RECURRENCE OF QUADRATIC DIFFERENTIALS FOR HARMONIC MEASURE.

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ABSTRACT. We consider random walks on the mapping class group that have finite first moment with respect to the word metric, whose support generates a non-elementary subgroup and contains a pseudo-Anosov map whose invariant Teichmüller geodesic is in the principal stratum of quadratic differentials. We show that a Teichmüller geodesic typical with respect to the harmonic measure for such random walks, is recurrent to the thick part of the principal stratum. As a consequence, the vertical foliation of such a random Teichmüller geodesic has no saddle connections.

1. INTRODUCTION

Let S be an orientable surface of finite type. Let $\mathrm{Mod}(S)$ be the mapping class group of orientation preserving diffeomorphisms of S modulo isotopy. Let $\mathcal{T}(S)$ be the Teichmüller space of marked conformal structures on S, and the moduli space $\mathcal{M}(S)$ of Riemann surfaces is the quotient $\mathcal{T}(S)/\mathrm{Mod}(S)$. Let $\mathcal{Q}(S)$ be the space of unit area quadratic differentials, which may be identified with the unit cotangent bundle of $\mathcal{T}(S)$. A unit area quadratic differential determines a unit area flat metric, and we shall only ever consider flat metrics which have unit area. We shall write π for the projection map $\pi\colon \mathcal{Q}(S) \to \mathcal{T}(S)$ which sends a quadratic differential to the underlying Riemann surface. For punctured surfaces, the quadratic differentials in $\mathcal{Q}(S)$ are assumed to be meromorphic with poles only at punctures and with every puncture a simple pole. The space $\mathcal{Q}(S)$ is stratified by the order of the zeros of the quadratic differential; the principal stratum $\mathcal{Q}_{pr}(S)$ consists of those quadratic differentials whose zeros are all simple. For the remainder of this paper we shall assume that we have fixed the surface S, and so we shall omit it from our notation, and just write $\mathcal{T}$ for $\mathcal{T}(S)$, and so on.

The ϵ-thick part of Teichmüller space $\mathcal{T}$, which we shall denote $\mathcal{T}(\epsilon)$, is the collection of all conformal structures corresponding to hyperbolic metrics in which no simple closed curve has length less than ϵ. The complement $\mathcal{T} \setminus \mathcal{T}_\epsilon$ is called the ϵ-thin part of Teichmüller space. The thick part $\mathcal{T}(\epsilon)$ is mapping class group invariant, and we shall write $\mathcal{M}(\epsilon)$ for the quotient, which is a subset of moduli space $\mathcal{M}$ and is called the ϵ-thick part of moduli space. The ϵ-thick part of a connected component of a stratum is the set of all quadratic differentials in the component for which the q-length of all saddle connections is at least ϵ. The principal stratum is connected, and we shall write $\mathcal{Q}_{pr}(\epsilon)$ for the ϵ-thick part of the principal stratum. Maskit [Mas85] showed that the existence of a short curve in a hyperbolic metric implies that the curve is short in any compatible unit area flat metric. More precisely, given $\epsilon > 0$, there is an $\epsilon' > 0$, such that for any hyperbolic metric X in the thin part $\mathcal{T} \setminus \mathcal{T}(\epsilon)$, and for any flat metric in the same conformal class as X, the length of any curve in the flat metric metric is at most ϵ', i.e. $\pi^{-1}(\mathcal{T} \setminus \mathcal{T}(\epsilon)) \subset \mathcal{Q} \setminus \mathcal{Q}(\epsilon')$. A simple closed curve α either has a unique geodesic representative in the flat metric, which is a concatenation of saddle connections, or else there is a maximal flat cylinder on S foliated by parallel closed geodesics. In the latter case, the boundary curves of the cylinder will contain singularities, and hence saddle connections. In either case, the existence of a short curve in the

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flat metric implies the existence of a short saddle connection. However, even if there are no short simple closed curves, there may be arbitrarily short saddle connections.

In summary, the thick part of a strata of quadratic differentials has a projection into moduli space that is contained in a thick part of moduli space, i.e. for any $\epsilon > 0$ there is an $\epsilon' > 0$ such that $\pi(\mathcal{Q}_{pr}(\epsilon')) \subset \mathcal{T}(\epsilon)$. We remark however, that any point in $\mathcal{T}$ has a pre-image in $\mathcal{Q}$ which contains points which do not lie in the thick part of the principal strata.

By the Thurston classification, mapping classes are periodic, reducible or pseudo-Anosov. A pseudo-Anosov map g has a unique invariant Teichmüller geodesic γ_g. Given a point $X \in \gamma_g$ there is a unique quadratic differential q at X in the direction of γ_g. If the invariant Teichmüller geodesic is given by a quadratic differential that lies in the principal stratum, then we say that the pseudo-Anosov map is in the principal stratum.

We consider random walks on the mapping class group $\text{Mod}(S)$ that have finite first moment with respect to word metric and whose support generates a non-elementary subgroup of $\text{Mod}(S)$, i.e. the subgroup generated by the support of the initial distribution contains a pair of pseudo-Anosov maps with distinct stable and unstable measured foliations. In independent work, Maher [Mah11] and Rivin [Riv08] showed that the probability that a random walk gives a pseudo-Anosov map tends to 1 in the length of the sample path, and in particular, the invariant foliations of pseudo-Anosov elements do not contain saddle connections. As a refinement of these results, we showed the following in [GM17], answering a question of Kapovich and Pfaff [KP15]:

Theorem 1.1. *Let S be a connected orientable surface of finite type, whose Teichmüller space $\mathcal{T}(S)$ has complex dimension at least two. Let μ be a probability distribution on $Mod(S)$ such that*

(1) *μ has finite first moment with respect to d_{Mod},*
(2) *$Supp(\mu)$ generates a non-elementary subgroup H of $Mod(S)$, and*
(3) *The semigroup generated by $Supp(\mu)$ contains a pseudo-Anosov g such that the invariant Teichmüller geodesic γ_g for g lies in the principal stratum of quadratic differentials.*

Then, for almost every bi-infinite sample path $\omega = (w_n)_{n\in\mathbb{Z}}$, there is positive integer N such that for all $n \geqslant N$ the mapping class w_n is a pseudo-Anosov map in the principal stratum, that is its invariant Teichmüller geodesic is given by a quadratic differential with simple zeros and poles. Furthermore, almost every bi-infinite sample path determines a unique Teichmüller geodesic γ_ω with the same limit points as the bi-infinite sample path, and this geodesic also lies in the principal stratum.

In this note, we prove the following recurrence result, answering a further question of Algom-Kfir, Kapovich and Pfaff [AKKP17]:

Theorem 1.2. *Let S and μ satisfy the hypothesis of Theorem 1.1. Then there exists $\epsilon(S,\mu) > 0$ such that almost every bi-infinite sample path $\omega = (w_n)_{n\in\mathbb{Z}}$ determines a unique Teichmüller geodesic γ_ω in the principal stratum of quadratic differentials with the same limit points in $PMF(S)$ as ω, and moreover γ_ω is recurrent to the ϵ-thick part of the principal stratum.*

Recurrence to the thick part of the moduli space $\mathcal{M}$ is shown in Kaimanovich-Masur [KM96] and does not require the extra hypothesis that the subgroup generated by $\text{Supp}(\mu)$ contains a pseudo-Anosov in the principal stratum. With this extra hypothesis, Theorem 1.2 is a finer recurrence statement and implies their result. A consequence of Theorem 1.2 and [Mas92, Theorem 1] is the following refinement of Theorem 1.1.

Corollary 1.3. *Let S and μ satisfy the hypothesis of Theorem 1.1. Then almost every bi-infinite sample path ω determines a unique Teichmüller geodesic γ_ω in the principal stratum of quadratic differentials with the same limit points as ω, and the vertical and horizontal projective measured foliations corresponding to γ_ω are uniquely ergodic with no vertical and horizontal saddle connections.*

This corollary follows from the fact that if a quadratic differential has a saddle connection which is contained in a leaf of the horizontal or vertical foliations, then the length of this saddle connection tends to zero in one direction along the geodesic, and so the geodesic cannot be recurrent to the thick part of a strata. Corollary 1.3 implies that if one passes from measured foliations to measured laminations then the lamination given by γ_ω are principal i.e., they have all complementary regions ideal triangles or once-punctured monogons.

The proof of the recurrence result, Theorem 1.2, follows from the fellow traveling discussion in Section 2 below and the ergodicity of the shift map on $\mathrm{Mod}(S)$.

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2. Fellow traveling and thickness

Let $\mathcal{Q}_{\rm pr}$ be the principal stratum of quadratic differentials. Let $\mathcal{Q}_{\rm pr}(\epsilon)$ be the set of principal quadratic differentials q for which every saddle connection β on q satisfies $\ell_q(\beta) \geqslant \epsilon$ in the induced unit area flat metric on S. We shall write $\overline{\mathcal{Q}}_{pr}$ for the quotient of $\mathcal{Q}_{pr}$ by the mapping class group.

A quadratic differential q determines a Teichmuller geodesic γ in $\mathcal{T}$, and we shall write $\widetilde{\gamma}$ for the corresponding image of q in Q under the geodesic flow, which projects down to γ. Given a quadratic differential q, we shall parameterize the corresponding geodesic by setting $q(0) = q$ and $\gamma(0) = \pi(q(0))$. We shall write γ_t for the point in $\mathcal{T}$ distance t along the geodesic in $\mathcal{T}$, and $q(t)$ for the corresponding point in $\widetilde{\gamma}$, so $\gamma(t) = \pi(q(t))$.

We say a Teichmüller geodesic γ is *recurrent in $\mathcal{M}$ in the forward direction* if there is a compact set K in $\mathcal{M}$, and a sequence of points $t_n \to \infty$, such that $\gamma(t_n) \in K$. For any compact set in $\mathcal{M}$ there is an $\epsilon > 0$ such that K is contained in the ϵ-thick part of $\mathcal{M}$, so recurrent in $\mathcal{M}$ implies recurrence to $\mathcal{M}(\epsilon)$ for some $\epsilon > 0$. Masur [Mas92] showed that if γ is recurrent in $\mathcal{M}$, then γ has a uniquely ergodic vertical foliation. We say a Teichmüller geodesic γ is *recurrent in $\overline{\mathcal{Q}}_{pr}$ in the forward direction* if there is a compact set K in $\overline{\mathcal{Q}}_{pr}$, and a sequence of points $t_n \to \infty$, such that $q(t_n) \in K$. Any compact set in $\overline{\mathcal{Q}}_{pr}$ is contained in $\overline{\mathcal{Q}}_{pr}(\epsilon)$ for some $\epsilon > 0$, so recurrence in $\overline{\mathcal{Q}}_{pr}$ implies recurrence to the thick part $\overline{\mathcal{Q}}_{pr}(\epsilon)$, for some sufficiently small ϵ. Recurrence in $\overline{\mathcal{Q}}_{pr}$ implies recurrence in $\mathcal{M}$, and furthermore recurrence in $\overline{\mathcal{Q}}_{pr}$ implies that the vertical foliation of γ contains no saddle connections, as the length of a vertical saddle connection tends to zero as $t \to \infty$.

Proposition 2.1. *Suppose that a Teichmüller geodesic γ, determined by a quadratic differential $q_0 \in \mathcal{Q}_{pr}(\epsilon)$, is recurrent to $\overline{\mathcal{Q}}_{pr}(\epsilon)$ in both the forwards and backwards directions. Suppose τ_n is sequence of Teichmüller geodesic segments that R-fellow travel γ for distance d_n such that the midpoints X_n of τ_n are within Teichmüller distance R of X_0 and $d_n \to \infty$. Let q_n be the quadratic differential at X_n corresponding to γ_n. Then there exists $\epsilon' > 0$, depending on q_0 and R, and a subsequence n_k with $k \in \mathbb{Z}$ such that $q_{n_k} \in \mathcal{Q}_{pr}(\epsilon')$ as $k \to \pm\infty$.*

Proof. As the Teichmüller geodesic γ is recurrent to the thick part $\overline{\mathcal{Q}}_{pr}(\epsilon)$, it is also recurrent to a thick part of $\mathcal{M}(\epsilon_1)$ for some $\epsilon_1 > 0$. By work of Masur [Mas92], as the Teichmüller geodesic is recurrent in both directions, this implies that both the vertical and horizontal foliations are uniquely ergodic. As $\overline{\mathcal{Q}}_{pr}$ is open, we may choose an open neighbourhood U of $\{q_t \mid t \in (-R, R)\}$ in $\overline{\mathcal{Q}}_{pr}$ which is contained in $\overline{\mathcal{Q}}_{pr}$, and whose closure $K = \overline{U}$ is also contained in $\overline{\mathcal{Q}}_{pr}$, and is compact. In particular, there is an $\epsilon_2 > 0$ such that $K \subset \overline{\mathcal{Q}}_{pr}(\epsilon_2)$.

By convergence on compact sets, one can pass to a subsequence of τ_n's that converges to bi-infinite Teichmüller geodesic γ' whose vertical and horizontal foliations have intersection number zero with the vertical and horizontal foliations (F_s, F_u) of γ. Hence, the vertical and horizontal foliations of γ' are also F_s and F_u. Since a Teichmüller geodesic with this foliation data has to be unique, $\gamma' = \gamma$. In particular, by passing to a subsequence we get that $q_{n_k} \to q_0(s)$ for some $s \in (-R, R)$. So the tail of the sequence q_{n_k} must consists of quadratic differentials in $K \subset \overline{\mathcal{Q}}_{\mathrm{pr}}$ and moreover in $\overline{\mathcal{Q}}_{\mathrm{pr}}(\epsilon')$ as $k \to \infty$ proving the proposition. $\square$

Let g be a pseudo-Anosov map whose invariant Teichmüller geodesic γ_g is in the principal stratum. Also suppose that ϵ has been chosen small enough such that γ_g is contained in $\mathcal{Q}_{\mathrm{pr}}(\epsilon)$.

Proposition 2.2. *Given a pseudo-Anosov element g and a constant R, there is an $\epsilon > 0$, such that if γ is a Teichmüller geodesic which has sequences T_n, d_n for $n \in \mathbb{N}$ such that*

(1) *$T_n, d_n \to \infty$ as $n \to \infty$, and*
(2) *there are mapping classes h_n such that the geodesic $\gamma_n = h_n(\gamma_g)$ has a segment that R-fellow travels γ_t over the time interval $(T_n - d_n, T_n + d_n)$.*

Then there is a subsequence n_k such that $q_{T_{n_k}} \in \mathcal{Q}_{pr}(\epsilon)$.

Proof. Pulling back by h_n^{-1}, the sequence of geodesic segments $g_n = h_n^{-1}(\gamma(T_n - d_n, T_n + d_n))$ satisfy the hypothesis of Proposition 2.1 with respect to the geodesic γ_g which is recurrent by the virtue of being thick. The proposition then follows from Proposition 2.1. $\square$

3. Random walks and recurrence

We recall some terminology and results from [GM17]. For a point $X \in \mathcal{T}(S)$ and $r > 0$ let $B_r(X)$ be the ball of radius r centred at X. Let γ be a Teichmüller geodesic. For points X and Y on γ let $\Gamma_r(X, Y)$ be the set of Teichmüller geodesics that pass through $B_r(X)$ and $B_r(Y)$. By work of Rafi [Raf14], if X and Y lie in the thick part $\mathcal{M}(\epsilon)$, then there is an R, that depends on r and ϵ, such that every geodesic in $\Gamma_r(X, Y)$ fellow travels with constant R the geodesic segment $[X, Y]$ of γ.

Now let g be a pseudo-Anosov element in $\mathrm{Supp}(\mu)$ such that $\mu^{(j)}(g) > 0$ for some $j \in \mathbb{N}$ and the invariant Teichmüller geodesic γ_g is in the principal stratum of quadratic differentials. Without loss of generality, we choose a base-point X on γ_g. Following the proof of [GM17, Theorem 1.1], for all $k \in \mathbb{N}$ large enough let Ω_k be the set of bi-infinite sample paths $\omega = (w_n)_{n \in \mathbb{Z}}$ such that the sequence $w_n X$ converges to uniquely ergodic foliations F_+ and F_- as $n \to \infty$ and $n \to -\infty$ respectively and the Teichmüller geodesic $\gamma(F_-, F_+)$ is contained in $\Gamma_r(g^{-k}X, g^k X)$.

Let ν be the harmonic measure and $\hat{\nu}$ be the reflected harmonic measure. Let $\sigma : \mathrm{Mod}^{\mathbb{Z}} \to \mathrm{Mod}^{\mathbb{Z}}$ be the shift map. Following the proof of [GM17, Theorem 1.1], we get the following result

Proposition 3.1. *Let S and μ satisfy the hypothesis of Theorem 1.1. For any large k and for almost every bi-infinite sample path ω, there is a sequence of times $n_j \to \infty$ as $j \to \infty$ such that $\sigma^{n_j}(\omega) \in \Omega_k$.*

Since a countable intersection of full measure sets has full measure we get that

Proposition 3.2. *Let S and μ satisfy the hypothesis of Theorem 1.1. For almost every bi-infinite sample path ω there is a sequence $m_k \to \infty$ as $k \to \infty$ such that $\sigma^{m_k}(\omega) \in \Omega_k$ for all k large enough.*

Now we get to the proof of the main recurrence result, Theorem 1.2:

Proof of Theorem 1.2. By Proposition 3.2, for almost every sample path $\omega = (w_n)$ there exists a sequence m_k such that γ_ω fellow travels $w_{m_k}(\gamma_g)$ between $w_{m_k} g^{-k} X$ and $w_{m_k} g^k X$. Equivalently, the geodesics $w_{m_k}^{-1}(\gamma_\omega)$ fellow travels γ_g between $[g^{-k}X, g^k X]$. The distances $d_{\mathcal{T}}(g^{-k}X, g^k X)$ form a sequence that tends to infinity as $k \to \infty$. So by Proposition 2.1, a further subsequence of quadratic differentials given by the midpoints of the fellow travelling segments of $w_{m_k}^{-1}(\gamma_\omega)$ are in $\overline{\mathcal{Q}}_{\mathrm{pr}}(\epsilon)$. Thus γ_ω is recurrent to $\overline{\mathcal{Q}}_{\mathrm{pr}}(\epsilon)$. $\square$

Proof of Corollary 1.3. By Theorem 1.2, for almost every sample path ω the tracked Teichmüller geodesic γ_ω is recurrent to the thick part $\overline{\mathcal{Q}}_{\mathrm{pr}}(\epsilon)$. The projection to moduli space $\mathcal{M}$ of γ_ω is then recurrent to the thick part $\mathcal{M}(\epsilon')$ for some $\epsilon' > 0$. By Masur's theorem [Mas92], the vertical foliation F_s of γ_t is uniquely ergodic. Moreover, recurrence to $\overline{\mathcal{Q}}_{pr}(\epsilon)$ implies that F_s has no vertical saddle connections $\square$

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