Lecture 4

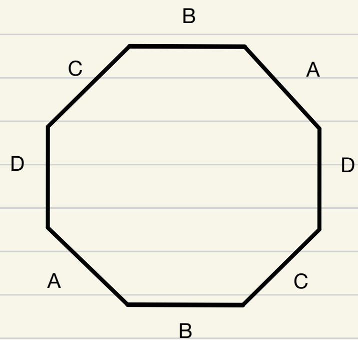
Mapping class groups continued + $SL(2,\mathbb{R})$-action

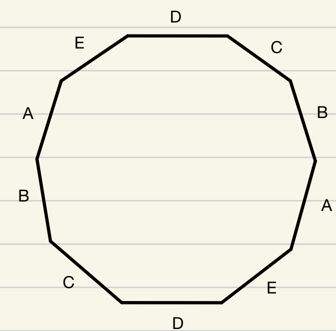
Recap

Nielsen - Thurston classification of mapping classes

- finite order $\rightarrow$ Riemann surface automorph.
- reducible $\rightarrow$ a power fixes a multi-curve
- pseudo-Anosov $\rightarrow$ rich dynamics
  invariant measured foliations

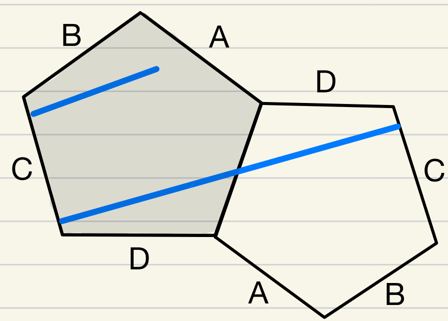
Examples





order 8 symmetry

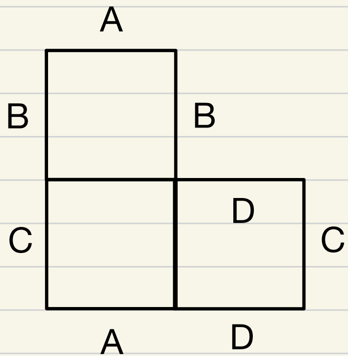
order 10 symmetry



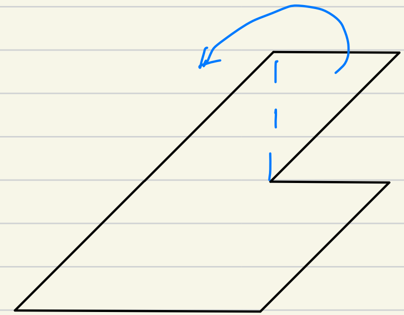
order 5 symmetry

Examples

square-tiled L shaped table



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



but not a scissors move on cylinder C

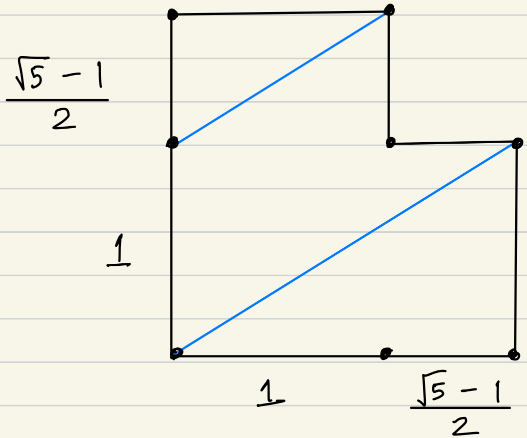
Exercise: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ gives a scissors move on cylinders B and C to get back original surface;

mapping class twists in core curves of horizontal cylinders

The product $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ gives a pseudo-Anosov.

Examples

pseudo-Anosov on a golden L



Exercise: show that

$$\begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 1 \end{bmatrix}$$ gives a twist

in horizontal cylinders

Hint: the diagonals in blue have same slopes.

Then $$\begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1+\sqrt{5}}{2} & 1 \end{bmatrix}$$ gives a pseudo-Anosov.

Veech groups $SL(X,q)$

Recall $SL(2,\mathbb{R})$ action on components of quadratic strata.

$\longrightarrow$ $SL(2,\mathbb{R})$ acts on charts to $\mathbb{C}=\mathbb{R}^2$ and takes transl / half-transl to transl / half-transl hence descends.

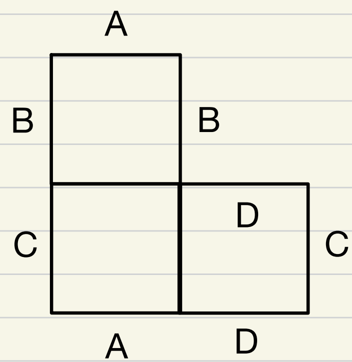
$$SL(X,q) = \{ A \in SL(2,\mathbb{R}) \text{ s.t } Aq = q \}$$

up to scissors congr.

subgrp of Mod, discrete subgrp of $SL(2,\mathbb{R})$.

Examples

square-tiled L shaped table



we already know that $SL(X,\omega)$ contains

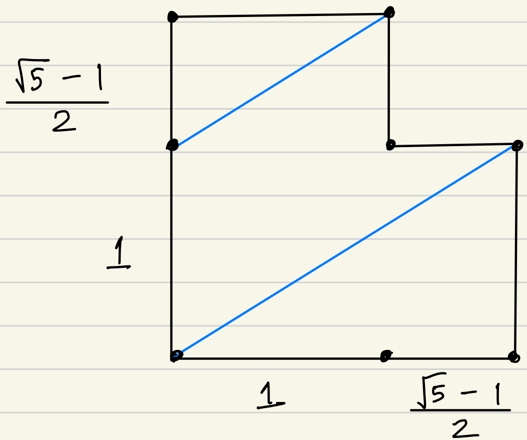
$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$SL(X,\omega) = \Gamma_2$ congruence 2 subgrp of $SL(2,\mathbb{Z})$

In general, square tiled surfaces are branched covers of flat tori, so $SL(X,\omega) < SL(2,\mathbb{Z})$.

"rational points" of an alg. variety

Examples



$SL(X, \omega)$ for the golden L

$= (2, 5, \infty)$ Hecke triangle group

Saddle connections and periods

Suppose that q is a quadratic diff.

An arc γ on the transl./half-transl. surface is a saddle connection if

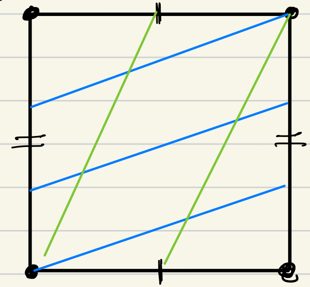
- the interior of γ embeds in $S - Z$;
- the endpoints of γ are contained in Z; and
- γ is a geodesic in the flat metric.

Choosing $\sqrt{q}$, the period of a saddle connection is defined as

$$\mathrm{per}(\gamma) = \int_\gamma \sqrt{q}$$

Saddle connection periods

Square torus



period (blue) = $3+i$

period (green) = $1+2i$

Saddle connection periods (without multiplicities) are given by primitive lattice points in $\mathbb{Z} \oplus i\mathbb{Z} \subset \mathbb{C} = \mathbb{R}^2$

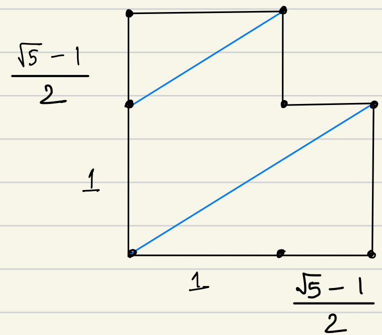
Counting asymptotics very interesting

$$\# \text{ of primitive points in } B(x_0, R) \sim \frac{6}{\pi^2} (\pi R^2)$$

Siegel-Veech const.

Saddle connection periods

Golden L



Saddle connection periods thought of as vectors in $\mathbb{R}^2$

$$\Lambda_\omega = SL(X,\omega)\cdot 1 \text{ union } SL(X,\omega)\cdot \frac{\sqrt{5}-1}{2}$$

where $SL(X,\omega)$ is the $(2,5,\infty)$ Hecke triangle group.

Counting asymptotics $\sim \frac{3\pi}{10} R^2$

Siegel-Veech constant

Coordinates on a stratum component

Let $(\underbrace{S}_{\text{surface}}, \underbrace{Z}_{\text{finite set of points}})$ be the top. data required by a stratum component

Fix a $\mathbb{Z}$-basis $\{\alpha_1, \ldots, \alpha_n\}$ for $H_1(S, Z)$.

The period map

$$q \longrightarrow \{ \mathrm{per}_{\sqrt{q}}(\alpha_1), \ldots, \mathrm{per}_{\sqrt{q}}(\alpha_n) \}$$

define local co-ordinates in a stratum component.

$SL(2,\mathbb{R})$-action

Overall Picture :

$$\begin{tikzcd}
SL(2,\mathbb{R})\cdot q \arrow[r, hook] \arrow[d] & QTeich(\underline{k}) \arrow[d] \\
SO(2,\mathbb{R})\backslash SL(2,\mathbb{R})\cdot q = \mathbb{D}_q \arrow[r, hook] & Teich(S_{g,n})
\end{tikzcd}$$

called a Teichmüller disc ; it is an isometrically embedded hyperbolic disc in $Teich(S_{g,n})$.

Teichmüller metric = Kobayashi metric

$SL(2,\mathbb{R})$ action

Theorem (Smillie):

$SL(2,\mathbb{R})$ orbit of q closed $\Longleftrightarrow$ $SL(X,q)$ is a lattice. //

Such a flat surface is called a lattice (Veech) surface; the closed curve of moduli space it generates a Teichmüller curve.

$SL(2,\mathbb{R})$ action

Eskin - Mirzakhani - Mohammadi : $SL(2,\mathbb{R})$ orbit closures are cut out in period co-ordinates by linear (homogeneous) equations with real coefficients

Non - homogeneous analogue of the Margulis - Ratner orbit classification theory.

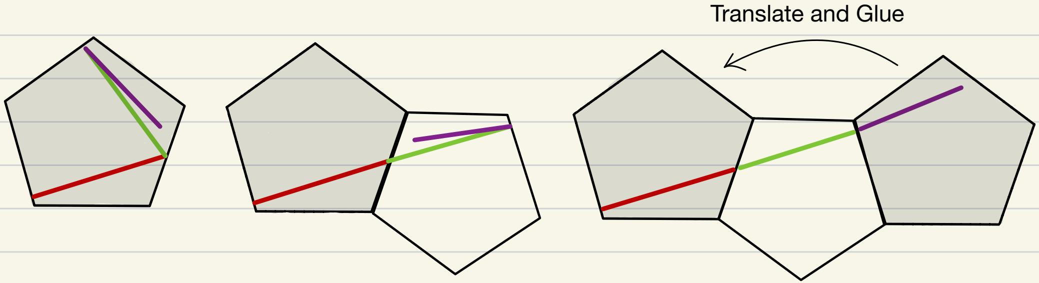
$SL(2,\mathbb{R})$ action

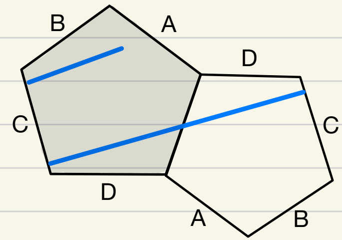
Eskin - Mirzakhani: Every $SL(2,\mathbb{R})$ invariant ergodic measure is supported on some orbit closure and is in the Lebesgue measure class.

Non-homogeneous analogue of Ratner measure classification theorem.

Filip: orbit closures are algebraic subvarieties of the moduli spaces.

Veech dichotomy





F_θ straight line foliation in direction θ.

$SL(X,q)$ lattice implies F_θ is either completely periodic or uniquely ergodic

Next Week

Sub-actions of the $SL(2,\mathbb{R})$ action, mainly

Teichmüller flow.