Lecture 5

$SL(2,\mathbb{R})$ action

Recap

1 Period co-ordinates

2 Overall Picture :

$$
\begin{tikzcd}
SL(2,\mathbb{R}) \cdot q \arrow[r, hook] \arrow[d] & QTeich(\underline{k}) \arrow[d] \\
SO(2,\mathbb{R}) \backslash SL(2,\mathbb{R}) \cdot q = \mathbb{D}_q \arrow[r, hook] & Teich(S_{g,n})
\end{tikzcd}
$$

$SL(2,\mathbb{R})$ action

Theorem (Smillie):

discrete subgrp of $SL(2,\mathbb{R})$ s.t $\mathrm{vol}(SL(2,\mathbb{R})/SL(X,q)) < \infty$

$SL(2,\mathbb{R})$ orbit of q closed $\iff$ $SL(X,q)$ is a lattice. //

Such a flat surface is called a lattice (Veech) surface; the closed curve of moduli space it generates a Teichmüller curve.

$SL(2,\mathbb{R})$ action

Eskin - Mirzakhani - Mohammadi : $SL(2,\mathbb{R})$ orbit closures are cut out in period co-ordinates by linear (homogeneous) equations with real coefficients

Non-homogeneous analogue of the Margulis - Ratner orbit classification theory.

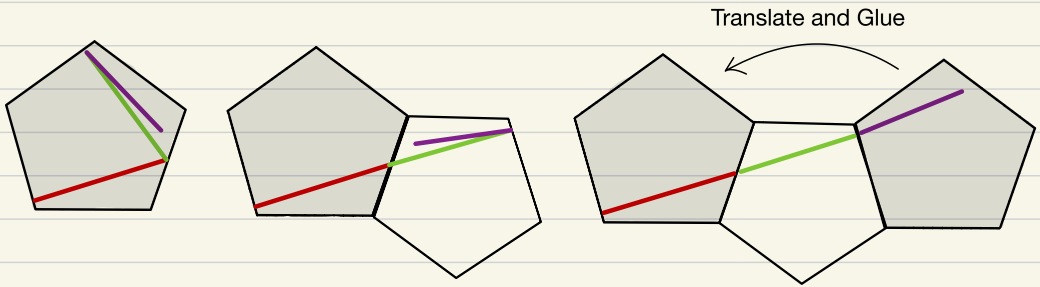
Filip : orbit closures are algebraic subvarieties of the moduli spaces.

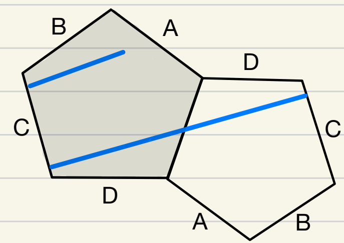
$SL(2,\mathbb{R})$ action

Eskin - Mirzakhani: Every $SL(2,\mathbb{R})$ invariant ergodic measure is supported on some orbit closure and is in the Lebesgue measure class.

Non-homogeneous analogue of Ratner measure classification theorem.

Veech dichotomy





F_θ straight line foliation in direction θ.

$SL(X,q)$ lattice implies F_θ is either completely periodic or uniquely ergodic

Masur-Veech measure

Fix a $\mathbb{Z}$-basis $\{\alpha_1, \ldots, \alpha_n\}$ for $H_1(S, \mathbb{Z})$.

The period map

$$q \longrightarrow \{ \mathrm{per}_{\sqrt{q}}(\alpha_1), \ldots, \mathrm{per}_{\sqrt{q}}(\alpha_n) \}$$

define local co-ordinates in a stratum component.

Thus differentials in a stratum component can be thought of as elements in relative cohomology

Masur-Veech measure

Normalize the Lebesgue measure in relative cohomology so that the co-volume of the integral lattice is 1.

Different choices of period co-ordinates differ by unimodular action on relative cohomology.

So the measure is well defined and is $SL(2,\mathbb{R})$ invariant.

called the Masur-Veech measure

Sub-actions of the $SL(2,\mathbb{R})$-action

g_t : diagonal action $\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$

Teichmüller geodesic flow

h_s : horocycle flow $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$

r_θ : rotations $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Sub-actions of the $SL(2,\mathbb{R})$-action

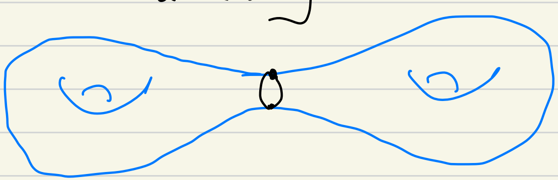
Lot is known for g_t, h_s in the homogeneous setting

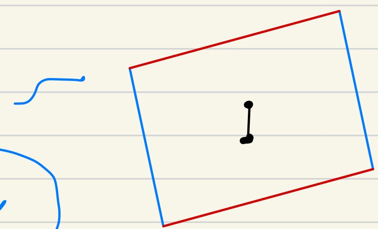
Teichmüller setting $\longrightarrow$ inhomogeneity

A flat surface q is ϵ-thin if it contains a saddle connection α s.t $|\mathrm{per}_q(\alpha)| < \epsilon$.

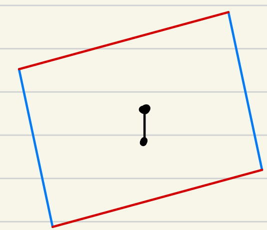
Geometry around thin q is different from geometry around thick q.

Schematically





$\xrightarrow[\text{the slit}]{\text{glue across}}$



Sub-actions of the $SL(2,\mathbb{R})$-action

Think $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$

Rank 1 versus higher rank $SL(d,\mathbb{R})/SL(d,\mathbb{Z})$

Thick part is "negatively curved"

Thin parts have product metric features and complicated intersections. Think $SL(d,\mathbb{R})/SL(d,\mathbb{Z})$, $d>2$

By and large, rank 1 behaviour prevails.

Teichmüller flow

g_t : Teichmüller geodesic flow behaves analogous to the homogeneous setting in rank 1.

g_t is ergodic w.r.t the Masur-Veech measure

g_t is exponentially mixing satisfying Moore-Ratner type decay of correlations

— Avila-Gouezel-Yoccoz

Avila-Gouezel

in fact, has spectral gap for its action on the appropriate class of functions.

Horocycle flow

h_s : horocycle flow differs significantly from the homogeneous setting

: examples of exotic h_s orbit closures and h_s invariant measures

— Chaika - Smillie - Weiss .

Sample theorem demonstrating analogies for Teich. flow

Discussion related Sullivan, Masur log-laws, Khintchine theorems

Continued fractions : Given $x \in [0,1]$, write it as

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

Gauss: For Leb-almost every x

$$\lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \infty$$

Sample theorem demonstrating analogies for Teich. flow

Khintchine : for any $\epsilon > 0$

$$\lim_{n\to\infty} \mathrm{Leb}\left\{ x \in [0,1] \text{ such that } \left| \frac{a_1+\dots+a_n}{n\log n} - \frac{1}{\log 2} \right| > \epsilon \right\} = 0$$

Borel-Bernstein : For Leb-almost every $x \in [0,1]$

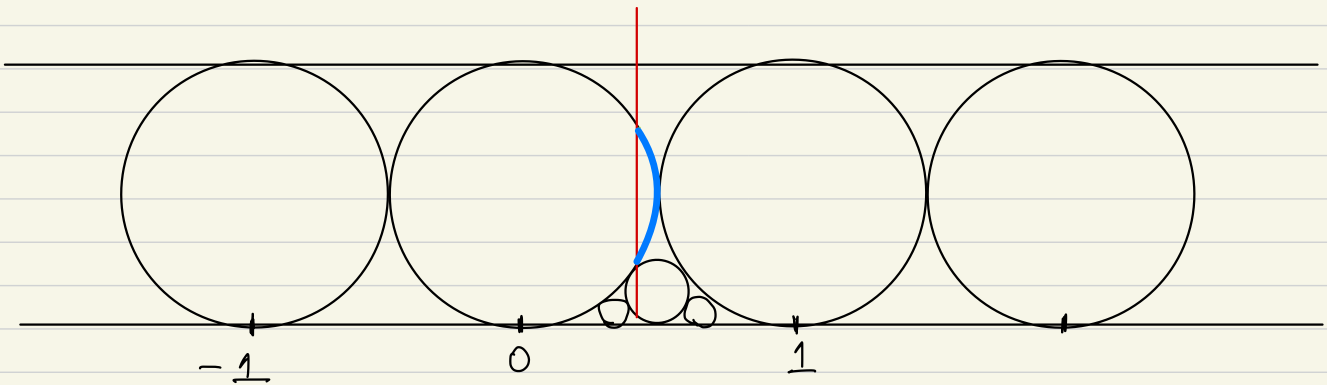
$a_n > n(\log n)(\log\log n)$ for ∞ many n.

Diamond - Vaaler : For Leb-almost every $x \in [0,1]$

$$\lim_{n\to\infty} \frac{a_1+\dots+a_n - \max_{k\le n} a_k}{n\log n} = \frac{1}{\log 2}$$

Sample theorem demonstrating analogies for Teich. flow

Connection to hyperbolic geometry of the modular surface



Excursions in horoballs : $E_1(x) = \ell_{\partial H}($ blue segment $)$

Similarly E_2, E_3, ...

Turns out $E_k(x) = a_k(x)$

up to fixed additive error.

Sample theorem demonstrating analogies for Teich. flow

Continuous time versions of along typical hyperbolic geodesics of all results on previous slides.

Exact results along typical Teichmüller geodesics on $SL(2,\mathbb{R})$ orbit closures

Limits related to Siegel-Veech constants.

Next time

Kontsevich - Zorich cocycle and Lyapunov exponents