$, so

conjugate to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

Classification in $SL(2,\mathbb{Z}) = \mathrm{Mod}(S_1)$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

fixed point for Möbius action is $z = i$

symmetry of the square torus in $SL(2,\mathbb{Z})$

$$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \qquad \frac{-1}{1+z} = z$$

fixed point for Möbius action is cube root of unity

symmetry of the hexagonal torus.

2) $|Tr(A)| = 2 \Rightarrow \lambda = \lambda^{-1}$

to be accurate supposed to be in PSL . .

so $\lambda = 1$

$\Rightarrow$ parabolic action

fixed point for Möbius action in $\mathbb{Q}$ = eigendirection has rational slope.

Classification in $SL(2,\mathbb{Z}) = \mathrm{Mod}(S_1)$

3) $|\mathrm{Tr}(A)| > 2 \implies \lambda, \lambda^{-1}$ real

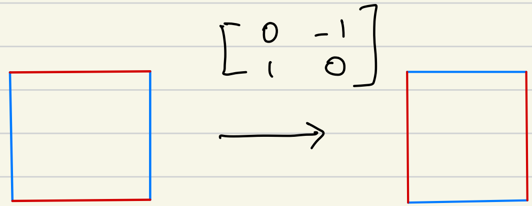
Exercise: slopes of eigen-directions are fixed points in $\mathbb{R}$ for Möbius action.

Note λ are quadratic irrationals

$\implies$ eigendirections have irrational slopes

so pair of irrational foliations on S_1 with one stretched and the other contracted.

Topological descriptions

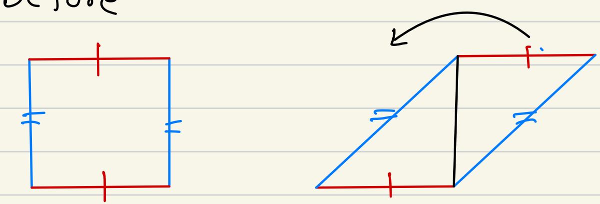
1) 

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

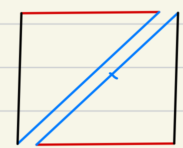
symmetry of the square torus

2) scissors congruence

before



after

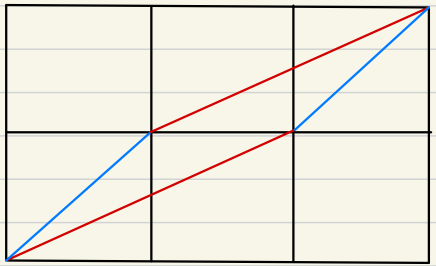


$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

marking change: red curve unchanged, black is the new blue $\Longrightarrow$ Dehn twist in the red $(1,0)$ curve.

Topological descriptions

3



Anosov map
of the torus

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

completely work out stable / unstable foliations

build Markov partitions

Nielsen Thurston classification

The classification in $SL(2,\mathbb{Z})$ "generalises".

Theorem: Any mapping class f is either

1) finite order: in this case, there exists a Riemann surface str such that f can be realised as an automorphism ~ in fact, any finite group
Nielsen realisation problem.

2) reducible: some power of f fixes a multi-curve

Nielsen Thurston classification

3) pseudo-Anosov:

No power of f fixes any simple closed curve

There exists a quadratic differential q s.t

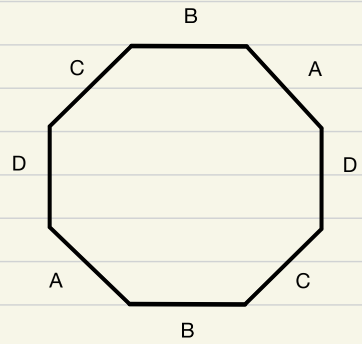
f can be realised as an action of

$\begin{pmatrix} \lambda(f) & 0 \\ 0 & \lambda(f)^{-1} \end{pmatrix}$ on the half-transl. surface

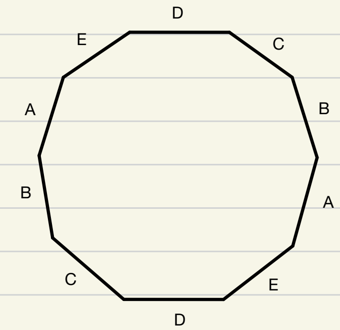
$\lambda(f)$ is called the stretch factor of f

the horizontal and vertical foliations are measured foliations with rich dynamics (minimal, uniq. erg)

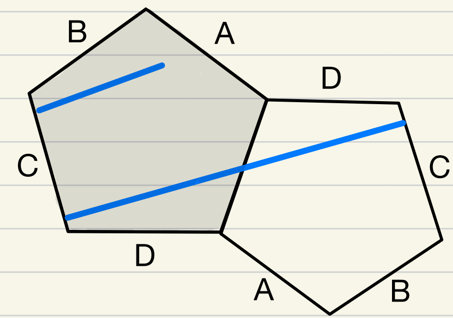
Examples



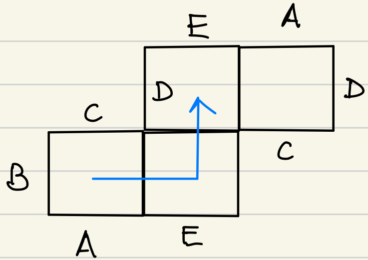
order 8 symmetry



order 10 symmetry



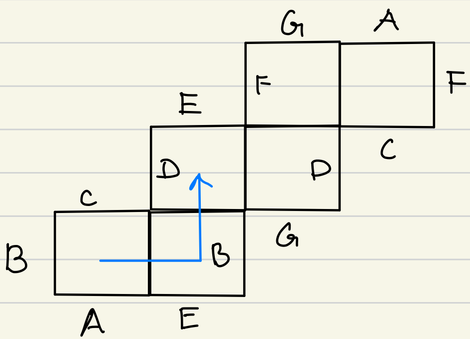
Examples



Stratum

Order

Exercises

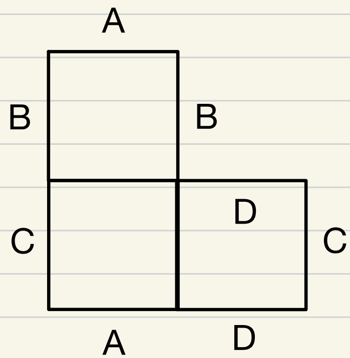


Stratum

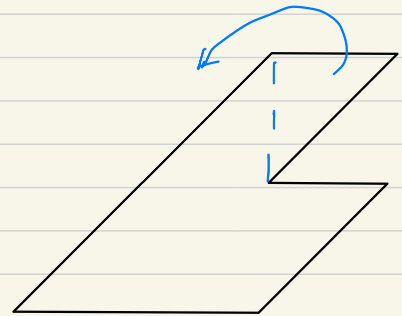
Order

Examples

square-tiled L shaped table



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



but not a scissors move on cylinder C

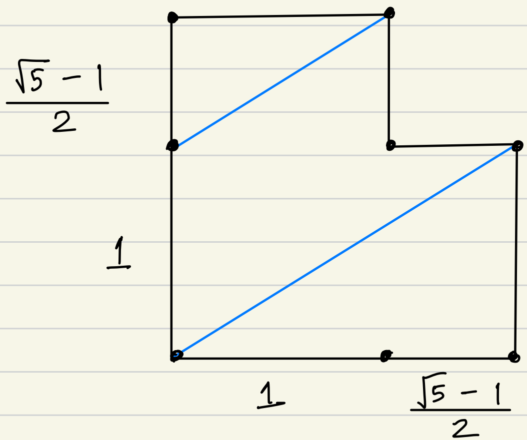
Exercise: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ gives a scissors move on cylinders B and C to get back original surface;

mapping class twists in core curves of horizontal cylinders

The product $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ gives a pseudo-Anosov.

Examples

pseudo-Anosov on a golden L



Exercise: show that

$$\begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 1 \end{bmatrix}$$

gives a twist in horizontal cylinders

Hint: the diagonals in blue have same slopes.

Then $\begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1+\sqrt{5}}{2} & 1 \end{bmatrix}$ gives a pseudo-Anosov.

Veech groups $SL(X,q)$

Recall $SL(2,\mathbb{R})$ action on components of quadratic strata.

$\longrightarrow$ $SL(2,\mathbb{R})$ acts on charts to $\mathbb{C}=\mathbb{R}^2$ and takes transl / half-transl to transl / half-transl hence descends.

$$SL(X,q) = \{ A \in SL(2,\mathbb{R}) \text{ s.t } Aq = q \}$$

up to scissors congr.

subgrp of Mod, discrete subgrp of $SL(2,\mathbb{R})$.

Next Week

$SL(2,\mathbb{R})$ action

— Masur-Veech-Smillie measure

— Teichmüller curves + Smillie's result

curves in $\mathcal{M}_g$

— Orbit closures

"analogous" to Margulis-Ratner theory

notable work of Eskin-Mirzakhani-Mohammadi, Filip.