THE FLOW GROUP OF ROOTED ABELIAN OR QUADRATIC DIFFERENTIALS

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Abstract. We define the flow group of any component of any stratum of rooted abelian or quadratic differentials (those marked with a horizontal separatrix) to be the group generated by almost-flow loops. We prove that the flow group is equal to the fundamental group of the component. As a corollary, we show that the plus and minus modular Rauzy–Veech groups are finite-index subgroups of their ambient modular monodromy groups. This partially answers a question of Yoccoz.

Using this, and recent advances on algebraic hulls and Zariski closures of monodromy groups, we prove that the Rauzy–Veech groups are Zariski dense in their ambient symplectic groups. Density, in turn, implies the simplicity of the plus and minus Lyapunov spectra of any component of any stratum of quadratic differentials. Thus, we establish the Kontsevich–Zorich conjecture.

1. Introduction

Moduli spaces of abelian or quadratic differentials consist of Riemann surfaces endowed with abelian differentials or, respectively, meromorphic quadratic differentials with at most simple poles. By integration along appropriate relative cycles, these moduli spaces are endowed with local complex (orbifold) charts known as *period coordinates*. The usual identification of $\mathbb{C}$ with $\mathbb{R}^2$ gives rise to a natural $\mathrm{SL}(2,\mathbb{R})$-action on period coordinates. The resulting diagonal action is known as the *Teichmüller flow*.

Moduli spaces are a meeting ground of many mathematical disciplines. A very deep example of this, which is moreover relevant for our work, is as follows. Fix any differential q and form its $\mathrm{SL}(2,\mathbb{R})$ orbit closure. This is dynamically defined, yet is an algebraic variety [Fil16]. In period coordinates the orbit closure is cut out by homogeneous linear equations, with (real) algebraic coefficients [EMM15].

In general, one stratifies a space of differentials by fixing various topological and combinatorial data such as the genus of the underlying surface S, the number and character of the singularities, and so on. The resulting strata are not necessarily connected; the classification of their components is known [KZ03; Lan08; CM14].

Suppose now that $\mathcal{C}$ is such a *stratum component* of abelian or quadratic differentials. There is a forgetful map from $\mathcal{C}$ to $\mathcal{M}(S)$: the moduli space of Riemann surface structures on S. Both $\mathcal{C}$ and $\mathcal{M}(S)$ are orbifolds; both have manifold covers which we will need.

For $\mathcal{M}(S)$ this story is classical. Briefly, points in $\mathcal{M}(S)$ are in fact equivalence classes; taking the universal cover breaks these classes apart. This gives

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the Teichmüller space $\mathcal{T}(S)$ which is homeomorphic to an open ball in $\mathbb{R}^{6g-6}$; here $g = \text{genus}(S)$. The deck group of this covering is the mapping class group $\text{Mod}(S)$.

To obtain a manifold cover of $\mathcal{C}$ we consider *rooted* differentials. A choice of root is simply a horizontal unit tangent vector at a singularity. The choice of root removes any symmetry of the differential and so unwraps the orbifold locus. The resulting finite cover is a manifold which may not be connected. For instance, differentials with roots at zeroes of different orders lie in different components. We fix a connected component of this cover and denote it by $\mathcal{C}^{\text{root}}$.

The maps from the previous paragraphs give us the following sequence of homomorphisms:

$$\pi_1(\mathcal{C}^{\text{root}}) \to \pi_1^{\text{orb}}(\mathcal{C}) \to \pi_1^{\text{orb}}(\mathcal{M}(S)) = \text{Mod}(S) \xrightarrow{\rho} \text{Aut}(H_1(S;\mathbb{Z})) \cong \text{Sp}(2g,\mathbb{Z})$$

Here the third map, ρ, is the symplectic representation of the mapping class group: the action of $\text{Mod}(S)$ on the homology of S. We call the image of $\pi_1^{\text{orb}}(\mathcal{C})$, inside the mapping class group, the *modular monodromy group*. The image of the modular monodromy group under ρ is known as the *monodromy group*.

The (modular) monodromy groups are "topological offspring" of the stratum component $\mathcal{C}$.

In an attempt to relate the topology and dynamics of $\mathcal{C}$ we ask the following: to what extent can $\pi_1^{\text{orb}}(\mathcal{C})$ be "detected" by the Teichmüller flow? More precisely, let $U \subseteq \mathcal{C}$ be a contractible open set missing the orbifold locus. Fix a base-point $q_0 \in U$. Consider the Teichmüller trajectories that start and end in U. For each, we connect its endpoints to q_0 inside of U to get a loop based at q_0. As U is contractible, the resulting based homotopy class is independent of the choices we made inside of U. We call these based homotopy classes *almost-flow loops*. The *flow group* of $\mathcal{C}$ associated with the pair (U, q_0) is the subgroup of $\pi_1^{\text{orb}}(\mathcal{C})$ generated by all such loops. The question can then be stated as:

Question 1.1. *For any stratum component $\mathcal{C}$ of the moduli space of abelian or quadratic differentials, is the flow group of $\mathcal{C}$ equal to $\pi_1^{\text{orb}}(\mathcal{C})$?*

This is a version of a question of Yoccoz [Yoc10, Section 9.3].[1]

This question can also be stated for rooted differentials by defining the flow group analogously. As $\mathcal{C}^{\text{root}}$ is a manifold, the open set U can be any contractible open set in $\mathcal{C}^{\text{root}}$. A positive answer to this question is our main result:

Theorem 5.6. *Let $\mathcal{C}^{\text{root}}$ be any component of a stratum of the moduli space of rooted abelian or quadratic differentials. Let $q_0 \in \mathcal{C}^{\text{root}}$ be any base-point and U any contractible open set containing q_0. Then the flow group of $\mathcal{C}^{\text{root}}$ associated with the pair (U, q_0) is equal to $\pi_1(\mathcal{C}^{\text{root}}, q_0)$.*

Since $\mathcal{C}^{\text{root}}$ is a finite cover of $\mathcal{C}$, this theorem shows that the answer to Question 1.1 is yes, at least up to finite index.

Through the zippered rectangles construction, the Teichmüller flow on $\mathcal{C}^{\text{root}}$ can be coded combinatorially by the reduced Rauzy diagram. In the course of the proof of Theorem 5.6, we prove

Theorem 5.2. *Let $\mathcal{D}^{\text{red}}$ be the reduced Rauzy diagram for $\mathcal{C}^{\text{root}}$. Then the natural homomorphism $\pi_1(\mathcal{D}^{\text{red}}) \to \pi_1(\mathcal{C}^{\text{root}})$ is surjective.*

[1] Yoccoz asks if the image of the flow group in $\text{Mod}(S)$ is all of $\text{Mod}(S)$. However, what is meant here is the modular monodromy group [Mat21].

Taking a further image to $\operatorname{Mod}(S)$, this answers up to finite index, a weaker version of Yoccoz's question. The flow group and some of its applications to Teichmüller dynamics are also discussed by Hamenstädt [Ham18, Section 4.2].

For strata of abelian differentials, previous work by Calderon and Calderon–Salter also allows us to explicitly compute the image of the flow group inside of $\operatorname{Mod}(S)$ and of $\operatorname{Aut}(H_1(S;\mathbb{Z}))$ (or some larger group, such as $\operatorname{Aut}(H_1(S,Z;\mathbb{Z}))$), up to finite index [Cal20; CS19a; CS19b; CS20].

Cocycles. Fix $\mathcal{C}$, a stratum component. Given a bundle over $\mathcal{C}$, the Teichmüller flow gives us a natural cocycle. The most studied of these is the *Kontsevich–Zorich cocycle*. This can be lifted to a connected component $\mathcal{TC}$ of the Teichmüller space of abelian or quadratic differentials (the choice of $\mathcal{TC}$ is, in general, not unique [Cal20; CS20]).

In more detail, we define a vector bundle over $\mathcal{TC}$ with a suitable fibre. In the abelian case, this fibre is the first cohomology of the underlying topological surface; in the quadratic case, it is the first cohomology of the orientation double cover. By Poincaré-duality, it is also possible to use the corresponding homology groups as the fibre.

The $\mathrm{SL}(2,\mathbb{R})$-action induces a trivial dynamical cocycle on this vector bundle. By modding out by the mapping class group, the vector bundle descends to a bundle over $\mathcal{C}$ known as the *Hodge bundle*; similarly the cocycle descends to the *Kontsevich–Zorich cocycle* [KZ97; Kon97]. In the quadratic case, the cocycle then naturally splits into two distinct symplectically orthogonal blocks, usually referred to as the *plus* (or *invariant*) and *minus* (or *anti-invariant*) pieces.

Many interesting dynamical properties of abelian or quadratic differentials can be written in terms of the Lyapunov exponents of the Kontsevich–Zorich cocycle. An important example are the deviations of ergodic averages of the linear flow on almost every abelian or quadratic differential [Zor97; EKZ14]. In fact, when the Lyapunov spectrum of the Kontsevich–Zorich cocycle is simple, these deviations can be precisely described.

Kontsevich–Zorich conjectured that the Lyapunov spectrum is simple for all abelian stratum components [Zor97, Conjecture 2; Zor99, page 1499]. Their conjecture extends naturally to the quadratic case as follows. We form the branched orientation double cover. The homology of the cover splits into the plus and minus eigenspace for the involution; the $\mathrm{SL}(2,\mathbb{R})$ action preserves this splitting. Simplicity is conjectured in both pieces [Zor18].

For the abelian case, this conjecture was established in the famous work by Avila–Viana [AV07b]. The quadratic case is known for many stratum components but not in full generality [Tre13; Gut17].

Our paper establishes simplicity in all cases; as discussed below, our proof relies on certain machinery of these previous authors, but is independent of their theorems.

Rauzy–Veech groups. The Rauzy–Veech groups are subgroups of the symplectic group generated by the matrices (in a preferred basis) induced by evaluating these cocycles over based loops in the Rauzy diagram. It follows from Theorem 5.2 that Rauzy–Veech groups have finite index in the corresponding monodromy groups. We leverage this finiteness with standard techniques of splitting zeroes [AV07b; Gut19] to extend from simpler strata to more complicated ones in order to prove the following.

Theorem 9.1. *The Rauzy–Veech groups for all components of all abelian strata are Zariski dense in their ambient symplectic groups. The same holds for the plus and minus Rauzy–Veech groups for all components of all quadratic strata.*

The groups of Theorem 9.1 that arise, by splitting singularities, from abelian strata are known to be finite index inside the ambient symplectic groups (over $\mathbb{Z}$) and hence Zariski dense. This was done by Avila–Matheus–Yoccoz [AMY18] for abelian hyperelliptic components and by the fourth author [Gut19; Gut17] for all other components mentioned above. Using our techniques, and again replying on certain machinery from previous work, our Theorem 9.1 gives Zariski density in all cases.

By the work of Benoist [Ben97], Zariski density of an appropriate Rauzy–Veech group implies that the monoids associated with the Kontsevich–Zorich cocycles are "rich" in the sense of the simplicity criterion of Avila–Viana [AV07a; AV07b]. As a consequence of Theorem 9.1, we can apply the Avila–Viana criterion to prove the Kontsevich–Zorich simplicity conjecture.

Theorem 10.1. *The Kontsevich–Zorich cocycle has a simple spectrum for all components of all strata of abelian differentials. The plus and minus Kontsevich–Zorich cocycles also have a simple spectrum for all components of all strata of quadratic differentials.*

As mentioned before, simplicity was known for all abelian [AV07b] and some quadratic stratum components [Gut17]. It is also known for the principal stratum of quadratic differentials by different methods through the recently announced solution by Eskin–Mirzakhani–Rafi of the Furstenberg problem for random walks on the mapping class group. However, we have claimed the known results as our proof is self-contained and is uniform across all stratum components.

With Theorem 5.6 in hand, we can compute the Kontsevich–Zorich cocycle over any loop in $\mathcal{C}^{\mathrm{root}}$ and not just along the Teichmüller flow. This additional flexibility implies that, for Zariski density, we can always consider a monodromy group instead of a Rauzy–Veech group.

For the monodromy groups of abelian differentials, and also for the monodromy groups induced by the minus piece of the cocycle for quadratic differentials, we directly apply some of Filip's results to obtain Zariski density [Fil17, Corollary 1.7]. For the monodromy groups induced by the plus piece of the cocycle, we need to discuss *algebraic hulls*.

The algebraic hull of the Kontsevich–Zorich cocycle restricted to a linear invariant suborbifold can be thought of as the smallest algebraic group into which the cocycle can be measurably conjugated. As such, the hull is both an algebro-geometric and an ergodic-theoretic object. Eskin–Filip–Wright showed that the algebraic hull is as large as it can be, namely it equals the stabiliser of the tautological plane (that is, the cohomology classes spanned by the real and imaginary parts of the differential) in the Zariski closure of the monodromy group [EFW18, Theorem 1.1].

The plus piece of the Kontsevich–Zorich cocycle does not meet the tautological plane. The stabiliser then equals the Zariski closure of the monodromy, and hence so does the algebraic hull. This result, together with Filip's classification of the possible Lie algebra representations of algebraic hulls [Fil17, Theorem 1.2], allows us to show that the Zariski closure of the monodromy group corresponding to the plus piece is $\mathrm{Sp}(2g, \mathbb{R})$ by a simple dimension count.

Finally, we remark that, just as Eskin–Filip–Wright's theorem shows that the algebraic hull is as large as it can be, Theorem 5.6 shows that the flow group is also as large as it can be. Thus, for stratum components, Theorem 5.6 can be considered as a dynamical analogue of Eskin–Filip–Wright's result.

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2. Strategies

We outline the steps and the key ideas in our proofs.

Passing to rooted differentials. The dynamical issues considered here are stable under passing to a finite cover of the given stratum component $\mathcal{C}$. We pass to the space $\mathcal{C}^{\mathrm{root}}$ of *rooted differentials*: differentials marked with a horizontal separatrix (or, equivalently, a horizontal unit tangent vector at a marked point). The reasons are two-fold. Unlike $\mathcal{C}$, the cover $\mathcal{C}^{\mathrm{root}}$ is a manifold, which simplifies various transversality (and fundamental group) arguments. Also, a generic rooted differential admits a description via a zippered rectangles construction.

Zippered rectangles and the based loop theorem. The *zippered rectangles construction* is originally due to Veech [Vee82] for abelian stratum components and due to Boissy–Lanneau [BL09] for quadratic stratum components. Parameter spaces of zippered rectangles, where the length of the base-arc is normalised, define contractible open sets in $\mathcal{C}^{\mathrm{root}}$ which we call *polytopes*. The union of the polytopes is dense in $\mathcal{C}^{\mathrm{root}}$. However, the complement of their union is complicated; the polytopes do not give a cell structure on $\mathcal{C}^{\mathrm{root}}$. For instance, there are compact arcs in $\mathcal{C}^{\mathrm{root}}$ that intersect polytope faces infinitely many times. See Appendix A.1 for an explicit example and relevant discussions. As a result, the *based loop theorem*, which we explain below, cannot be deduced from naïve transversality arguments.

Fortunately, as discussed by Yoccoz [Yoc10, Proposition in Section 9.3], the subset of rooted differentials that do *not* admit any zippered rectangle construction is contained in the (codimension two) set of differentials that have both a vertical and a horizontal saddle connection. Thus, any based loop $\gamma\colon [0,1] \to \mathcal{C}^{\mathrm{root}}$ can be homotoped to be disjoint from such differentials.

After this homotopy, we can cover the image of γ by finitely many reasonably nice charts. Unfortunately, these may not be contained in the interior of any of the polytopes defined above. We arrange matters so that the boundaries of these charts are codimension-one embedded submanifolds. A further homotopy makes γ transverse to these boundaries remaining covered by the charts. Since a chart may not be contained in the interior of a polytope, the lengths of the base-arcs in these charts need not be normalised. We fix the required normalisation as follows.

Given a sufficiently small subsegment of γ, where the base-arcs are not normalised, we apply the (forward or backward, as needed) Teichmüller flow. This replaces (via homotopy) the subsegment of γ by two segments contained in the flow and one segment contained in the interior of a polytope. Doing this finitely many times, we homotope γ to be a concatenation of segments which are forward or backward Teichmüller segments or completely contained inside a polytope. This is our *based loop theorem*, namely Theorem 4.23.

Rauzy induction and the Teichmüller flow. The combinatorial information of a zippered rectangles construction is an irreducible generalised permutation [BL09]. Also, there are various associated parameters such as the dimensions of the rectangles and the heights of the various zippers. The combinatorics together with the parameters uniquely specify the differential.

If we apply the forward Teichmüller flow, the base-arc grows until it violates the normalisation. At this point we pass to the largest base-arc strictly contained in the original base-arc. In this way we obtain a new irreducible generalised permutation as well as new parameters. We call a single such operation a *Rauzy–Veech move*.

The collection of all these moves gives a renormalisation procedure known as the *Rauzy–Veech renormalisation* or the *Rauzy–Veech induction*. It was originally defined by Rauzy and Veech for abelian differentials [Rau79; Vee82] and by Boissy–Lanneau [BL09] for quadratic differentials. Applying the Teichmüller flow, we obtain a sequence of pairs of combinatorics and parameters. Thus, the Rauzy–Veech renormalisation gives a coding for the Teichmüller flow.

We encode this as an "automaton" (a directed graph) as follows. The vertices are equivalence classes of irreducible generalised permutations suited to $\mathcal{C}^{\mathrm{root}}$. Two permutations π and π' are equivalent if we can precompose with a permutation σ to obtain $\pi \circ \sigma = \pi'$. There is a directed edge from $[\pi]$ to $[\rho]$ if some representative of the latter arises from a single Rauzy–Veech move. This automaton is called the *reduced Rauzy diagram*. Since the Teichmüller flow is ergodic, as shown by Masur [Mas82] and Veech [Vee82; Vee86], it follows that the reduced Rauzy diagram is *strongly connected*: there is a directed path from any vertex to any other vertex. By accelerating the renormalisation, we can derive a coding that has the properties that the Avila–Viana criterion stated below requires.

Flow groups and the fundamental group. There is a natural homomorphism from the fundamental group of the reduced Rauzy diagram (as an undirected graph) to the fundamental group of $\mathcal{C}^{\mathrm{root}}$. By leveraging the based loop theorem and Rauzy–Veech sequences for Teichmüller segments, we show that the homomorphism is surjective. This partially answers a question of Yoccoz [Yoc10, Remark in Section 9.3].

We use the based loop theorem, and the above surjectivity, to show that the flow group is equal to the fundamental group of $\mathcal{C}^{\mathrm{root}}$. See Theorem 5.5 and Theorem 5.6. In other words, at the level of the fundamental group, the Teichmüller flow captures the topology of $\mathcal{C}^{\mathrm{root}}$, and hence the topology of $\mathcal{C}$, up to finite index.

Cocycle simplicity. By a criterion of Avila–Viana [AV07a; AV07b], simplicity of natural integrable cocycles, such as the Kontsevich–Zorich cocycle, boils down to the existence of a coding with an almost product structure and a notion of "richness" of the cocycle. As we indicated earlier, a coding with the required integrability and distortion properties can be achieved by accelerating the Rauzy–Veech renormalisation. This was done by Avila–Gouëzel–Yoccoz [AGY06] for abelian differentials and by Avila–Resende [AR12] for quadratic differentials. See Section 6 for more details. The remaining task, and the crux of the problem, is to show the richness of the cocycle. The required richness was established by Avila–Viana [AV07b] for abelian stratum components by a direct computation.

In general, to obtain the richness condition for a symplectic cocycle it is enough to establish the Zariski density of an appropriate group inside the symplectic group; using work of Benoist [Ben97] it implies the above notion of richness (Zariski density is, in fact, strictly stronger [AMY18, Appendix A]).

For the Kontsevich–Zorich cocycle, the relevant group is the Rauzy–Veech group. Its Zariski density was proved for hyperelliptic components by Avila–Matheus–Yoccoz. In fact, their result is stronger as they show it to be an explicit finite index subgroup of its ambient symplectic group [AMY18, Theorem 1.1]. This finite index result was extended by the fourth author to all abelian stratum components and to quadratic stratum components that have abelian components on their boundary [Gut19, Theorem 1.1; Gut17, Theorem 1.1].

Our main result, namely Theorem 5.6 stating that the flow group equals the fundamental group of $\mathcal{C}^{\mathrm{root}}$, is crucial to achieve the Zariski density of the Rauzy–Veech group of any stratum component. Indeed, it allows us to compute the cocycle along any loop in $\mathcal{C}^{\mathrm{root}}$ instead of only along almost T. This extra flexibility is significant since we do not have to restrict to directed loops in the reduced Rauzy diagram. In recent work [Fil17], Filip gives a finite list of possible Zariski closures of the monodromy of a linear invariant suborbifold. From this description, he also derives the fact that the Zariski closure of the monodromy restricted to the symplectic block that contains the tautological plane is the full symplectic group for this block. Combined with this fact, our Theorem 5.6 directly yields simplicity for abelian components.

A quadratic stratum component lifts to a linear invariant suborbifold of its orientation double-cover and hence Filip's result applies to this situation. The involution on the orientation double-cover splits the Kontsevich–Zorich cocycle into two symplectically orthogonal blocks, usually referred to as the *plus* (or *invariant*) piece and the *minus* (or *anti-invariant*) pieces. The minus piece contains the tautological plane. Again by Filip's corollary, the Zariski closure for the minus cocycle is the full symplectic group. Simplicity of the minus cocycle follows directly from combining this with Theorem 5.6.

It remains to tackle the plus cocyle. Here, we exploit our extra flexibility to build a dimension argument that eliminates all but the full symplectic group as the Zariski closure. We carry out the dimension argument first for components of minimal strata and hyperelliptic components with two zeros to conclude Zariski density for the monodromy groups of these components. This implies the Zariski density of their Rauzy–Veech groups as they are finite index in the monodromy groups (a consequence of Theorem 5.2). We then deal with a few remaining low genera components by using a well-known criterion for Zariski density [PR14]. Finally, we extend the density to Rauzy–Veech groups of all quadratic components by standard techniques of surgery/splitting zeroes. The density allows us to apply the Avila–Viana criterion to conclude the proof of the Kontsevich–Zorich conjecture in full generality.

3. Preliminaries

3.1. **Moduli spaces of abelian and quadratic differentials.** A connected, oriented surface S of finite type, that is, with finite genus and finitely many marked points, can be equipped with a conformal/complex structure by charts to the complex plane and holomorphic transition functions. The Teichmüller space of S is the space of marked conformal structures on S. The mapping class group $\mathrm{Mod}(S)$ is the group of orientation preserving diffeomorphisms of S modulo isotopy. The mapping class group acts on the Teichmüller space by changing the marking. The quotient $\mathcal{M}$ is the moduli space of Riemann surfaces homeomorphic to S.

The cotangent bundle to Teichmüller space is the space of (marked) meromorphic quadratic differentials on S with at most simple poles. The zeroes or poles of the differential must lie at the marked points. The quotient by the mapping class group is the moduli space $\mathcal{Q}$ of quadratic differentials. The space $\mathcal{Q}$ is stratified by the orders at the marked points and components of the strata are classified by the following combinatorial and algebraic invariants:

(1) The singularity data which can be encapsulated as follows. Let $Z \subseteq S$ be a non-empty and finite set of points; we set $n = |Z|$. Let $\kappa\colon Z \to \{-1, 0\} \cup \mathbb{N}$ be any function so that $\sum \kappa(z) = 4g - 4$. The points $z \in Z$ with $\kappa(z) = -1$ are called *simple poles* and these have to be the marked points of S. The points with $\kappa(z) = 0$ are called *regular points*. To ensure generality, we allow finitely many additional points in S to be marked as regular points.
(2) Abelian or quadratic, that is, whether the vertical foliations of differentials in the component are orientable or not. In the abelian case, the function κ is even at every $z \in Z$, so it is common to consider $\kappa/2$ instead of κ as the function giving the singularity data. We will follow this convention.
(3) Hyperelliptic or non-hyperelliptic (when possible), that is, whether differentials in the component have some rotational symmetry of order two with $2g + 2$ fixed points [Lan04].
(4) Odd or even spin (only for abelian components for which $\kappa(z)$ is even for each $z \in Z$), which is defined as the Arf invariant of a specific quadratic form [Joh80; Zor08, Appendix C].
(5) Regular or irregular (when possible), which can be distinguished by the dimension of a cohomology group corresponding to a specific divisor [CM14].

For the reader's convenience, we state the complete classification of abelian and quadratic stratum components in Section 8.1.

We note that, in general, a stratum component is an *orbifold*. We refer to the book by Boileau–Maillot–Porti [BMP03] for background on orbifolds and their fundamental groups, although in most of our exposition we will only consider the fundamental groups of actual manifolds.

Remark 3.2. *The extent to which $q \in \mathcal{C}$ is* marked *varies in different expositions. For us, we are assuming that $\mathcal{C}$ is as small as possible; so we have forgotten the marking by S and forgotten the marking by Z. However, this does not mean that we erase marked regular points—we only erase their names, as well as the names of all poles and zeros. Thus, travelling around a loop in $\mathcal{C}$, a pair of points $z, z' \in Z$ may be permuted. We deduce that, while there is no map from $\pi_1^{\mathrm{orb}}(\mathcal{C}) \to \mathrm{Sym}(Z)$, for a loop in $\mathcal{C}$ there is a well-defined conjugacy class in* $\mathrm{Sym}(Z)$.

3.3. **SL(2, $\mathbb{R}$)-action.** We fix a stratum component $\mathcal{C}$ and let q be a differential in $\mathcal{C}$. By integrating a square-root of q we get charts from S to $\mathbb{C}$ with transition functions that are translations (or half-translations), that is, transition functions that are of the form $z \to z + c$ (or $z \to \pm z + c$).

The action of the group $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^2 = \mathbb{C}$ can be restricted to the charts. As the transition functions are translations (or half-translations), the $\mathrm{SL}(2, \mathbb{R})$-action preserves the form of the transition functions. As a result, it descends to an action on the differentials. As the classifying invariants are also preserved, the $\mathrm{SL}(2, \mathbb{R})$ orbit of any differential in $\mathcal{C}$ is contained in $\mathcal{C}$.

Fixing a basis for the relative homology of (S, Z), we can compute periods. The period of a basis element is the integral of a square root of q over an arc in the homology class of the element. The periods define local charts on $\mathcal{C}$. By the famous work of Eskin–Mirzakhani–Mohammadi [EMM15], closures of $\mathrm{SL}(2,\mathbb{R})$-orbits inside $\mathcal{C}$ are suborbifolds cut out by linear equations (with real coefficients and no constant terms) in the period coordinates. Such an orbit closure is called a *linear invariant suborbifold.*

The diagonal part of the $\mathrm{SL}(2,\mathbb{R})$-action defines *the Teichmüller flow.*

3.4. **Monodromy groups.** The canonical projection $\mathcal{C} \to \mathcal{M}$ associates to a differential the underlying conformal structure. So we may consider in $\pi_1^{\mathrm{orb}}(\mathcal{M}) = \mathrm{Mod}(S)$ the image of $\pi_1^{\mathrm{orb}}(\mathcal{C})$ under the induced map on the orbifold fundamental groups. The image group $\mathrm{MMon}(\mathcal{C})$ is called the *modular monodromy group* of $\mathcal{C}$.

The mapping class group $\mathrm{Mod}(S)$ has a natural action on the (absolute) integral homology $H_1(S;\mathbb{Z})$. The action preserves the symplectic form on $H_1(S;\mathbb{Z})$ given by the algebraic intersection. As a result, $\mathrm{Mod}(S)$ admits a representation to the automorphism group of $H_1(S;\mathbb{Z})$ that preserves the symplectic form. The restriction of this symplectic representation to $\mathrm{MMon}(\mathcal{C})$ gives us a subgroup of the symplectic group which we call the *monodromy group* $\mathrm{Mon}(\mathcal{C})$ of $\mathcal{C}$.

3.5. **Rooted differentials.** For this article, we need to pass to a finite manifold cover of $\mathcal{C}$. We begin as follows.

Definition 3.6. *Suppose that $q \in \mathcal{C}$ is a differential. Let z be a zero, regular point, or pole of q. Let v be a unit tangent vector at z pointing along the horizontal foliation. We call the pair (q, v) a* rooted *differential.*

The usual difference between the order of a point and the total angle at a point allows us to show that the number of rootings of q is $4g-4+2|Z|$. However, some rootings of q may be equivalent to others when q has a symmetry.

Rooted differentials are intended to reproduce the notion of a *marked translation surface* that is widely used in the literature [Yoc10; Boi20]. We use $\mathcal{C}^{\mathrm{root}}$ to denote the space of rooted differentials.

4. Zippered rectangles

We now outline a procedure to pass from flat geometry to combinatorics with parameters. To do so, we exhibit a generic rooted quadratic differential as a collection of rectangles with gluings. This is a well-known construction, originally due to Veech [Vee82], called *zippered rectangles.* As we are interested in the topology (fundamental group) of $\mathcal{C}^{\mathrm{root}}$ and not just in the dynamics of the Teichmüller flow on $\mathcal{C}^{\mathrm{root}}$, we will present the full details of the construction for greater clarity.

There are several useful systems of parameters. We make use of the *singularity* parameters but there are other commonly used parameters such as *zipper* parameters introduced by Veech. We define the parameters and discuss how to move between them.

4.1. **The combinatorics.** A *saddle connection* for a quadratic differential is a flat geodesic that connects a pair of possibly distinct points in Z and is otherwise disjoint from Z. We say that a quadratic differential q has a *vertical vanishing coordinate* (respectively, *horizontal vanishing coordinate*) if q has a horizontal (respectively, vertical) saddle connection. Let $\mathcal{V} \subseteq \mathcal{C}^{\mathrm{root}}$ be the set of such. So $\mathcal{V}$ is a countable union of codimension-one loci.

Continuing in this way, we say that a quadratic differential q is *doubly vanishing* if q has both a horizontal and a vertical saddle connection. Let $\mathcal{W} \subseteq \mathcal{C}^{\mathrm{root}}$ be the set of such differentials. So $\mathcal{W}$ is a countable union of codimension-two loci.

A quadratic differential is said to be *vertically non-vanishing* (respectively, *horizontally non-vanishing*) if it has no horizontal (respectively, vertical) saddle connections. We will call a quadratic differential *doubly non-vanishing* if it has neither horizontal nor vertical saddle connections.

Given a rooted quadratic differential (q, v), let I_v be the horizontal separatrix defined by v, which may be finite if q has a vertical vanishing coordinate. Given a point $x \in I_v$, let $I(x)$ be the subarc of I_v from the base of the root to x.

Definition 4.2. *We say that $I(x)$ is a* base-arc *if*

(1) *the interior of $I(x)$ meets every leaf of the vertical foliation, and*
(2) *at least one of the two rays (going "up" or "down") perpendicular to I_v and starting at x hit a singularity in Z before hitting $I(x)$ a second time.*

The proof of the following lemma is analogous to the one given by Yoccoz in the abelian case [Yoc10, Proposition 5.6].

Lemma 4.3. *Let (q, v) be a rooted quadratic differential. If q is not doubly vanishing, then it admits a base-arc.*

Proof. Assume first that q has no vertical saddle connection. Thus the vertical foliation for q is minimal (as otherwise the closure of a vertical leaf would be a subsurface with boundary, containing saddle connections). Thus, the interior of any horizontal subarc of I_v will meet every leaf of the vertical foliation, so condition (1) in Definition 4.2 is met. Shortening the arc as needed, we can arrange condition (2) in Definition 4.2.

Assume instead that q has no horizontal saddle connection. Now the horizontal foliation for q is minimal. Thus, we can and do take a sufficiently long subarc of I_v so that its interior meets every vertical leaf. Then, condition (1) in Definition 4.2 is met. Making the arc longer as needed, we can arrange condition (2) in Definition 4.2. $\square$

Since I is simply connected, we can orient the vertical foliation in a small neighbourhood of I. We so orient the vertical foliation (locally) so that the "upwards direction" crosses I from right to left.

We consider the first return map to I, in q, defined by travelling along the vertical foliation. We travel in both directions (up and down) to find both the first return map and its "inverse". Let S_{t} be the (finite) set of points $x \in I$ where the upward leaf from x runs in to a singularity in Z before returning to I. We define S_{b} similarly. If q is horizontally non-vanishing, the sets S_{t} and S_{b} are disjoint, but this is not true in general. To distinguish between the points in S_{t} and S_{b}, we will add the labels t and b.

Let I_{t} be the components of $I - S_{\mathrm{t}}$. We call these the *top intervals*. Similarly, we define the *bottom intervals* I_{b} to be the components of $I - S_{\mathrm{b}}$. Again, since if q is horizontally non-vanishing, we have $I_{\mathrm{t}} \cap I_{\mathrm{b}} = \varnothing$, but this is not true in general. Thus, we will distinguish them by the labels t and b.

There is a fixed-point free involution τ on $I_{\mathrm{t}} \times \{\mathrm{t}\} \sqcup I_{\mathrm{b}} \times \{\mathrm{b}\}$ as follows. Any interval $(J, *)$, where $* \in \{\mathrm{t}, \mathrm{b}\}$, pairs with the interval $(J', *') = \tau(J, *)$ so that the first return map takes $(J, *)$ to $(J', *')$. Thus $|I_{\mathrm{t}}| + |I_{\mathrm{b}}|$ is even. We write $2d = |I_{\mathrm{t}}| + |I_{\mathrm{b}}|$.

We capture the above information, combinatorially, as follows. Let $\mathcal{A}$ be a set of d *letters*. Let $\ell = |I_\mathrm{t}|$ and $m = |I_\mathrm{b}|$. Note that the sets I_t and I_b are ordered by how the intervals appear along I. We use this to index the intervals: $I_\mathrm{t} \times \{\mathrm{t}\} = \{J_i\}_{i=1}^{\ell}$ and $I_\mathrm{b} \times \{\mathrm{b}\} = \{J_i\}_{i=\ell+1}^{\ell+m}$. We define $\pi\colon \{1, 2, \ldots, 2d\} \to \mathcal{A}$ to be any two-to-one map with the following property: for all $a \in \mathcal{A}$, if $\{i, j\} = \pi^{-1}(a)$ then $\tau(J_i) = J_j$. Associated with π is a fixed-point free involution σ of $\{1, 2, \ldots, 2d\}$ where $\sigma(i) = j$ implies $\pi(i) = \pi(j)$. Maps π of the above type were first considered by Danthony–Nogueira [DN88; DN90], and by Boissy–Lannaeu [BL09] and are known as *generalised permutations*. As shown by Boissy–Lanneau [BL09, Theorems A–D], the generalised permutations π that arise in the above construction are *irreducible*. Moreover, the set of generalised permutations that arise from $\mathcal{C}^{\mathrm{root}}$ is known as the *Rauzy class* of $\mathcal{C}^{\mathrm{root}}$. We denote the Rauzy class by $\mathcal{R}(\mathcal{C}^{\mathrm{root}})$. Moreover, as we vary over all stratum components and choices of rootings, all irreducible generalised permutations arise from this construction. See the article by Boissy–Lanneau [BL09] for a combinatorial definition of irreducibility and a proof of these facts.

We refer to the letters $\pi(1), \ldots, \pi(\ell)$ as the *top letters* for π. Similarly, we refer to the letters $\pi(\ell+1), \ldots, \pi(\ell+m)$ as the *bottom letters*. Any letter that is both a top letter and a bottom letter is called a *translation letter*. Any letter that is only a top letter (or only a bottom letter) is called a *flip letter*. We explain the terminology below.

We say that π is a *abelian permutation* if it has no flip letters. We say that π is a *quadratic permutation* it has (at least one) top flip letter and (at least one) bottom flip letter. All generalised permutations that arise in the construction above are of one of this two types.

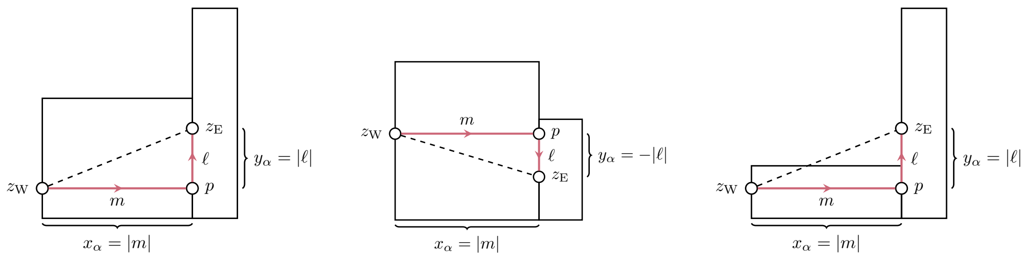
From now on, will eschew the terminology "generalised permutation" and collectively refer to abelian and quadratic permutations simply as *permutations*.

4.4. **The rectangles.** Let $\alpha \in \mathcal{A}$ and let $\pi^{-1}(\alpha) = \{i, \sigma(i)\}$. There is an associated rectangle $R = R_\alpha$ with the following properties.

(1) The horizontal sides of R are exactly $R \cap I = J_i \cup J_{\sigma(i)}$.
(2) With the exception of one rectangle, each vertical side of R contains exactly one singularity. The exceptional rectangle is either $R_{\pi(\ell+m)}$ or $R_{\pi(\ell)}$ depending on whether the right endpoint of I is in S_t or S_b, respectively.

4.5. **The zippers.** We now define the zippers. Let $p \in S_\mathrm{t}$. By definition, the perpendicular ray that goes up from p hits a singularity before it can return to I. We call the resulting vertical segment $Z(p)$ a *top zipper*. Similarly, we define *bottom zippers* to be the segments of the perpendicular rays that go down from points in S_b and hit a singularity before they return to I.

4.6. **Singularity parameters.** We have fixed an orientation on the surface. Let $R = R_\alpha$ be the rectangle with letter $\alpha \in \mathcal{A}$. Recall that every rectangle R has two vertical sides and two horizontal sides. Laying R out in the plane we call its sides the *east*, *north*, *west*, and *south* sides. By construction, the east and west sides of R lie in vertical leaves ℓ_E and ℓ_W that meet Z, the set of singularities, in exactly one point before returning to I. We also lay out ℓ_E and ℓ_W in the plane. Let z_E and z_W be the images of the singularities in ℓ_E and ℓ_W, as they lie in the plane. Also by construction, at least one of z_E or z_W lies in (the closure of) a vertical side of R.



(A) The curve γ turns left at p and can be tightened to a saddle connection.

(B) The curve γ turns right at p and can be tightened to a saddle connection.

(C) The curve γ may not be able to be tightened to a saddle connection.

FIGURE 4.7. Three cases of singularity coordinates.

Breaking symmetry, suppose that z_{W} lies in the west side of R, not just in ℓ_{W}. Let m be the horizontal spanning arc of R, which has one endpoint at z_{W}. Let p be the endpoint of m lying in ℓ_{E}. We call p the *projection* of z_{W} to ℓ_{E}. We define the *singularity width* of the letter α to be

$$x_\alpha = |m|$$

that is, the unsigned length of m. Note that x_α is exactly the width of $R = R_\alpha$.

If $p = z_{\mathrm{E}}$, we define the *singularity height* of the letter α to be $y_\alpha = 0$. Otherwise, let ℓ be the bounded segment of $\ell_{\mathrm{E}} - \{p, z_{\mathrm{E}}\}$. We orient the path $\gamma = m \cup \ell$ away from z_{W}. Note that γ turns right or left at p depending on the position of z_{E} in ℓ_{E}. This turning is defined due to the orientation on q; also it is independent of the choices made. We now define y_α the *singularity height* of the letter α. We take the magnitude of y_α to be

$$|y_\alpha| = |\ell|$$

We take the sign of y_α to be positive if and only if γ turns left at p

Remark 4.8. *The singularity height y_α is* not, *in general, the height of $R = R_\alpha$. We discuss this point further below.*

Remark 4.9. *For all but one rectangles $R = R_\alpha$, the points z_E and z_W lie in its east and west sides, respectively. When this happens, the path γ defined above can be tightened to give a saddle connection in q. The parameter $x_\alpha + iy_\alpha$ is then (up to global change of sign) the period of γ. However, there are (abelian and quadratic) differentials where γ is not homotopic (relative to its endpoints) to a saddle connection Figure 4.7. This accounts for the complexity of the definition of the singularity height y_α.*

Note that the horizontal edges of rectangles representing the top letters (correctly repeating the flip letters) are arcs whose union is exactly the base-arc I. The same holds for the bottom letters. We deduce the *width equality*

$$\sum_{k=1}^{\ell} x_{\pi(k)} = \sum_{k=\ell+1}^{\ell+m} x_{\pi(k)}. \tag{4.10}$$

Again, the left sum is over the top while the right is over the bottom. Since every translation letter appears exactly once on each side, we deduce from the width equality that $\sum x_\alpha = \sum x_\beta$; here α ranges over the top flip letters and β ranges over the bottom flip letters.

4.11. **Zipper parameters.** Breaking symmetry, let $p \in S_{\mathrm{t}} \times \{\mathrm{t}\}$ where we assume $p \neq r \times \{\mathrm{t}\}$ if $r \in S_{\mathrm{t}}$. Let $Z(p)$ be a top zipper based at p. By a slight abuse of notation, think of p as a point in I. Let $R_{\pi(i)}$ for $i \leqslant \ell$ be the rectangle to the left of $Z(p)$. Then the horizontal coordinate of p, that is the distance of p from the left-endpoint of I, is given by

$$x(p) = \sum_{j=1}^{i} x_{\pi(j)}.$$

The height of $Z(p)$ is given by

$$h(Z(p)) = \sum_{j=1}^{i} y_{\pi(j)}.$$

and we require this to be positive. This gives us the *top zipper inequalities*

$$\sum_{j=1}^{i} y_{\pi(j)} > 0 \tag{4.12}$$

for all $i < \ell$.

Similarly, if $Z(p)$ for $p \in S_{\mathrm{b}} \times \{\mathrm{b}\}$ and $p \neq r \times \{\mathrm{b}\}$ if $r \in S_{\mathrm{b}}$ is a bottom zipper and $R_{\pi(i)}$ for $i \geqslant \ell + 1$ then the horizontal coordinate is

$$x(p) = \sum_{j=\ell+1}^{i} x_{\pi(j)}$$

and the height is

$$h(Z(p)) = \sum_{j=\ell+1}^{i} y_{\pi(j)}.$$

Here we require the height $h(Z(p))$ to be negative. This gives us the *bottom zipper inequalities*

$$\sum_{j=\ell+1}^{i} y_{\pi(j)} < 0 \tag{4.13}$$

for all $\ell + 1 \leqslant i < \ell + m$.

It remains to consider the right endpoint r. The zipper height of $Z(r)$ gives us a linear relation in the y parameters. Note that the above equalities express the height $h(Z(r))$ in two ways; namely

$$h(Z(r)) = \sum_{j=1}^{\ell} y_{\pi(j)}$$

and

$$h(Z(r)) = \sum_{j=\ell+1}^{\ell+m} y_{\pi(j)}.$$

We deduce the *height* equality

$$\sum_{k=1}^{\ell} y_{\pi(k)} = \sum_{k=\ell+1}^{\ell+m} y_{\pi(k)}. \tag{4.14}$$

This is equivalent to $\sum y_\alpha = \sum y_\beta$, where α ranges over the top flip letters and β ranges over the bottom flip letters.

The height and width equalities are essentially identical. Thus, the dimensions of the space of x and y parameters are equal; they are $|\mathcal{A}|$ in the abelian case and $|\mathcal{A}| - 1$ in the quadratic case.

4.15. **Rectangle parameters.** For all rectangles $R = R_\alpha$, at least one of the points z_{E} and z_{W} lie in its east and west sides, respectively. Breaking symmetry, suppose that α is a top letter and z_{E} lies in its east side. Let $Z(p)$ for $p \in S_{\mathrm{t}} \times \{\mathrm{t}\}$ be the zipper with end point z_{E}. If α is a translation letter then there is a zipper $Z(p')$ for $p' \in S_{\mathrm{b}} \times \{\mathrm{b}\}$ with endpoint z_E such that the union $Z(p) \cup Z(p')$ is the east side of R_α. Recall that the heights of bottom zippers are negative. Hence, the height $h(R_\alpha)$ satisfies

$$h(R_\alpha) = h(Z(p)) - h(Z(p')). \tag{4.16}$$

If α is a flip letter instead then there is a zipper $Z(p')$ for $p' \in S_{\mathrm{t}} \times \{\mathrm{t}\}$ with end point z_E such that the union $Z(p) \cup Z(p')$ is the east side of R_α. The height $h(R_\alpha)$ is then

$$h(R_\alpha) = h(Z(p)) + h(Z(p')). \tag{4.17}$$

A similar discussion follows if α is a bottom letter.

4.18. **Polytopes in $\boldsymbol{\mathcal{C}^{\mathrm{root}}}$.** Because of the flexibility in choosing base-arcs, a rooted differential can have (infinitely) many different zippered rectangles constructions. For example, suppose that q is a doubly non-vanishing rooted differential. Then the vertical foliation for q is minimal; to see this, note that otherwise the closure of a vertical leaf would be a subsurface with boundary, containing saddle connections. Thus, for this q any subarc $I \subseteq I_v$ satisfying condition (2) in Definition 4.2 can serve as a base-arc.

To remove this ambiguity from the combinatorics, an additional base-arc normalisation must be imposed, as follows. Let $\mathcal{R}$ be the Rauzy class of $\mathcal{C}^{\mathrm{root}}$. Fix an irreducible permutation $\pi \in \mathcal{R}$.

Definition 4.19. *We define the set P_π of* parameters *for π to be the pairs $(x, y) \in \mathbb{R}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{A}}$ satisfying*

1. *the width and height equalities* (4.10) *and* (4.14)*,*
2. *the positivity condition $x_\alpha > 0$ for all α,*
3. *the zipper inequalities* (4.12) *and* (4.13)*, and*
4. *the* base-arc normalisation
$$1 < \sum_{i=1}^{\ell} x_{\pi(i)} < 1 + \min\{x_{\pi(\ell)}, x_{\pi(\ell+m)}\}.$$

The pair of inequalities in (4) are the promised restrictions on the length of the base-arc. Any zippered rectangles construction arising from parameters in this way is called *(base-arc) normalised.*

Let q be a doubly non-vanishing rooted differential. Let $Z(q, v)$ be the subset of x along the separatrix I_v such that $I(x)$ is a base-arc.

Lemma 4.20. *Let q be a doubly non-vanishing rooted differential. Then the base of the root is the only accumulation point of $Z(q, v)$ in I_v.*

Proof. Suppose that a point $x \in Z(q, v)$ is an accumulation point of $Z(q, v)$ and that x is not the root. As q is doubly non-vanishing, there is no upper bound on the lengths of possible base-arcs. Hence, $Z(q, v)$ contains a point x' such that the base-arc $I(x')$ is longer than $I(x)$. But then the first return map to $I(x')$ along the vertical has infinitely many intervals, which is a contradiction. See Yoccoz's lectures notes for more details [Yoc10, Section 3.1]. □

It follows that $Z(q, v)$ contains a point r whose base-arc is the shortest among those whose length is at least one. The base-arc $I(r)$ is then the unique subarc of I_v whose zippered rectangle construction yields an irreducible permutation with parameters that satisfy condition (4) of Definition 4.19.

The above procedure applies to rooted quadratic differentials off of a (somewhat complicated) measure zero set. For each such, it gives a pair (combinatorics, parameter).

The opposite direction is provided by Boissy and Lanneau [BL09, Lemma 2.12]. Suppose that π in $\mathcal{R}(\mathcal{C}^{\mathrm{root}})$ is an irreducible permutation. Suppose that (x, y) is any parameter in P_π. Then, by placing a marked point at the origin of $\mathbb{C}$, by laying out an arc on the positive real axis, by laying down rectangles, and gluing according to the associated zipper lengths, Boissy and Lanneau build a quadratic differential; the details are somewhat subtle.

We call this differential $q_\pi(x, y)$; we use $q_\pi \colon P_\pi \to \mathcal{C}^{\mathrm{root}}$ to denote the resulting map. We call the image $\mathcal{C}_\pi = q_\pi(P_\pi) \subseteq \mathcal{C}^{\mathrm{root}}$ a *polytope*. For any doubly non-vanishing rooted differential q in $\mathcal{C}_\pi$, if $(\pi, (x, y))$ are its normalised combinatorics and parameters, then $q_\pi(x, y) = q$.

On the other hand, it may happen that the polytopes arising from distinct permutations coincide as sets. More precisely, consider the following equivalence relation on permutations. Two permutations π and π' are *equivalent through re-indexing* if there is a permutation $p \in \mathrm{Sym}(\mathcal{A})$ such that $\pi' = p \circ \pi$. As letter re-indexing by p does not affect the geometric construction, it follows that if π and π' are equivalent then $\mathcal{C}_\pi = \mathcal{C}_{\pi'}$.

Lemma 4.21. *For any irreducible permutation $\pi \in \mathcal{R}$, the map q_π is a homeomorphism from P_π onto $\mathcal{C}_\pi$. Moreover, if π and π' are not equivalent then $\mathcal{C}_\pi \cap \mathcal{C}_{\pi'} = \varnothing$. The union of the sets $\mathcal{C}_\pi$ is dense in $\mathcal{C}^{\mathrm{root}}$.*

Proof. Fix π and a parameter $(x_\alpha, y_\alpha)_{\alpha \in \mathcal{A}}$. For every letter $\alpha \in \mathcal{A}$ we are given an arc γ_α connecting a pair of singularities whose period is exactly $x_\alpha + iy_\alpha$. Boissy–Lanneau show that these periods give coordinates in $\mathcal{C}^{\mathrm{root}}$ [BL09, Lemma 2.12]. Since q_π is linear in these periods, it is continuous and injective. Since P_π and $\mathcal{C}_\pi$ have the same dimension, the map q_π is a homeomorphism onto its image.

Thus, for any non-equivalent π and π' the intersection $\mathcal{C}_\pi \cap \mathcal{C}_{\pi'}$ is open. Hence, if it is non-empty, it must contain a doubly non-vanishing differential. However, this contradicts that uniqueness of the permutation given by the zippered rectangles construction.

Furthermore, every doubly non-vanishing differential q in $\mathcal{C}^{\mathrm{root}}$ lies in some $\mathcal{C}_\pi$, so we are done.

□

The above lemma shows that we can pass unambiguously from a typical differential (such as a doubly non-vanishing differential) to combinatorics and parameters.

4.22. **Based loops in $\mathcal{C}^{\mathrm{root}}$.** The main result of this article is the following theorem stating that every based loop in $\mathcal{C}^{\mathrm{root}}$ can almost be straightened out into a concatenation of Teichmüller geodesic segments.

Theorem 4.23. *Let $\mathcal{C}^{\mathrm{root}}$ be a stratum component of the moduli space of rooted quadratic differentials. Let q_0 be a base-point in $\mathcal{C}^{\mathrm{root}}$. Let $\gamma\colon [0,1] \to \mathcal{C}^{\mathrm{root}}$ be a loop based at q_0. Then, up to a homotopy relative to the base-point, the loop γ can be written as a finite concatenation of paths that are either*

- *(forward or backward) Teichmüller geodesic segments; or*
- *contained inside some polytope.*

Proof. We fix a base-point q_0 in $\mathcal{C}^{\mathrm{root}}$. For convenience, we assume that q_0 is doubly non-vanishing and hence contained in some polytope.

Recall that the set $\mathcal{W} \subseteq \mathcal{C}^{\mathrm{root}}$ of doubly vanishing rooted differentials is a countable union of codimension-two loci. Moreover, if $q \in \mathcal{C}^{\mathrm{root}} - \mathcal{W}$, then it admits a base-arc by Lemma 4.3.

Let q be a differential in $\mathcal{C}^{\mathrm{root}} - \mathcal{W}$ and I be a base-arc. This gives a permutation π and singularity parameters (x, y).

Since the singularity parameters are coordinates for $\mathcal{C}^{\mathrm{root}} - \mathcal{W}$, there exists an open set in $\mathcal{C}^{\mathrm{root}}$ containing q given by the zippered rectangles construction with underlying permutation π. Note that the base-arc I may not be normalised, so q may not belong to $\mathcal{C}_\pi$. However, the only condition that the parameters (x, y) may not satisfy to belong to $\mathcal{C}_\pi$ is

$$1 < \sum_{k=1}^{\ell} x_{\pi(k)} < 1 + \min\{x_{\pi(\ell)}, x_{\pi(\ell+m)}\},$$

which can be forced to hold by applying the Teichmüller flow. Thus, there exists $t(q) \in \mathbb{R}$ such that $g_{t(q)} q \in \mathcal{C}_\pi$.

We now define $U(q) \subseteq \mathcal{C}^{\mathrm{root}}$ to be a contractible open set around q obtained by varying the parameters (x, y) by a very small amount so that $g_{t(q)} U(q) \subseteq \mathcal{C}_\pi$. That is, with respect to the parameters (x, y), the set $U(q)$ is a *box* with sides parallel to the coordinate planes. Therefore, $\partial U(q)$ is a union of finitely many codimension-one embedded submanifolds (with boundary) in $\mathcal{C}^{\mathrm{root}}$.

The locus $\mathcal{V}$ of vanishing rooted differentials, that is, rooted differentials with a horizontal or vertical saddle connection, can be covered by countably many relatively open codimension-one charts. Hence, we may apply a homotopy (relative to q_0) to arrange that γ is transverse to $\mathcal{V}$. This can be done by using standard techniques in differential topology [Hir94, Theorem 2.5, page 78]. After this, the loop γ is disjoint from $\mathcal{W}$.

Now, the boxes $(U(\gamma(s)))_s$ cover γ, so, by compactness, there exists a finite collection $s_0, \ldots, s_n \in [0,1]$ such that $(U(\gamma(s_j)))_j$ covers γ. Let $U_j = U(\gamma(s_j))$.

We now perform a further homotopy (relative to q_0) supported in the union of the boxes. Again appealing to standard techniques [Hir94, Theorem 2.5, pp. 78], we now have that γ is transverse to the sides of the boxes U_j and is again transverse to $\mathcal{V}$.

We obtain that γ intersects ∂U_j only finitely many times and, therefore, that $\gamma^{-1}(U_j)$ is a finite union of intervals in $[0,1]$ for each k. All such intervals are open, except for possibly two intervals $[0, s)$ and $(s', 1]$. If these two intervals exist in $\gamma^{-1}(U_j)$, we replace them by their union $[0, s) \cup (s', 1]$.

Now, we select a minimal subcollection $J_0, \ldots, J_m$ of these sets that covers $[0,1]$. Thus, $\gamma(J_k)$ is contained inside some U_j, which we denote by V_k. Observe that the

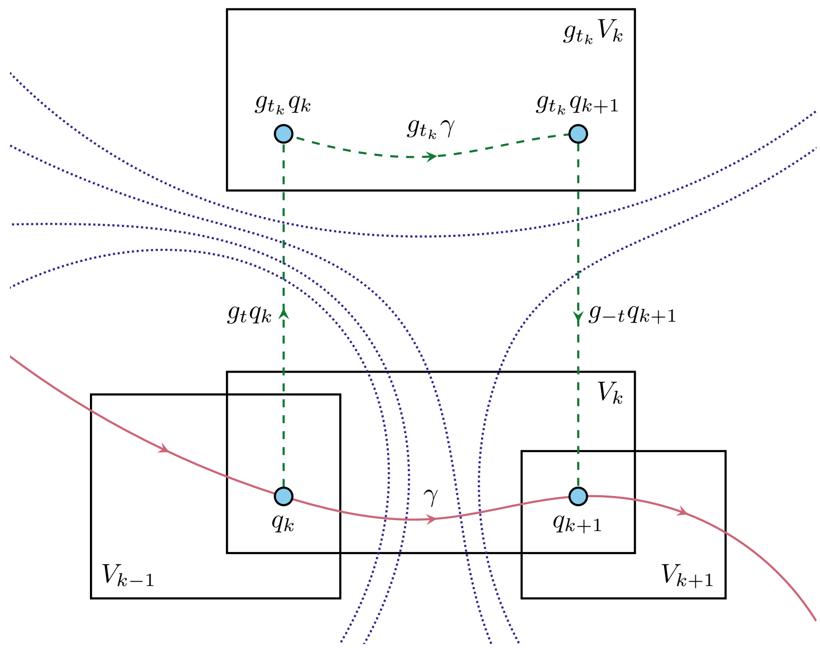


FIGURE 4.24. Illustration of the proof of Theorem 4.23. Part of the loop γ is depicted as a solid curve. The dotted lines represent the boundaries of the polytopes. Unlike the boxes V_{k-1} and V_{k+1}, the box V_k is not contained inside a polytope, so the Teichmüller flow must be applied to it. The resulting segment δ_k is shown as dashed curve.

list $V_0, \ldots, V_m$ may contain repetitions. By setting $J_{m+1} = J_0$, we assume that the indices are chosen so that $J_k \cap J_{k+1}$ is a non-empty open interval or a set of the form $[0, s) \cup (s', 1]$. Without loss of generality, we can assume that $0 \in J_m \cap J_0$, since this can be arranged by covering J_0 and J_m with smaller intervals and rearranging the indices.

Since γ is transverse to $\mathcal{V}$, we have that $\gamma(s)$ lies in $\mathcal{V}$ for at most countably many $s \in [0, 1]$. Thus, there exists a doubly non-vanishing quadratic differential q_{k+1} in the image of each set $\gamma(J_k \cap J_{k+1})$ (with $q_{m+1} = q_0$). Hence, we obtain a sequence of times $0 = s_0 \leqslant s_1 \leqslant \cdots \leqslant s_m = 1$ such that the closed intervals $[s_k, s_{k+1}] \subseteq [0, 1]$ cover $[0, 1]$ and $\gamma(s_k) = q_k$.

Since V_k is equal to one of the U_j by construction, there exists a real number t_k such that $g_{t_k}V_k$ is completely contained inside some polytope $\mathcal{C}_{\pi_k}$. Let δ_k be the path starting at q_k and ending at q_{k+1} given by the concatenation of the paths

- $g_t q_k$ for $t \in [0, t_k]$;
- $g_{t_k}\gamma(s)$ for $s \in [s_k, s_{k+1}]$; and
- $g_{-t}q_{k+1}$ for $t \in [-t_k, 0]$.

if $t_k \geqslant 0$ or

- $g_{-t}q_k$ for $t \in [0, -t_k]$;
- $g_{t_k}\gamma(s)$ for $s \in [s_k, s_{k+1}]$; and
- $g_t q_{k+1}$ for $t \in [t_k, 0]$.

if $t_k \leqslant 0$. See Figure 4.24 for an illustration of this proof.

The union of the arcs δ_k and $\gamma_k = \gamma|[s_k, s_{k+1}]$ bounds a disc in $\mathcal{C}^{\mathrm{root}}$ foliated by the arcs $g_t\gamma_k$ where $t \in [0, t_k]$. In particular, δ_k is homotopic to γ_k, relative to the endpoints.

Let δ be the concatenation of the paths $(\delta_k)_k$. By construction, δ is a closed curve, homotopic to γ. Moreover, the pieces $g_{t_k}\gamma(s)$ for $s \in [s_k, s_{k+1}]$ in the concatenation are the only paths that are not (forward or backward) Teichmüller geodesic segments. This concludes the proof of the theorem. □

Remark 4.25. *We do not attempt to make optimal choices to reduce the length of geodesic pieces in the concatenation for δ. A simple way to reduce these lengths is to choose the normalised zippered rectangle construction whenever q admits one. Thus, if $q \in \mathcal{C}^{\mathrm{root}} - \mathcal{W}$ is contained in some polytope, then we can choose the box $U(q)$ to be contained inside the same polytope and set $t(q) = 0$ for such boxes. On the other hand, when q does not lie in any polytope we can choose the length of the base-arc to be as close to 1 as possible, so $t(q)$ is as small as possible. See Appendix A.4 for a concrete example of the construction.*

4.26. **Rauzy–Veech induction and the Teichmüller flow.** We will now define Rauzy–Veech induction on zippered rectangles. The induction is defined by passing to the smaller base-arc with length $|I| - \min\{x_{\pi(\ell)}, x_{\pi(\ell+m)}\}$.

Let π be an irreducible permutation in $\mathcal{R}$. Let (x, y) be singularity parameters for a zippered rectangle construction with underlying permutation π. Let $\alpha = \pi(\ell)$ and $\beta = \pi(\ell+m)$. Since π is irreducible, $\alpha \neq \beta$ and we will assume that $x_\alpha \neq x_\beta$. Breaking symmetry, suppose that $x_\alpha > x_\beta$. In this case, we say that the *top letter wins*. We set the new width parameters as

$$x'_\alpha = x_\alpha - x_\beta,$$

and $x'_\rho = x_\rho$ for all $\rho \neq \alpha$. Similarly we set the new height parameters as

$$y'_\beta = y_\beta + y_\alpha,$$

and $y'_\rho = y_\rho$ for all $\rho \neq \beta$. The parameter transformations can be encoded in terms of a matrix. Let $E = (e_{rs})_{r,s\in\mathcal{A}}$ be the $\mathcal{A} \times \mathcal{A}$ elementary matrix with ones along the diagonal, $e_{\alpha\beta} = 1$ and all other entries zero. Then, $Ex' = x$ and $E^T y = y'$.

To define the new permutation we consider the two cases:

(1) α is a translation letter; or
(2) α is a flip letter.

Suppose α is a translation letter and let $\pi(j) = \alpha$ for some $\ell + 1 \leqslant j < \ell + m$. We then set

- $\pi'(i) = \pi(i)$ for all $i \leqslant j$,
- $\pi'(j+1) = \beta$, and
- $\pi'(i) = \pi(i-1)$ for all $i > j+1$.

Suppose now that α is a flip letter and let $\pi(j) = \alpha$ for some $1 \leqslant j < \ell$. We set the top indices to range from 1 to $\ell + 1$ and the bottom indices to range from $\ell + 2$ to $\ell + m$ and then set

- $\pi'(i) = \pi(i)$ for all $i < j$,
- $\pi'(j) = \beta$,
- $\pi'(i) = \pi(i-1)$ for all $i > j$.

With the above definitions, We set $R_{\mathrm{t}}(\pi, x, y) = (\pi', x', y')$. If $x_\alpha < x_\beta$ instead, then we say that the *bottom letter wins*. The definition of R_{b} is analogous.

The codimension-one locus $x_\alpha = x_\beta$ is contained in $\mathcal{V}$. The induction is undefined for it.

Rauzy–Veech induction makes the Rauzy class $\mathcal{R} = \mathcal{R}(\mathcal{C}^{\mathrm{root}})$ into a directed graph. The vertices of $\mathcal{D}$ are irreducible permutations in $\mathcal{R}$ with an arrow from permutation π to π' if $\pi' = R_{\mathrm{t}}(\pi)$ or $R_{\mathrm{b}}(\pi)$. A component $\mathcal{D}$ of this graph is called a *Rauzy diagram*.

We now explain the coding of the Teichmüller flow using normalised parameters and Rauzy–Veech induction.

Let $\mathcal{C}_\pi = q_\pi(P_\pi)$ be the polytope given by normalised singularity parameters for an irreducible permutation π. Let $\alpha = \pi(\ell)$ and $\beta = \pi(\ell + m)$.

We say that q is a *forward-tied* differential for π if there exists a sequence $q_n = q_\pi(\pi, x_n, y_n)$ converging to q for which the length of the base-arc I tends to $1 + \min\{x_{\pi(\ell)}, x_{\pi(\ell+m)}\}$ from below. Similarly, we say q is *backward-tied* for π if it is the limit of a sequence where the length of the base-arc tends to 1 from above.

The period of the base-arc is linear in period coordinates. Therefore, the backward-tied differentials are contained in a codimension-one locus in $\mathcal{C}^{\mathrm{root}}$. It will become clear using Rauzy–Veech induction that the forward-tied differentials are also contained in a codimension-one locus in $\mathcal{C}^{\mathrm{root}}$.

Let $\mathcal{F}(\pi)$ and $\mathcal{B}(\pi)$ be the set of forward-tied and backward-tied differentials for π. We call these the *flow faces*. Let q be a forward-tied differential. Suppose that there is a sequence $q_n = q_\pi(\pi, x_n, y_n)$ converging to q for which the parameters x_n and y_n are bounded away from zero and infinity respectively. Then the sequence (x_n, y_n) converges to some (x, y) in the closure of P_π such that all its widths $x_\alpha > 0$ and heights y_α are bounded. It follows that q is contained in the interior of $\mathcal{F}(\pi)$ and the map q_π extends from P_π to such parameters. We can similarly characterise differentials in the interior of $\mathcal{B}(\pi)$ and extend q_π to such parameters.

Let q be a forward-tied differential in the interior of $\mathcal{F}(\pi)$ and further suppose that $x_\alpha > x_\beta$. The Rauzy–Veech induction on (π, x, y) is then defined and let $(\pi', x', y') = R_{\mathrm{t}}(\pi, x, y)$. Let I be the base-arc for $q = q_\pi(\pi, x, y)$ and I' the base-arc for $q = q_{\pi'}(\pi', x', y')$. Note that

$$|I'| = |I| - x_\beta = 1 + \min\{x_\alpha, x_\beta\} - x_\beta = 1.$$

This means that q' is a backward-tied differential for π'.

We conclude that the interiors of $\mathcal{F}(\pi) \cap \{x_\alpha > x_\beta\}$ and $\mathcal{F}(\pi) \cap \{x_\alpha < x_\beta\}$ are identified with corresponding subsets of $\mathcal{B}(R_{\mathrm{t}}(\pi))$ and $\mathcal{B}(R_{\mathrm{b}}(\pi))$, respectively.

Let $q \in \mathcal{C}^{\mathrm{root}} - \mathcal{V}$ and let $g_t q$ for $t \geqslant 0$ be the Teichmüller geodesic ray through q. Since q is doubly non-vanishing, it is contained in some polytope $\mathcal{C}_\pi$ and its parameters satisfy $x_{\pi(\ell)} \neq x_{\pi(\ell+m)}$. Let I be the base-arc in q. Then the length of the base-arc in $g_t q$ is $e^t |I|$. Therefore, there is some time $t_1 > 0$ such that $g_{t_1} q \in \mathcal{F}(\pi)$. By Rauzy induction, $g_{t_1} q$ is also contained in $\mathcal{B}(\pi')$ where $\pi = R_*(\pi)$ where $* = \mathrm{t}$ or $* = \mathrm{b}$, depending on which of $x_{\pi(\ell)}$ and $x_{\pi(\ell+m)}$ is larger. Let $\alpha', \beta' \in \mathcal{A}$ be the last top and bottom letters of π', respectively. As $g_{t_1} q$ is also doubly non-vanishing, the widths corresponding to α' and β' do not coincide. So there is time $t_2 > t_1$ such that $g_{t_2} q$ is contained in $\mathcal{F}(\pi')$. This description continues iteratively.

It is possible that the Rauzy diagram $\mathcal{D}$ contains permutations π and π' that are equivalent under some re-indexing $p \in \mathrm{Sym}(\mathcal{A})$. For such permutations, $\mathcal{C}_\pi = \mathcal{C}_{\pi'}$. Note then that the permutations arising from π and π' by Rauzy–Veech induction are also pairwise equivalent under the same re-indexing p. It follows that p induces a symmetry of $\mathcal{D}$ as a directed graph. We call the quotient of $\mathcal{D}$ by all such symmetries the *reduced Rauzy diagram* and denote it by $\mathcal{D}^{\text{red}}$. It is then clear that we should use the quotient graph $\mathcal{D}^{\text{red}}$ to code Teichmüller flow on $\mathcal{C}^{\text{root}}$.

The above coding of Teichmüller flow has an immediate combinatorial consequence. By the work of Masur and Veech [Mas82; Vee82], the Teichmüller flow on $\mathcal{C}^{\text{root}}$ is ergodic for the Masur–Veech measure. By ergodicity, a positive measure set in $\mathcal{C}_\pi - \mathcal{V}$, visits every $\mathcal{C}_\sigma$ under Teichmüller flow. This implies that the reduced Rauzy diagram $\mathcal{D}^{\text{red}}$ is a strongly connected graph, that is, there is a directed path between any pair of vertices. It then follows that each component of $\mathcal{D}$ is also strongly connected. See the article by Boissy–Lanneau for more details [BL09].

The following three lemma are standard facts in the theory of Rauzy–Veech sequences and are often implicitly used. We include proof sketches for completeness.

Lemma 4.27. *For any doubly non-vanishing rooted differential (q, v) and any $T > 0$ the geodesic segment $[q, g_T q]$ crosses finitely many flow faces.*

Proof. By applying Teichmüller flow, we may assume that (q, v) is contained in some $\mathcal{B}_0 = \mathcal{B}(\pi)$. Let $\mathcal{B}_1, \mathcal{B}_2, \ldots$ be the sequence of backward faces the geodesic segment $[q, g_T q]$ crosses. Let $0 < t_k \leqslant T$ be the monotonically increasing sequence of times such that $g_{t_k} q$ is contained in $\mathcal{B}_k$.

Recall that $Z(q, v)$ is the set of points x in I_v such that $I(x)$ is a base-arc. Identifying I_v with the positive real axis, it follows from the definitions that each point e^{-t_k} is contained in $Z(q, v)$.

By Lemma 4.20, the intersection $Z(q, v) \cap [e^{-T}, 1)$ is finite. Hence, the sequence t_k is finite and we are done. □

Remark 4.28. *It follows from the previous lemma that the curve δ we construct in the proof of Theorem 4.23 crosses finitely many flow faces.*

Let ζ be a finite Rauzy–Veech sequence that starts at π and ends at π'. Let $P_\zeta \subseteq P_\pi$ be the parameters of differentials in $\mathcal{C}_\pi$ whose Rauzy–Veech sequence begins with ζ. By inductively using the definition of Rauzy–Veech induction, it follows that the set P_ζ is a convex open subset of P_π. In particular, $\mathcal{C}_\zeta = q_\pi(P_\pi)$ is path connected.

We say that a Teichmüller segment $[q, g_t q]$ is a ζ-segment if $q \in \mathcal{C}_\zeta, g_t q \in \mathcal{C}_{\pi'}$ and the Rauzy–Veech sequence of $[q, g_t q]$ is ζ.

Lemma 4.29. *Let ζ be a finite Rauzy–Veech sequence that starts at π and ends at π'. Then any pair of ζ-segments are isotopic in $\mathcal{C}^{\text{root}}$ through ζ-segments.*

Proof. As $[q, g_t q]$ is a ζ-segment, it follows that there exists an open set U in $q_\pi(P_\zeta)$ centred at q such that $g_t U$ is contained in $\mathcal{C}_{\pi'}$. Any ζ-segment with an endpoint in U is thus homotopic to $[q, g_t q]$. Indeed, for any q' in U, we can connect q to q' by an arc contained in U. We flow the arc for time t to get an arc in $\mathcal{C}_{\pi'}$. We thus have a homotopy between $[q, g_t q]$ and $[q', g_t q']$. We then do a further homotopy from $[q', g_t q']$ and $[q', g_{t'} q']$. The lemma then follows from the path connectedness of $q_\pi(P_\zeta)$. □

Lemma 4.30. *Let U be an open set contained in some polytope $\mathcal{C}_\pi$. Then there exists a Rauzy–Veech sequence θ (that depends on U) starting from π such that, for every $q \in \mathcal{C}_\theta$, the Teichmüller segment in $\mathcal{C}_\pi$ containing q intersects U.*

Proof. Let Δ be the standard simplex in $\mathbb{R}^{\mathcal{A}}$. Let $p : P_\pi \to \Delta$ be the projection $(x, y) \to x/\|x\|_1$.

By iteration, the transformation on parameters induced by a Rauzy–Veech sequence ζ is encoded by a non-negative matrix B_ζ, that is, if (x, y) is in P_ζ then the new parameters $x^{(\zeta)}$ are related to x by $B_\zeta x^{(\zeta)} = x$. Suppose ζ ends at π'. It also follows that $p(B_\zeta P_{\pi'}) = p(P_\zeta)$.

It then suffices to show that there is a Rauzy–Veech sequence θ such that $p(B_\theta P_{\pi'})$ is contained in $p(q_\pi^{-1}U)$, where π' is the permutation that θ ends at. This is a standard fact but we will include a brief justification for completeness.

By Masur's [Mas82] and Veech's [Vee82] solution of the Keane conjecture or even more strongly by Kerckhoff–Masur–Smillie [KMS86], we can find $q \in U$ with a uniquely ergodic vertical foliation or more strongly giving a recurrent Teichmüller ray. Let ζ_n be the Rauzy–Veech sequence of length n for q and B_n the corresponding matrix. Also, let π_n denote the permutation at its end. Since the vertical foliation is uniquely ergodic, the nested sequence $p(B_n P(\pi_n))$ converges to $p(q_\pi^{-1}(q))$ as $n \to \infty$. Since all such sets are polytopes inside the standard simplex Δ, there is some n large enough such that $p(B_n P_{\pi_n})$ is contained inside $p(q_\pi^{-1}U)$. We make take θ to be ζ_n to conclude the proof.

□

5. The flow group is the fundamental group

Let $\pi_1(\mathcal{D}^{\mathrm{red}}, \pi)$ be the fundamental group based at π of $\mathcal{D}^{\mathrm{red}}$ as an undirected graph. Let q_0 be a point in $\mathcal{C}_\pi$.

To simplify notation, we will fix a component of $\mathcal{D}$ evenly covering $\mathcal{D}^{\mathrm{red}}$. We will continue to refer to a vertex in $\mathcal{D}^{\mathrm{red}}$ as an irreducible permutation when in fact it is an equivalence class of vertices related by letter re-indexing. Note that every directed path in $\mathcal{D}^{\mathrm{red}}$ is realised by an actual Rauzy–Veech sequence in $\mathcal{D}$. Thus, we will use the actual Rauzy–Veech sequences in $\mathcal{D}$ to concatenate sensibly.

Proposition 5.1. *There is a natural homomorphism*

$$\pi_1(\mathcal{D}^{\mathrm{red}}, \pi) \to \pi_1(\mathcal{C}^{\mathrm{root}}, q_0).$$

Proof. Let σ be any irreducible permutation in $\mathcal{D}^{\mathrm{red}}$. As $\mathcal{D}^{\mathrm{red}}$ is strongly connected, we can choose a directed path $\xi(\sigma)$ from π to σ and a directed path $\xi'(\sigma)$ from σ to π. We choose empty paths for $\xi(\pi)$ and $\xi'(\pi)$.

Every loop κ in $\mathcal{D}^{\mathrm{red}}$ based at π can be written as a concatenation of alternating forward and backward paths. Breaking symmetry, suppose the odd indexed paths are forward paths and the even indexed paths are backward paths. We may then write the concatenation as $\kappa_1 \kappa_2^{-1} \kappa_3 \kappa_4^{-1} \cdots$.

Let σ_i and τ_i be the beginning and ending permutations for κ_i. Then we have the string of relations $\pi = \sigma_1, \tau_1 = \sigma_2, \tau_2 = \sigma_3$, etc. The concatenation for κ is then equal to the concatenation $\lambda_1 \lambda_2^{-1} \lambda_3 \lambda_4^{-1} \cdots$, where each λ_i is a loop based at π given by $\lambda_i = \xi(\sigma_i)\, \kappa_i\, \xi'(\tau_i)$.

It thus suffices to associate a loop in $\mathcal{C}^{\mathrm{root}}$ based at q_0 with a directed loop κ in $\mathcal{D}^{\mathrm{red}}$ based at π. Observe that an actual Rauzy–Veech sequence in $\mathcal{D}$ representing κ might end at a permutation π' equivalent to π. However, this will not matter as $\mathcal{C}_{\pi'} = \mathcal{C}_\pi$ in that case. Let q be a point in $\mathcal{C}_\pi$. As $\mathcal{C}_\pi$ is a polytope and hence contractible, any pair of paths in $\mathcal{C}_\pi$ from q_0 to q are homotopic relative to the endpoints. We fix one such path and call it η_q. By η_q^{-1}, we mean the reverse path from q to q_0. We now choose any κ-segment γ_κ. By Lemma 4.29, any two choices of γ_κ are homotopic. Let q and q' in $\mathcal{C}_\pi$ be the beginning and end points of γ_κ. We then map κ to the based loop $\eta_q\, \gamma_\kappa\, \eta_{q'}^{-1}$ in $\mathcal{C}^{\mathrm{root}}$.

It remains to show that this map is a homomorphism. Let κ' and κ'' be two directed loops based at π. Let $\kappa = \kappa'\kappa''$ and let γ_κ be a κ-segment. Let q and q'' in $\mathcal{C}_\pi$ be the beginning and the end points of γ_κ. We deduce that there exists a point q' in $\mathcal{C}_\pi$ on γ_κ such that if we write $\gamma_\kappa = [q, q'] \cup [q', q'']$ then $\gamma_{\kappa'} = [q, q']$ is a κ'-segment and $\gamma_{\kappa''} = [q', q'']$ is a κ''-segment. Then $\eta_q \, \gamma_\kappa \, \eta_{q''}^{-1}$ is homotopic to $(\eta_q \, \gamma_{\kappa'} \, \eta_{q'}^{-1})(\eta_{q'}\gamma_{\kappa''}\eta_{q''}^{-1})$ and so we conclude that the map is a homomorphism. □

As a consequence of Theorem 4.23, we prove

Theorem 5.2. *Let $\mathcal{C}^{\mathrm{root}}$ be a component of a stratum of rooted abelian or quadratic differentials. Let π be a permutation in $\mathcal{D}^{\mathrm{red}}$. Let q_0 be a base-point in $\mathcal{C}^{\mathrm{root}}$ contained in the polytope $\mathcal{C}_\pi$. Then the natural homomorphism*

$$\pi_1(\mathcal{D}^{\mathrm{red}}, \pi) \to \pi_1(\mathcal{C}^{\mathrm{root}}, q_0)$$

is surjective.

Proof. By Theorem 4.23, a loop γ based at q_0 is homotopic to a finite concatenation of paths γ_i where each γ_i is either a (forward or backward) Teichmüller geodesic segment or is contained inside a polytope. Breaking symmetry, we may assume γ is the concatenation $\gamma_1\gamma_2 \cdots \gamma_k$ where the odd indexed γ_i are contained in a polytope and the even indexed γ_i are (forward or backward) Teichmüller segments. By Lemma 4.27, the Teichmüller segments γ_{2i} give us finite Rauzy–Veech sequences ζ_{2i} such that

- ζ_2 starts at π; and
- successive ζ_{2i} can be concatenated as undirected paths in $\mathcal{D}$.

The concatenation $\zeta_2\zeta_4 \cdots$ descends to a loop κ in $\mathcal{D}^{\mathrm{red}}$ based at π. As before, the actual sequence in $\mathcal{D}$ might end at a permutation π' equivalent to π but that will not matter as $\mathcal{C}_{\pi'} = \mathcal{C}_\pi$. As in the proof of Proposition 5.1, the loop κ can be written as a concatenation of (forward and backward) loops based at π. The surjectivity follows. □

Remark 5.3. *As the components of $\mathcal{D}$ evenly cover $\mathcal{D}^{\mathrm{red}}$, it follows from Theorem 5.2 that there is a finite cover of $\mathcal{C}^{\mathrm{root}}$ that corresponds to components of $\mathcal{D}$. We denote this cover by $\mathcal{C}^{\mathrm{lab}}$. This cover is easy to describe intrinsically in the case of an abelian stratum component but its description for a quadratic stratum component is an interesting question.*

Let q_0 be a base-point in $\mathcal{C}^{\mathrm{root}}$ and U be a contractible open set around q_0. For every $q \in U$, we choose a path η_q from q_0 to q inside U. As U is contractible, any choice of η_q is homotopic to any other choice relative to their end-points. We take the convention that η_q^{-1} is η_q in reverse connecting q to q_0.

Let γ be a Teichmüller segment that begins at some $q \in U$ and ends at some $q' \in U$. The concatenation $\eta_q\gamma\eta_{q'}^{-1}$ is a loop in $\mathcal{C}^{\mathrm{root}}$ based at q_0. We call such loops *almost-flow loops* (based at q_0).

Definition 5.4. *The flow group $G(U, q_0)$ is the subgroup of $\pi_1(\mathcal{C}^{\mathrm{root}}, q_0)$ generated by the almost-flow loops based at q_0.*

We first prove the following theorem.

Theorem 5.5. *Let $\mathcal{C}^{\mathrm{root}}$ be a component of a stratum of rooted abelian or quadratic differentials. Let q_0 be a base-point contained in some polytope $\mathcal{C}_\pi$. Then*

$$G(\mathcal{C}_\pi, q_0) = \pi_1(\mathcal{C}^{\mathrm{root}}, q_0).$$

Proof. In the proof of Theorem 5.2, we showed that images of directed loops in $\pi_1(\mathcal{D}^{\mathrm{red}}, \pi)$ generate $\pi_1(\mathcal{C}^{\mathrm{root}}, q_0)$. So it suffices to show that every directed loop ζ in $\mathcal{D}^{\mathrm{red}}$ based at π is realised by an almost-flow loop in $G(\mathcal{C}_\pi, q_0)$. We choose any q in $\mathcal{C}_\zeta$ and let $[q, g_t q]$ be a ζ-segment that it gives. By definition, $q' = g_t q$ is also contained in $\mathcal{C}_\pi$. Then the based loop $\eta_q \gamma \eta_{q'}^{-1}$ realises the loop ζ. This concludes the proof of the theorem. □

As a corollary, we obtain one of our main results.

Theorem 5.6. *For any base-point q_0 in $\mathcal{C}^{\mathrm{root}}$ and any contractible open set U containing q_0*

$$G(U, q_0) = \pi_1(\mathcal{C}^{\mathrm{root}}, q_0).$$

Proof. Let q be a point in U. As U is contractible, it follows that $G(U, q) \cong G(U, q_0)$. Since the union of polytopes is dense in $\mathcal{C}^{\mathrm{root}}$, we may then assume that q_0 is contained in some polytope $\mathcal{C}_\pi$. Suppose $V \subseteq U$ is a smaller contractible open set that contains q_0. By definition of the flow groups, $G(V, q_0)$ is a subgroup of $G(U, q_0)$. So we may assume that U is also contained in $\mathcal{C}_\pi$. It now suffices to show that any directed loop ζ in $\mathcal{D}^{\mathrm{red}}$ based at π can be written as a word in almost-flow loops in $G(U, q_0)$.

By Lemma 4.30, there is a Rauzy–Veech sequence θ starting from π such that for any $q \in \mathcal{C}_\theta$ the Teichmüller segment in $\mathcal{C}_\pi$ containing q intersects U. Note then that the same is true for any finite extension $\theta\zeta$.

As $\mathcal{D}$ is strongly connected, we may extend θ to assume that it also ends at π. If π' is equivalent to π, we also get a loop θ' based at π' so that θ and θ' descend to the same loop in $\mathcal{D}^{\mathrm{red}}$. Let q be a differential in $\mathcal{C}_{\theta\theta}$. As the Teichmüller segment in $\mathcal{C}_\pi$ containing q intersects U, we may assume q is in U.

By definition, there exists a time $t > 0$ such that the Teichmüller segment $[q, g_t q]$ is a $\theta\theta$-segment. We may decompose $[q, g_t q]$ as $[q, g_s q] \cup [g_s q, g_t q]$ such that both segments $[q, g_s q]$ and $[g_s q, g_t q]$ are θ-segments. In particular, $q' = g_s q$ is also contained in $\mathcal{C}_\theta$. As the Teichmüller segment in $\mathcal{C}_\pi$ containing q' intersects U, by tweaking s we may assume that q' is also contained in U. Let $\gamma_u = [q, q']$. Then $\eta_q \gamma_u \eta_{q'}^{-1}$ is an almost-flow loop in $G(U, q_0)$ that realises θ.

Now, let ζ be an oriented loop in $\mathcal{D}^{\mathrm{red}}$ based at π. We consider the oriented loop $\xi = \theta\zeta\theta$. As a sequence in $\mathcal{D}$, the loop ζ could terminate at some π' equivalent to π. In this case we mean the concatenation $\theta\zeta\theta'$, where θ' is a loop in $\mathcal{D}$ based at π' that descends to the same loop in $\mathcal{D}^{\mathrm{red}}$ as θ. We will assume first that ζ is a loop based at π in $\mathcal{D}$. The argument below extends to the case where ζ ends instead at π' by concatenating the appropriate arcs.

Let $\gamma_\xi = [q, g_t q]$ be a ξ-segment. As the Teichmüller segment in $\mathcal{C}_\xi$ containing q intersects U, we may choose q to be in U. We then write γ_ξ as a concatenation $[q, g_s q] \cup [g_s q, g_t q]$ where $[q, g_s q]$ is a $\theta\zeta$-segment and $[g_s q, g_t q]$ is a θ-segment. Let $q' = g_s q$ and $q'' = g_t q$. As $[q', q'']$ is a θ-segment, q' is contained in $\mathcal{C}_\theta$. The Teichmüller segment inside $\mathcal{C}_\pi$ containing q' intersects U. So by changing s, we may assume that q' is also contained in U.

The Teichmüller segment $\gamma' = [q, q']$ then begins and ends in U. So the directed loop $\theta\zeta$ is realised by the almost-flow loop $\eta_q \gamma' \eta_{q'}^{-1}$ in $G(U, q_0)$.

As we already established that θ is realised by an element in $G(U, q_0)$, we deduce that ζ must also be realised by an element in $G(U, q_0)$. This concludes the proof of the corollary.

□

6. Dynamics of the Teichmüller flow

6.1. **Coding formalism.** Let Π be a finite or countable set. We consider the symbolic space $\Sigma = \Pi^{\mathbb{Z}}$ endowed with the left shift map S. Suppose $w \in \Pi^m$ is a finite word. The (forward) cylinder $\Sigma(u)$ induced by u is defined as $\Sigma(u) = \{a \in \Sigma$ such that $a_k = u_k$ for $k = 0, \ldots, m-1\}$. Given another word $v \in \Pi^n$, we write $uv \in \Pi^{m+n}$ for the concatenation of u and v.

Definition 6.2. *We say that an* S*-invariant probability measure* μ *has* bounded distortion *if there exists a constant* $K > 0$ *such that, for any finite words* $u \in \Pi^m$ *and* $v \in \Pi^n$*,*

$$\frac{1}{K}\mu(\Sigma(u))\mu(\Sigma(v)) \leqslant \mu(\Sigma(uv)) \leqslant K\mu(\Sigma(u))\mu(\Sigma(v)).$$

The bounded distortion property allows us to treat the symbolic space "almost" as a Bernoulli shift. For this reason, it is also called an *approximate product structure*. Since μ is shift-invariant, the previous definition would not change if we used backward or centred cylinders instead of forward cylinders.

We will now describe how the Teichmüller flow can be coded by such a symbolic setup. Let π be an irreducible generalised permutation in $\mathcal{D}^{\mathrm{red}}$. For a backward-tied differential q in $\mathcal{B}(\pi)$, let $(q, g_t q)$, for some $t > 0$, be the longest Teichmüller segment entirely contained in $\mathcal{C}_\pi$. It follows that $g_t q$ is contained in $\mathcal{F}(\pi)$. Let ζ be a finite Rauzy–Veech sequence starting at π. Let $\mathcal{S}_\zeta$ be the differentials q in $\mathcal{B}(\pi)$ for which the Teichmüller segment above is contained in $\mathcal{C}_\zeta$.

Recall that Δ is the standard simplex in $\mathbb{R}^{\mathcal{A}}$ and $p : P_\pi \mapsto \Delta$ is the projection $(x, y) \to x/\|x\|_1$. Let B_θ denote the matrix of a Rauzy–Veech sequence θ. As indicated in the proof of Lemma 4.30, it is possible to find a sequence θ from π to π such that $p(B_\theta P_\pi)$ is compactly contained in $p(P_\pi)$. Let $\overline{p} : (x, y) \mapsto y$ be the map that records the height parameters. By extending θ to a longer loop based at π, we may also assume that $\overline{p}(P_\pi)$ is compactly contained in $\overline{p}(B_\theta P_\pi)$. We fix this θ once and for all and consider $\mathcal{S}_\theta$. By finessing θ further, we may assume that it is *neat*, that is, if $\theta = \zeta\eta$ and $\theta = \eta'\zeta$ then $\zeta = \theta$. In the coding that we consider, the set $\mathcal{S}_\theta$ will serve as a transverse section to the Teichmüller flow.

We recall the bounded distortion theorem for Rauzy–Veech induction. This theorem states that that there exists a constant $K \geqslant 1$, that depends only on the topology of the surface, and a countable collection of finite Rauzy–Veech sequences ζ from π to π such that

- for the map $p(q_\pi^{-1}(\mathcal{B}(\pi))) \to p(q_\pi^{-1}(\mathcal{B}(\pi)))$ given by $x \mapsto B_\zeta\, x/\|B_\zeta\, x\|_1$, its Jacobian $\mathcal{J}$ satisfies
$$\frac{1}{K}\mathcal{J}(x_1) < \mathcal{J}(x_2) < K\mathcal{J}(x_1)$$
for any pair of points $(x_1, y_1), (x_2, y_2) \in q_\pi^{-1}(\mathcal{B}(\pi))$;
- no ζ contains θ, that is, ζ cannot be written as a concatenation $\eta\theta\eta'$; and
- up to excising a set of differentials of measure zero,
$$\bigcup_\zeta \mathcal{B}(\zeta) = \mathcal{B}(\pi).$$

For more details, we refer the reader to the article by Avila–Gouëzel–Yoccoz [AGY06, Section 4] for abelian differentials, and by Avila–Resende [AR12, Sections 4 and 5] for quadratic differentials.

We stress that the Rauzy–Veech sequences considered here are sequences in $\mathcal{D}^{\mathrm{red}}$. To be precise with constraints that involve concatenations, they should be imposed over all lifts in $\mathcal{D}$. For instance, fixing a lift of π in $\mathcal{D}$, the second constraint should be read as finding a loop ζ in $\mathcal{D}$ that returns to π and does not contain a lift of θ. Alternatively, the coding we describe is a coding of the flow lifted to $\mathcal{C}^{\mathrm{lab}}$ that is "equivariant" with respect to the Deck group for the covering $\mathcal{C}^{\mathrm{lab}} \to \mathcal{C}^{\mathrm{root}}$.

With this setup, the first return maps under Teichmüller flow to $\mathcal{S}_\theta$ are given by Rauzy–Veech sequences of the form $\theta\zeta\theta$. Note then that, by tweaking the constant K, the Jacobian of the map $p(q_\pi^{-1}(\mathcal{S}_\theta)) \mapsto p(q_\pi^{-1}(\mathcal{S}_\theta))$ given by $x \mapsto B_{\theta\zeta\theta}\, x / \|B_{\theta\zeta\theta}\, x\|_1$ satisfies

$$\frac{1}{K}\mathcal{J}(x_1) < \mathcal{J}(x_2) < K\mathcal{J}(x_1)$$

for each ζ and for any pair of points $(x_1, y_1), (x_2, y_2) \in q_\pi^{-1}(\mathcal{S}_\theta)$. In fact, the Jacobian property is enforced by the stronger property of the matrix that, up to a constant, that depends only on the topology of the surface, all columns have the same norm. More precisely, $\|B_{\theta\gamma\theta}\, x\|_1 \asymp \|B_{\theta\gamma\theta}\, x'\|_1$ for any $(x, y), (x', y') \in q_\pi^{-1}(\mathcal{S}_\theta)$.

Moreover, we have that, up to excising a set of zero measure,

$$\mathcal{S}_\theta = \bigcup_\zeta \mathcal{S}_{\theta\zeta\theta}. \tag{6.3}$$

We thus have a countable full measure partition of $\mathcal{S}_\theta$ into sets $\mathcal{S}_{\theta\zeta\theta}$ such that each $\mathcal{S}_{\theta\zeta\theta}$ is the image of smooth map $\phi_\zeta \colon \mathcal{S}_\theta \mapsto \mathcal{S}_\theta$. Each map ϕ_ζ is diffeomorphic onto its image and its inverse is a *uniformly expanding Markov map* in the sense of Avila–Gouëzel–Yoccoz [AGY06, Definition 2.2]. We assemble the inverses of ϕ_ζ into a map Φ on $\mathcal{S}_\theta$.

For such an expanding map Φ, there exists a unique Φ-invariant absolutely continuous probability measure ν, which is automatically ergodic and even mixing [Aar97; AGY06, Section 2]. In the case of the Teichmüller flow, the measure ν is the restriction of the Masur–Veech measure on $\mathcal{C}^{\mathrm{root}}$.

We now set Π as the countable set of sequences ζ above. The partition given by (6.3) induces an equivariant bijection between (Σ, S) and $(\bigcup_{\zeta\in\Pi} \mathcal{S}_{\theta\zeta\theta}, \Phi)$. The measure μ that we will consider on Σ is the unique S-invariant probability measure rendering this bijection a measure-theoretic conjugation. The bounded distortion inherited by ν from the Jacobians becomes equivalent to the bounded distortion of μ given by Definition 6.2.

6.4. **Return times.** A function $\xi : \Sigma \mapsto \mathbb{R}$ is *Hölder* if there exists a non-negative exponent $\alpha < 1$ and a constant $C > 0$ such that for any finite sequence u and any $a, b \in \Sigma(u)$

$$|\xi(a) - \xi(b)| \leqslant C\alpha^m,$$

where m is the length of u.

A *roof function* is a Hölder function $\xi \colon \Sigma \mapsto \mathbb{R}_{>0}$. A suspension of the symbolic space is a space that is homeomorphic to $\Sigma \times [0, 1]$ with the identification $(a, 0) \sim (\mathrm{S}(a), 1)$. The roof function equips the suspension with a flow ψ which flows in the interval direction and satisfies $\psi_{\xi(a)}(a, 0) = (\mathrm{S}(a), 1)$.

Definition 6.5. *A roof function is said to have* exponential tails *if there exists* $h > 0$ *such that*

$$\int_\Sigma e^{h\xi} d\mu < \infty.$$

The property of having exponential tails implies, in particular, that the volume of the suspension with respect to the local product measure $d\mu\, dt$ is finite. We will discuss this integrability in more detail when we talk of cocycles.

Note that under the measure-theoretic conjugacy, the section $\mathcal{S}_\theta$ gets identified with $\Sigma \times \{0\}$. The function ξ we are interested in is the return time function to $\mathcal{S}_\theta$ under Teichmüller flow. It is easy to check that the return time on $\mathcal{S}_{\theta\zeta\theta}$ is given by $\xi(x) = \log \|B_{\theta\zeta\theta}\, x\|_1$. The measure $d\nu\, dt$ is the Masur–Veech measure on $\mathcal{C}^{\mathrm{root}}$. The coding of the Teichmüller flow can be summarised as follows.

Theorem 6.6. *There exists a countable set Π whose full shift (Σ, S), $\Sigma = \Pi^{\mathbb{Z}}$, carries an S-invariant probability measure with bounded distortion and a roof function ξ with exponential tails such that there exists a measure-theoretic conjugacy $f \colon \Sigma \times [0,1] \to \mathcal{C}^{\mathrm{root}}$ (where $\mathcal{C}^{\mathrm{root}}$ is equipped with the Masur–Veech measure) that satisfies $f \circ \psi_t = g_t \circ f$ for all $t \in \mathbb{R}$.*

6.7. **Cocycles.** A linear cocycle with values in $\mathrm{SL}(m, \mathbb{R})$ for the Teichmüller flow is a map $C \colon \mathcal{C} \times \mathbb{R} \mapsto \mathrm{SL}(m, \mathbb{R})$ satisfying

- $C(q, 0) = I$ where I is the identity matrix, and
- $C(q, s+t) = C(g_s q, t)\, C(q, s)$ for all $q \in \mathcal{C}$ and $s, t \in \mathbb{R}$.

The most well known example is the Kontsevich–Zorich cocycle which records the dynamics of the flow on the integral first homology group of the surface. Choosing charts on $\mathcal{C}$, one can choose a basis for a trivialisation of the surface homology in these charts. As the Teichmüller flow has Poincaré recurrence, one can consider the change of basis matrices as a flow trajectory returns to a chart. This is the Kontsevich–Zorich cocycle. As the flow preserves the intersection form on the homology, the cocycle takes values in the symplectic group over $\mathbb{Z}$.

Another example is the Rauzy–Veech cocycle defined on $\mathcal{C}^{\mathrm{lab}}$, which is the finite cover of $\mathcal{C}^{\mathrm{root}}$ corresponding to the cover $\mathcal{D} \to \mathcal{D}^{\mathrm{red}}$ defined in Remark 5.3. Here, the polytopes $\mathcal{C}_\pi$ carry preferred coordinates through the normalised width and height parameters. The itinerary through polytopes of a typical flow trajectory is recorded by the Rauzy–Veech sequence. The coordinate transformation of a Rauzy–Veech sequence is linear. This defines the Rauzy–Veech cocycle. For an abelian stratum component, one can associate to an irreducible permutation a natural spanning set for the absolute homology of the surface. With respect to these spanning sets the Kontsevich–Zorich cocycle lifted to $\mathcal{C}^{\mathrm{lab}}$ is the same as the Rauzy–Veech cocycle. Thus, the two can be studied simultaneously. This structure is not available for stratum components of quadratic differentials.

A linear cocycle for the Teichmüller flow is said to be *integrable* with respect to a finite flow invariant measure if for all t the functions $q \mapsto \log \|C(q,t)\|$ and $q \mapsto \log \|C(q,t)^{-1}\|$ are L^1 with respect to the measure. As our cocycles are integer valued, the condition $q \mapsto \log \|C(q,t)\|$ being L^1 suffices. The flow invariant measure we are interested in is the Masur–Veech measure on $\mathcal{C}$ and its lifts to $\mathcal{C}^{\mathrm{root}}$ and $\mathcal{C}^{\mathrm{lab}}$. The lift to $\mathcal{C}^{\mathrm{root}}$ is exactly the measure $d\nu\, dt$.

The Teichmüller flow is ergodic with respect to the Masur–Veech measure. Thus, if a cocycle is integrable, then Oseledets theorem applies: for almost every q and every non-zero vector $v \in \mathbb{R}^m$ the limit

$$\lim_{t \to \infty} \frac{1}{t} \log \frac{\|C(q,t)v\|_1}{\|v\|_1}$$

exists and depends only on v and not on q. Moreover, the limit can achieve up to m values: $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m$. This set of numbers is known as the *Lyapunov spectrum.*

We say a cocycle on $\mathcal{C}^{\mathrm{root}}$ is *locally constant* if the cocycle is constant over the cylinder sets in our coding. By choosing a trivialisation of the homology over each polytope $\mathcal{C}_\pi$, the Kontsevich–Zorich cocycle lifted to $\mathcal{C}^{\mathrm{root}}$ is locally constant. The Rauzy–Veech cocycle is locally constant by construction.

We normalise the invariant measure μ on the section $\mathcal{S}_\theta$ to be a probability measure. Let ζ be a symbol in Π and let $B_{\theta\zeta\theta}$ be the associated Rauzy matrix. By standard methods of computing volumes of images of projective linear maps, there exists a constant $C > 1$ that depends only on the topology of the surface such that

$$\frac{1}{C\|B_{\theta\zeta\theta}\|_1^{d-1}} < \mu(\mathcal{S}_{\theta\zeta\theta}) < \frac{C}{\|B_{\theta\zeta\theta}\|_1^{d-1}}.$$

See our previous article for more details [Bel+19, Lemma 5.11]. Thus, for the discrete Rauzy–Veech cocycle $q \mapsto B(q)$ on $\mathcal{S}_\theta$, we get that, up to a similar uniform multiplicative constant,

$$\int_{\mathcal{S}_\theta} \log \|B(q)\|_1 \, d\mu(q) \asymp \sum_{\zeta\in\Pi} \frac{\log \|B_{\theta\zeta\theta}\|_1}{\|B_{\theta\zeta\theta}\|_1^{d-1}}.$$

To estimate the integral above, we organise the sequences $\zeta \in \Pi$ by the L^1 norms of the matrices $B_{\theta\zeta\theta}$ considered on a multiplicative scale on $\mathbb{R}_+$. Recurrence estimates in the bounded distortion theorem for Rauzy–Veech sequences show that there exist constants $M > 1$ and $0 < c < 1$ that depend only on the topology of the surface such that, for the set $\Pi_n = \{\zeta \in \Pi : \|B_{\theta\zeta\theta}\|_1 \in [1, M^n)\}$,

$$\sum_{\zeta\in\Pi_n} \mu(\mathcal{S}_{\theta\zeta\theta}) > 1 - c^n.$$

It follows that $\sum_\zeta \log \|B_{\theta\zeta\theta}\|_1/\|B_{\theta\zeta\theta}\|_1^{d-1}$ is dominated by $\sum nc^n$ and, hence, that the discrete Rauzy–Veech cocycle is integrable. Note that up to an additive constant that depends only on the surface, the first return time $\xi(q)$ at any q in $\mathcal{S}_{\theta\zeta\theta}$ is $\log \|B_{\theta\zeta\theta}\|_1$. So the integrability of the Rauzy–Veech cocycle is equivalent to the finiteness of the Masur–Veech volume of $\mathcal{C}^{\mathrm{root}}$.

Using the above estimates for $\gamma \in \Pi_n$, it is straightforward to derive that, if a locally constant cocycle with values in $\mathrm{SL}(m, \mathbb{Z})$ is integrable with respect to the Masur–Veech measure, then it is integrable with respect to $(\mathcal{S}_\theta, \mu)$. As a result, the integrability over $(\mathcal{S}_\theta, \mu)$ of the plus and minus cocycles (that we define in Section 9) can be deduced from their integrability with respect to the Masur–Veech measure. In any case, we will give a direct verification of the integrability over $(\mathcal{S}_\theta, \mu)$ of the plus and minus cocycles in Section 10. To do that, we will need the following lemma.

We say that a cocycle C is *dominated by* a cocycle C' if there is a constant $K > 0$ that depends only on the surface such that the L^1-norms satisfy $\|C\|_1 \leqslant K\|C'\|_1$.

Lemma 6.8. *Suppose that C is a locally constant cocycle dominated by the Rauzy–Veech cocycle. Then C is integrable in either sense.*

Proof. The lemma follows directly from integrability of the Rauzy–Veech cocycle. □

Recall that a cocycle into $\mathrm{SL}(m, \mathbb{R})$ has a *simple* Lyapunov spectrum if its Lyapunov spectrum consists on m distinct numbers. We will now state a weaker version of the Avila–Viana criterion for simplicity of the Lyapunov spectrum. The actual criterion is more general [AV07a; AV07b, Theorem 7.1], but we state it specifically for our context.

Let γ_1 and γ_2 be almost-flow loops given by directed Rauzy–Veech sequences η_1 and η_2. Then the concatenation $\gamma_1\gamma_2$ is also realised by an almost-flow loop given by the Rauzy–Veech sequence $\eta_1\eta_2$. Thus, the almost-flow loops give us a monoid. By evaluating a locally constant cocycle for each almost-flow loop, we get a representation of the monoid into $\mathrm{SL}(m, \mathbb{R})$. As the cocyles we consider here are defined over $\mathbb{Z}$ and preserve a symplectic structure, this representation has an image in the symplectic group.

Criterion 6.9. *Let C be a locally constant integrable cocycle for the Teichmüller flow. If the associated monoid is Zariski dense in the symplectic group, then the Lyapunov spectrum is simple.*

As was previously mentioned, the criterion stated by Avila–Viana [AV07b, Theorem 7.1] does not require Zariski density. Instead, it has the weaker hypothesis of requiring the presence of *pinching* and *twisting* elements in the group generated by the monoid. It is a classical fact that a Zariski dense monoid gives a group that contains a pinching element. It follows from the work of Benoist [Ben97] that this group also contains elements that are twisting relative to the pinching element. As we directly establish Zariski density, we will omit the precise definitions of pinching and twisting, which are technical to state.

Furthermore, the criterion stated by Avila–Viana does not, strictly speaking, require the cocycle to be symplectic; it requires the cocycle to take values in the special linear group and satisfy pinching and twisting. Nevertheless, we state the criterion for symplectic cocycles as all of the cocycles that we consider are symplectic.

7. Rauzy–Veech groups

Recall that $\mathcal{C}^{\mathrm{lab}}$ is the cover of $\mathcal{C}^{\mathrm{root}}$ corresponding to the covering $\mathcal{D}^{\mathrm{red}} \to \mathcal{D}$, as defined in Remark 5.3. In the definitions that follow, we operate in $\mathcal{C}^{\mathrm{lab}}$.

7.1. Rauzy–Veech groups of abelian components.

Let $\mathcal{C}$ be an abelian stratum component. There is a natural spanning set for the absolute homology of the surface that one can associate to any permutation π in its Rauzy diagram $\mathcal{D}$ [AV07b; AMY18; Gut19]. For any loop δ in $\mathcal{D}$ based at π, one can define a matrix in $\mathrm{Sp}(2g, \mathbb{Z})$ by computing the linear action on absolute homology induced by δ, in terms of the preferred spanning set. The *Rauzy–Veech group* of π, denoted $\mathrm{RV}(\pi)$, is the subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ generated by such matrices. The matrix associated with any loop δ coincides with the Rauzy–Veech matrix B_δ but this is special for abelian stratum components.

7.2. Minus and plus pieces for quadratic stratum components.

Let $\mathcal{C}$ be a component of a stratum of quadratic differentials. There is a branched double cover $\widetilde{S}$ of S such that the lifts $\widetilde{q}$ for $q \in \mathcal{C}$ are abelian differentials on $\widetilde{S}$. This is often called *the orientation double cover of the quadratic differential.* The cover is branched over every zero of q with odd order and every pole of q. We will give the construction of the orientation double cover of a typical rooted quadratic differential shortly.

The differential $\widetilde{q}$ is symmetric with respect to an involution and the quotient is q. Viewed as an involution of $\widetilde{S}$, the induced linear action on $H^1(\widetilde{S}, \widetilde{Z}; \mathbb{Z})$ has eigenvalues $\{1, -1\}$. The $(+1)$-eigenspace is usually referred to as the plus (or invariant) piece and the (-1)-eigenspace is usually referred to as the minus (or anti-invariant) piece. By Poincaré-duality, we can also consider it as a splitting of the homology $H_1(\widetilde{S}, \widetilde{Z}; \mathbb{Z})$.

The Teichmüller flow on $\mathcal{C}$ defines a cocycle by its action on $H^1(\widetilde{S}, \widetilde{Z}; \mathbb{Z})$. The cocyle preserves the plus and minus eigenspaces. The plus Kontsevich–Zorich cocycle is its restriction to the plus piece. Similarly, we also get the minus Kontsevich–Zorich cocycle.

7.3. **Plus Rauzy–Veech groups and integrability of the plus cocycle.** As the absolute part of the plus piece is invariant under the involution, it is isomorphic to the absolute homology of S. Hence, the *plus Rauzy–Veech group* $\mathrm{RV}^+(\pi)$ can be defined in a similar way to the abelian case by associating to each irreducible quadratic permutation a preferred spanning set for the absolute homology of S. Then, for any loop δ in the Rauzy diagram $\mathcal{D}$ based at π, we may associate a matrix for the homology action induced by δ using the preferred spanning set. This matrix does not coincide in general with the Rauzy–Veech matrix B_δ and so we prefer to give a direct proof of the integrability of the plus cocycle.

More precisely, for each quadratic permutation in $\mathcal{D}$ there is a choice of a spanning set $\{c_\alpha\}_{\alpha\in\mathcal{A}}$ for the plus piece such that the matrix for the plus cocycle has a simple form in each Rauzy–Veech move. See the work by the fourth author [Gut17, Section 4.1] for the description of the spanning set.

The simple form of the matrix has the following description. Suppose α and β are top and bottom letters in π. Breaking symmetry, suppose that $x_\alpha > x_\beta$. Let $\delta = \sigma \to \tau$ be the Rauzy–Veech move dictated by the width constraint. If c_α and c_β have non-zero algebraic intersection then the matrix satisfies $C_\delta = I + M_{\alpha,\beta}$, where $M_{\alpha,\beta}$ is the matrix whose (α, β)-entry is one and zero otherwise. On the other hand, if c_α and c_β have zero algebraic intersection, then $C_\delta = I - M_{\alpha,\beta} - 2M_{\alpha,\alpha}$.

The explicit matrices give us a direct proof of the integrability of the plus cocycle. Inductively, the matrix for the cocycle can be defined for any finite Rauzy–Veech sequence δ as a product of matrices for individual Rauzy–Veech moves.

Lemma 7.4. *For any finite Rauzy–Veech sequence* δ

$$\|C_\delta\|_1 \leqslant \|B_\delta\|_1,$$

where B_δ *is the Rauzy–Veech matrix for* δ*.*

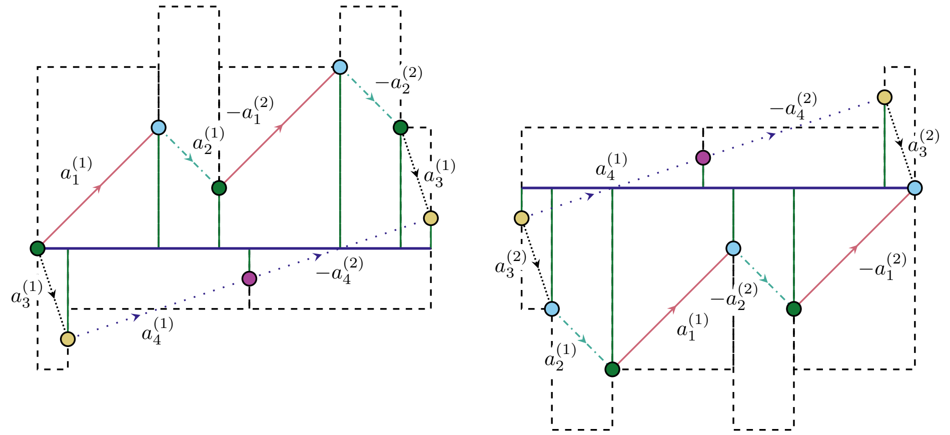
Proof. For any matrix C with coefficients c_{rs}, let $|C|$ be the non-negative matrix with coefficients $|c_{rs}|$.

Note that for an individual Rauzy move δ the plus cocycle satisfies $|C_\delta| = B_\delta$. Now let $\delta = \delta_1\delta_2 \dots \delta_k$ be a finite Rauzy–Veech sequence. We observe that

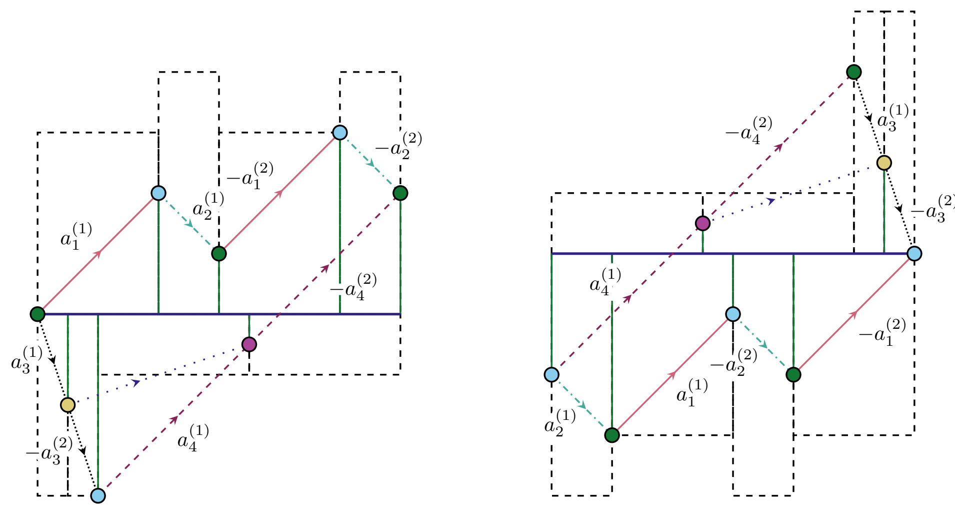
$$\|C_\delta\|_1 = \|C_{\delta_1} \dots C_{\delta_k}\|_1 \leqslant \||C_{\delta_1}| \dots |C_{\delta_k}|\|_1 = \|B_{\delta_1} \dots B_{\delta_k}\|_1 = \|B_\delta\|_1$$

□

As the above lemma shows, the plus cocycle is dominated by the Rauzy–Veech cocycle. The integrability of the plus cocycle now follows from Lemma 6.8.



(A) Original double cover.



(B) Double cover after a Rauzy move.

FIGURE 7.5. Example of the spanning set for the minus piece rendering the linear transformations coming from Rauzy moves equal to the Rauzy–Veech matrices. The original permutation is $\left(\begin{smallmatrix} 1 & 2 & 1 & 2 & 3 \\ 3 & 4 & 4 & & \end{smallmatrix}\right)$ representing the stratum $\mathcal{Q}(2,-1,-1)$, which becomes $\left(\begin{smallmatrix} 1 & 2 & 1 & 2 \\ 3 & 3 & 4 & 4 \end{smallmatrix}\right)$ after one bottom Rauzy move. In this case, the cycles in the spanning set can be tightened to saddle connections, so they are drawn in this manner. The general case is similar.

7.6. **Minus Rauzy–Veech groups and integrability of the minus cocycle.** The minus piece is in the kernel of the map induced on homology by the branched covering $\widetilde{S} \to S$. As a result, the minus cocycle has to be analysed directly in the orientation double cover of a quadratic differential. Here again, for each irreducible quadratic permutation there is a natural choice for a spanning set for the minus piece. Using this preferred set, the *minus Rauzy–Veech group* $\mathrm{RV}^-(\pi)$ can now be defined in a similar way to the other types of Rauzy–Veech groups. For rooted quadratic differentials that admit a zippered rectangles construction, we will explicitly construct their orientation double cover and then precisely describe the resulting matrices. These matrices preserve a specific alternating form defined by Avila–Resende [AR12, Equation 9].

Consider the arcs between singularities that we used to define singularity parameters. These arcs are a spanning set in the relative homology. Using these arcs, we can build a spanning set for the minus piece as follows.

To construct the orientation double cover of the rooted differential, we take two copies of the zippered rectangles. Let $1 \leqslant i \leqslant \ell + m$. As notation, a rectangle R_i will be denoted as $R_i^{(1)}$ in the first copy and $R_i^{(2)}$ in the second copy. The gluings are now constructed as follows.

- If $\pi(i)$ is a translation letter, then $R_i^{(1)}$ is glued to $R_{\sigma(i)}^{(1)}$ and $R_i^{(2)}$ is glued to $R_{\sigma(i)}^{(2)}$ as before.
- If $\pi(i)$ is a flip letter then $R_i^{(1)}$ is glued to $R_{\sigma(i)}^{(2)}$ by a translation.

The resulting abelian differential is the orientation double cover of the original quadratic differential. The involution rotates each rectangle by 180 degrees and maps it to the corresponding rectangle in the other copy.

Let $\alpha \in \mathcal{A}$ be a letter. Let a_α be the arc in the original quadratic differential oriented so that its period is $x_\alpha + y_\alpha$ where x_α, y_α are the singularity parameters. Let $a_\alpha^{(1)}$ and $a_\alpha^{(2)}$ be the lifts of a_α to the double cover. The spanning set for the minus piece in the relative homology of the orientation double cover is now defined as follows:

- Suppose α is a translation letter. Then let $A_\alpha = a_\alpha^{(1)} + a_\alpha^{(2)}$.
- Suppose α is a flip letter. Then let $A_\alpha = a_\alpha^{(1)} - a_\alpha^{(2)}$.

See Figure 7.5 for an illustration of these cycles.

It is straightforward to check that in a Rauzy–Veech move the linear change of these spanning sets is exactly encoded by the Rauzy–Veech matrix. Thus the Kontsevich–Zorich cocycle on the minus piece of the relative cohomology (by duality) coincides with the Rauzy–Veech cocycle. In particular, the L^1-norm of the restriction to the minus piece in absolute cohomology is dominated by the L^1-norm of the Rauzy–Veech matrix. Thus, the minus cocycle is integrable.

7.7. **Modular Rauzy–Veech groups.** By considering mapping classes instead of the homological actions induced by loops in the Rauzy diagram, we can define the *modular Rauzy–Veech groups*, the *plus modular Rauzy–Veech groups* and the *minus modular Rauzy–Veech groups.* These groups are then subgroups of mapping class groups and their images by the symplectic representations coincide with the corresponding Rauzy–Veech groups. They are denoted $\mathrm{MRV}(\pi)$, $\mathrm{MRV}^+(\pi)$ and $\mathrm{MRV}^-(\pi)$, respectively.

7.8. **Rauzy–Veech groups and monodromy groups.** Elementary theory of covering spaces together with our main theorems Theorem 4.23 and Theorem 5.5 implies that Rauzy–Veech groups are finite-index subgroups of appropriate monodromy groups. More precisely:

Corollary 7.9. *Let $\mathcal{C}$ be a component of a stratum of abelian or quadratic differentials and let π be an irreducible permutation that represents $\mathcal{C}$. Let $\widetilde{\mathcal{C}}$ be any finite manifold cover of $\mathcal{C}$ (which, in particular, can be taken to be either $\mathcal{C}^{\mathrm{root}}$ or $\mathcal{C}^{\mathrm{lab}}$). Then, the following groups are finite-index subgroups inside the following larger groups:*

1. $\pi_1(\widetilde{\mathcal{C}})$ *inside* $\pi_1^{\mathrm{orb}}(\mathcal{C})$*;*
2. *the (modular) monodromy group of $\widetilde{\mathcal{C}}$ inside the (modular) monodromy group of $\mathcal{C}$;*

(3) *if π is abelian, the (modular) Rauzy–Veech group of π inside the (modular) monodromy group of $\mathcal{C}$ of π;*
(4) *if π is quadratic, the (modular) plus (respectively, minus) Rauzy–Veech group of π inside the (modular) monodromy group of $\mathcal{C}$ corresponding to the plus (respectively, minus) piece of the homology.*

Proof. Since $\widetilde{\mathcal{C}}$ is a finite cover of $\mathcal{C}$, parts (1) and (2) follow from elementary theory of covering spaces.

Suppose that π is abelian. Then the modular Rauzy–Veech group of π is a subgroup of the modular monodromy group of $\mathcal{C}^{\mathrm{lab}}$. The push-forward of the modular Rauzy–Veech group to $\mathcal{C}^{\mathrm{root}}$ is exactly the image of the flow group $G(\mathcal{C}_\pi, q_0)$ inside the mapping class group. By Theorem 5.5, this image equals the modular monodromy group of $\mathcal{C}^{\mathrm{root}}$. By part (2), it is a finite-index subgroup of the modular monodromy group of $\mathcal{C}$. The rest of part (3) is obtained by applying the symplectic representation. Part (4) is obtained analogously. □

Part (3) of the previous corollary provides a partial answer (that is, up to finite index) to a question of Yoccoz [Yoc10, Remark in Section 9.3]. Part (4) extends these partial answers to analogous questions for quadratic stratum components.

Remark 7.10. *For abelian components, the overall structure of how various groups that we have considered fit together can be organised in the following commutative diagram:*

$$\begin{tikzcd}
\pi_1(\mathcal{D}) \arrow[r, two heads] \arrow[d, hook, "f.i."] & \pi_1(\mathcal{C}^{\mathrm{lab}}) \arrow[r, two heads] \arrow[d, hook, "f.i."] & \mathrm{MRV}(\pi) \arrow[r, two heads] \arrow[d, hook, "f.i."] & \mathrm{RV}(\pi) \arrow[d, hook, "f.i."] \\
\pi_1(\mathcal{D}^{\mathrm{red}}) \arrow[r, two heads] & G(\mathcal{C}_\pi) = \pi_1(\mathcal{C}^{\mathrm{root}}) \arrow[r, two heads] \arrow[d, hook, "f.i."] & \mathrm{MMon}(\mathcal{C}^{\mathrm{root}}) \arrow[r, two heads] \arrow[d, hook, "f.i."] & \mathrm{Mon}(\mathcal{C}^{\mathrm{root}}) \arrow[d, hook, "f.i."] \\
& \pi_1^{\mathrm{orb}}(\mathcal{C}) \arrow[r, two heads] & \mathrm{MMon}(\mathcal{C}) \arrow[r, two heads] \arrow[d, hook] & \mathrm{Mon}(\mathcal{C}) \arrow[d, hook] \\
& & \mathrm{Mod}(S) \arrow[r, two heads] & \mathrm{Sp}(2g, \mathbb{Z})
\end{tikzcd}$$

where "f.i." stands for "finite index", and recall that MMon *is the modular monodromy group and that* Mon *is the monodromy group. This diagram actually allows us to define the "most general" version of a Rauzy–Veech group, which is the image of the group homomorphism $\pi_1(\mathcal{D}) \to \pi_1(\mathcal{C}^{\mathrm{lab}})$. Theorem 5.2 shows that this group is actually equal to $\pi_1(\mathcal{C}^{\mathrm{lab}})$. Thus, any other version of Rauzy–Veech group can be obtained as the image of $\pi_1(\mathcal{C}^{\mathrm{lab}})$ by an appropriate group homomorphism.*

Similar commutative diagrams can also be stated for quadratic components by considering the images into the plus and minus pieces separately.

Combining Corollary 7.9 with the work of Calderon and Calderon–Salter [Cal20; CS19a; CS19b; CS20], we obtain a classification of the modular Rauzy–Veech groups and the Rauzy–Veech groups in relative homology, up to finite index, for genus at least five for non-hyperelliptic components. More precisely:

Corollary 7.11. *Let S be a topological surface of genus at least five. Let $\mathcal{C}$ be a non-hyperelliptic component of a stratum of abelian differentials on S whose set of marked points is Z. Let ϕ is the absolute framing induced by the horizontal vector field of a surface in $\mathcal{C}$. We have that:*

(1) *The modular Rauzy–Veech group in* $\mathrm{Mod}(S,Z)$ *is a finite-index subgroup of* $\mathrm{Mod}(S,Z)[\phi]$*, that is, of the stabiliser of* ϕ *inside* $\mathrm{Mod}(S,Z)$*.*
(2) *The Rauzy–Veech group in* $\mathrm{PAut}(H_1(S,Z;\mathbb{Z}))$ *is a finite-index subgroup of the kernel of the crossed homomorphism* $\Theta_\phi\colon \mathrm{PAut}(H_1(S,Z;\mathbb{Z})) \to H^1(S,\mathbb{Z}/2\mathbb{Z})$ *defined by Calderon–Salter [CS19b, Section 4].*

This classification was already known for hyperelliptic components, and in this case the index is known to be one [AMY18].

8. Classification of components and a reduction strategy

8.1. **Classification of the components of strata of abelian and quadratic differentials.** For the reader's convenience, we restate the complete classification of the components of abelian and quadratic strata.

Theorem 8.2 ([KZ03])**.** *The following is the classification of the components of the strata of abelian differentials (up to regular marked points).*

- *In genus one, the only stratum is* $\mathcal{H}(0)$*. It is non-empty, connected and hyperelliptic.*
- *In genus two, the only strata are* $\mathcal{H}(2)$ *and* $\mathcal{H}(1,1)$*. They are non-empty, connected and hyperelliptic.*
- *In genus three, the strata* $\mathcal{H}(4)$ *and* $\mathcal{H}(2,2)$ *have two components. One of them is hyperelliptic and the other one corresponds to odd spin structures. Every other stratum is non-empty and connected.*
- *Finally, for genus* g *at least four:*
  - *The stratum* $\mathcal{H}(2g-2)$ *has three components. One of them is hyperelliptic, and the other two are distinguished by even and odd spin structures.*
  - *The stratum* $\mathcal{H}(g-1,g-1)$ *can have two or three components depending on the parity of* g*. If* g *is even, it has two components. One of them is hyperelliptic, and the other one is not. If* g *is odd, it has three components. One of them is hyperelliptic, and the other two are distinguished by even and odd spin structures.*
  - *All other strata of the form* $\mathcal{H}(2\kappa_1,\dots,2\kappa_n)$ *have two components, distinguished by even and odd spin structures.*
  - *The remaining strata are non-empty and connected.*

Theorem 8.3 ([Lan08; CM14])**.** *The following is the classification of the components of the strata of quadratic differentials (up to regular marked points).*

- *In genus zero, every stratum is non-empty and connected.*
- *In genus one, the strata* $\mathcal{Q}(0)$ *and* $\mathcal{Q}(1,-1)$ *are empty. All other strata are nonempty and connected.*
- *In genus two, the strata* $\mathcal{Q}(4)$ *and* $\mathcal{Q}(3,1)$ *are empty. Moreover, the stratum* $\mathcal{Q}(2,2)$ *is non-empty, connected and hyperelliptic.*
- *In genus three, the strata* $\mathcal{Q}(9,-1)$*,* $\mathcal{Q}(6,3,-1)$ *and* $\mathcal{Q}(3,3,3,-1)$ *have two components, known as* regular *and* irregular *components.*
- *In genus four, the strata* $\mathcal{Q}(6,6)$*,* $\mathcal{Q}(6,3,3)$ *and* $\mathcal{Q}(3,3,3,3)$ *have three components. One of them is hyperelliptic, and the other two are known as* regular *and* irregular *components. Moreover, the strata* $\mathcal{Q}(12)$ *and* $\mathcal{Q}(9,3)$ *have two components, known as* regular *and* irregular *components.*
- *Finally, for genus at least two:*

– *The strata of the form $\mathcal{Q}(4j+2, 4k+2)$, $\mathcal{Q}(4j+2, 2k-1, 2k-1)$ and $\mathcal{Q}(2j-1, 2j-1, 2k-1, 2k-1)$ for $j, k \geqslant 0$ not contained in the previous list have two components. One of them is hyperelliptic and the other one is not.*
– *The remaining strata are non-empty and connected.*

8.4. **Adjacency of strata and a reduction strategy.** For abelian differentials [AV07b], the adjacency between components of different strata was exploited to show that Rauzy–Veech groups of simpler components are contained inside the Rauzy–Veech groups of more complex ones. We will exploit the same strategy to obtain the containment of Rauzy–Veech groups of quadratic stratum components.

We start with the notions of *simple extensions.*

Definition 8.5. *Let σ be an irreducible permutation. We say that a permutation τ is a* type preserving simple extension *of σ if τ is quadratic (respectively, abelian) when σ is quadratic (respectively, abelian) and τ can be obtained from σ by inserting a single letter α in such a way that:*

- *at most one occurrence of α is at the beginning of a row in τ; and*
- *no occurrence of α is at the end of a row in τ.*

Similarly, we have

Definition 8.6. *Let σ be an irreducible abelian permutation. We say that τ is a* type changing simple extension *of σ is τ is quadratic and τ can be obtained from σ by inserting a single top flip letter α and a single bottom flip letter β such that*

- *at most one occurrence of α (respectively, β) is at the beginning of a row in τ; and*
- *no occurrence of α (respectively, β) is at the end of a row in τ.*

As irreducible quadratic permutations already possess flip letters, there are no type changing extensions from a quadratic permutation to an abelian one.

The notion of simple extensions was original introduced by Avila–Viana [AV07b] in their proof of the Kontsevich–Zorich conjecture for abelian differentials. By expanding it to type changing extensions, the notion was extended to quadratic differentials by the fourth author [Gut17].

Note that if σ is an irreducible permutation and τ is a (type preserving or changing) simple extension of σ, then τ is also irreducible [Gut17, Lemma 3.2]. Also, note that the genera of the embodying strata of σ and τ are the same.

Let τ be a (type preserving or changing) simple extension of σ and suppose that ζ is a directed loop based at σ in the Rauzy diagram $\mathcal{D}$ that contains σ. It is then possible to shadow ζ by a Rauzy–Veech sequence starting from τ by requiring that the added letter (or letters) always lose if they participate in a Rauzy move [Gut17, Section 3]. It also follows from this description that the shadowing Rauzy–Veech sequence also returns to τ. This implies that $\mathrm{RV}(\sigma)$ is a subgroup of $\mathrm{RV}(\tau)$.

We now explain the geometric content underlying simple extensions. A type preserving simple extension allows us to split a singularity into a pair of singularities [Gut17, Lemma 5.1]. A type changing simple extension allows us to split a singularity in to three singularities at least one of which has odd order [Gut17, Corollary 5.2].

For quadratic stratum components which are our focus, the numerical invariant κ gets re-organised as follows;

- If $\kappa_1 \geqslant 1$, $\kappa_2, \ldots, \kappa_n \geqslant -1$ are integers and $\kappa_{1,1}, \kappa_{1,2} \geqslant -1$ are integers satisfying $\kappa_{1,1} + \kappa_{1,2} = \kappa_1$, then there exists a permutation σ whose embodying stratum is $\mathcal{Q}(\kappa_1, \ldots, \kappa_n)$ and a permutation τ whose embodying stratum is $\mathcal{Q}(\kappa_{1,1}, \kappa_{1,2}, \kappa_2, \ldots, \kappa_n)$ such that τ is a simple extension of σ.
- If $\kappa_1 \geqslant 0$, $\kappa_2 \ldots, \kappa_n \geqslant 1$ are integers and $\kappa_{1,1}, \kappa_{1,2}, \kappa_{1,3} \geqslant -1$ are integers which are not all even and satisfy $\kappa_{1,1} + \kappa_{1,2} + \kappa_{1,3} = 2\kappa_1$, then there exists a permutation σ whose embodying stratum is $\mathcal{H}(\kappa_1, \ldots, \kappa_n)$ and a permutation τ whose embodying stratum is $\mathcal{Q}(\kappa_{1,1}, \kappa_{1,2}, \kappa_{1,3}, 2\kappa_2, \ldots, 2\kappa_n)$ such that τ is a simple extension of σ.

These facts suggest the following strategy to show the Zariski density of every Rauzy–Veech group corresponding to the plus piece of quadratic differentials:

(1) Prove Zariski density for minimal quadratic strata, that is, for strata of the form $\mathcal{Q}(4g-4)$ for $g \geqslant 3$ with the exception of $\mathcal{Q}(12)^{\mathrm{reg}}$ and $\mathcal{Q}(12)^{\mathrm{irr}}$ (with $g = 4$) whose density has to be proved separately for technical reasons. Then use simple extensions to extend the density to every connected stratum with $g \geqslant 3$. For strata in $g \geqslant 3$ that have two components, one non-hyperelliptic and the other hyperelliptic, the non-hyperelliptic component can also be reached by such simple extensions from minimal strata. Note that a hyperelliptic component cannot arise by a simple extension of a non-hyperelliptic one. So we pass to the next case in our strategy.

(2) Prove Zariski density for every hyperelliptic component that has the form $\mathcal{Q}(4j+2, 4k+2)$ for $j, k \geqslant 1$. The rest of the hyperelliptic components arise by a string of simple extensions of these and hence we can extend density, except for those containing poles. Nevertheless, these last components arise as a string of simple extensions from the hyperelliptic component of minimal abelian strata.

    The remaining cases are all in genus four and lower and we will outline the strategy for those.

(3) The regular and irregular components in genus four arise by a simple extension from $\mathcal{Q}(12)$ or from $\mathcal{H}(6)$; the extensions from $\mathcal{H}(6)$ are already treated in previous work by the fourth author [Gut17, Table 1]. The density then extends to these.

(4) Prove Zariski density explicitly for $\mathcal{Q}(9, -1)^{\mathrm{irr}}$ in genus three. The remaining regular and irregular components in genus three can be handled by exhibiting a simple extension from $\mathcal{Q}(8)$ or from $\mathcal{H}(4)$; the extensions from $\mathcal{H}(4)$ are again already treated in previous work by the fourth author [Gut17, Table 1].

(5) Prove Zariski density explicitly for $\mathcal{Q}(5, -1)$ in genus two. The remaining non-hyperelliptic components all have at least three singularities and not all of these have even orders. So we may extend density from $\mathcal{H}(2)$ by using simple extensions.

(6) All quadratic strata in genus one have at least three singularities and not all of the singularities can have even orders. So we may extend density from $\mathcal{H}(0)$ by using simple extensions.

For abelian differentials, a similar strategy gives containment of Rauzy–Veech groups. As the Rauzy–Veech groups can be explicitly figured out in the base cases, they can then also be classified in all abelian cases using the containment. They turn out to be either the full symplectic group, or certain special finite index subgroups of the symplectic group. See the work by Avila–Matheus–Yoccoz

[AMY18] for hyperelliptic abelian components and by the fourth author [Gut19] for general abelian components. Plus and minus Rauzy–Veech groups for quadratic stratum components that can arise by a string of simple extensions from abelian base cases are also more tractable even though here a complete classification is not yet achieved [Gut17].

9. Zariski density

In this section we prove one of our main results.

Theorem 9.1. *The Rauzy–Veech groups for all components of all abelian strata are Zariski dense in their ambient symplectic groups. The same holds for the plus and minus Rauzy–Veech groups for all components of all quadratic strata.*

Being the full symplectic group or a finite index subgroup, Rauzy–Veech groups for abelian strata and quadratic components that arise by simple extensions from abelian strata are Zariski dense.

The Rauzy–Veech groups of the quadratic base cases are harder to track directly and here we bypass them. Instead, we leverage Filip's results [Fil17] to prove Zariski density of their monodromy groups. By Corollary 7.9, Rauzy–Veech groups are finite index in the monodromy groups. We deduce that they are Zariski dense. For clarity, we again refer the reader to the commutative diagram in Section 7. We then extend the density to all quadratic stratum components by simple extensions.

The proof presented here is self-contained and works for any (abelian or quadratic) connected component. A key ingredient is the work [Fil17] of Filip that gives a list of possible Zariski closures for algebraic hulls of linear invariant suborbifolds.

9.2. **Filip's results.** We briefly describe Filip's results [Fil17, Theorem 1.2 and Corollary 1.7] for the possible Zariski closures of the monodromy and algebraic hulls of a linear invariant suborbifold. Let $\mathcal{N}$ be a linear invariant suborbifold.

- If $p(T\mathcal{N})$ is the subbundle of the Kontsevich–Zorich cocycle that contains the tangent space to $\mathcal{N}$, then the monodromy has no zero exponents on $p(T\mathcal{N})$ and the closure of the monodromy for the action on $p(T\mathcal{N})$ is the full symplectic group [Fil17, Corollary 1.7]. This readily implies the Zariski density of the monodromy group of any abelian component, and, as detailed in Section 9.4, also implies the Zariski density of the monodromy group of the minus piece of any quadratic component.
- For strongly irreducible subbundles that do not contain the tautological plane, Filip shows that the Lie algebra representation of the corresponding piece of the algebraic hull must be, up to compact factors, one from the following list [Fil17, Theorem 1.2]:
  1. $\mathfrak{sp}(2g, \mathbb{R})$ in the standard representation,
  2. $\mathfrak{su}(p, q)$ in the standard representation or $\mathfrak{su}(p, 1)$ in any exterior power representation,
  3. $\mathfrak{so}(2n-1, 2)$ in the spin representation,
  4. $\mathfrak{so}^*(2n)$ in the standard representation, or $\mathfrak{so}(2n-2, 2)$ in either of the spin representations.

On the other hand, Eskin–Filip–Wright show that the algebraic hull coincides with the Zariski closure of the monodromy group for subbundles that do not contain the tautological plane [EFW18, Theorem 1.1]. Therefore, the previous theorem also classifies the Lie algebra representations of the Zariski closure of such monodromy groups.

9.3. **Zariski density for abelian components.** The Zariski density of the monodromy group for all abelian components follows directly from Filip's results, as the subbundle $p(T\mathcal{N})$ is the entire Hodge bundle.

9.4. **Zariski density for the minus piece.** Assume that $\mathcal{C}$ is a component of a stratum of the moduli space of quadratic differentials. Let $\widetilde{\mathcal{C}}$ be the linear invariant suborbifold consisting on the abelian differentials obtained by the orientation double cover of the quadratic differentials in $\mathcal{C}$. Let $\widetilde{S}$ be the underlying topological surface of the elements of $\widetilde{\mathcal{C}}$.

Let $p\colon H^1(\widetilde{S},\widetilde{Z};\mathbb{Z}) \to H^1(\widetilde{S};\mathbb{Z})$ be the restriction map to the absolute homology. By Filip's classification, the monodromy group acting on $p(T\widetilde{\mathcal{C}})$ is Zariski dense [Fil17, Corollary 1.7] in the full symplectic group. On the other hand, we have that $p(T\widetilde{\mathcal{C}})$ is exactly $H^1_-(\widetilde{S};\mathbb{Z})$, that is, the cohomology classes which are anti-invariant with respect to the action of ι. This can be seen by observing that if (X,ω) is an element of $\widetilde{\mathcal{C}}$, then $\iota^*\omega = -\omega$.

By Lefschetz-duality, this is equivalent to saying that the monodromy group is Zariski dense when acting on $H_1^-(\widetilde{S},\widetilde{Z};\mathbb{Z})$, so we obtain Zariski density for the minus piece of the homology.

9.5. **Zariski density for the plus piece.** It remains to show the Zariski density for the plus piece. We will first prove the Zariski density of the monodromy groups for minimal strata, hyperelliptic components with two singularities and some sporadic components. As Theorem 5.2 implies that the Rauzy–Veech groups are finite index in monodromy groups, it will follow that the Rauzy–Veech groups for these base components are also Zariski dense. We will then extend the density to the Rauzy–Veech groups of all components by simple extensions. In turn, this will imply that the monodromy groups of all components are Zariski dense.

For the base components, we will work directly on $H_1(S;\mathbb{Z})$ as it is isomorphic to $H_1^+(\widetilde{S};\mathbb{Z})$ in such a way that the corresponding monodromy groups are conjugate.

First observe that the monodromies in $H_1^+(S;\mathbb{Z})$ and $H^1_+(S;\mathbb{Z})$ are isomorphic by Poincaré duality. Let M denote this monodromy group. Note that $H^1_+(S;\mathbb{Z})$ does not contain the tautological plane. By Eskin–Filip–Wright [EFW18, Theorem 1.1], the algebraic hull coincides with the Zariski closure of the monodromy group M. This ensures that we can directly apply Filip's classification. Moreover, Treviño [Tre13, Theorem 1] proved that the plus Lyapunov spectrum contains no zero exponents.

Recall that the action of M on $H_1(S;\mathbb{R})$ is said to be *strongly irreducible* if no finite-index subgroup of M preserves a non-trivial vector subspace of $H_1(S;\mathbb{R})$. We will later show that this action is indeed strongly irreducible.

Assume for now that the action of M is strongly irreducible and let $\mathfrak{m}$ be the Lie algebra of the Zariski closure of M. Applying Filip's classification to $\mathfrak{m}$ and using the absence of zero exponents, we deduce that the possibilities for $\mathfrak{m}$ are, up to compact factors:

(1) $\mathfrak{sp}(2g,\mathbb{R})$ in the standard representation (degree $2g$, dimension $g(2g+1)$);
(2) $\mathfrak{su}(p,p)$ in the standard representation (degree $4p$, dimension $4p^2-1$);
(3) $\mathfrak{so}(2n-1,2)$ in the spin representation (degree 2^n, dimension $n(2n+1)$);
(4) $\mathfrak{so}(2n-2,2)$ in one of the spin representations (degree 2^{n-1}, dimension $n(2n-1)$); or

(5) $\mathfrak{so}^*(2n)$ in the standard representation for even n (degree $4n$, dimension $n(2n-1)$).

This list can be further refined by observing that the degree and dimension of the representation must match because of the strong irreducibility. In each of the above cases, we derive:

(1) $\dim_{\mathbb{R}} \mathfrak{sp}(2g, \mathbb{R}) = g(2g+1)$;
(2) $p = g/2$, so $\dim_{\mathbb{R}} \mathfrak{su}(g/2, g/2) = g^2 - 1$;
(3) $n = \log_2(2g)$, so $\dim_{\mathbb{R}} \mathfrak{so}(2n-1, 2) = \log_2(2g)\log_2(8g^2)$;
(4) $n = \log_2(4g)$, so $\dim_{\mathbb{R}} \mathfrak{so}(2n-2, 2) = \log_2(4g)\log_2(8g^2)$; and
(5) $n = g/2$, so $\dim_{\mathbb{R}} \mathfrak{so}^*(g/2, g/2) = g(g-1)/2$.

Note that possibilities (2)–(4) require the genus to be even. Thus, if g is odd and if the action of M is strongly irreducible then $\mathfrak{m}$ has to be $\mathfrak{sp}(2g, \mathbb{R})$. Hence, for odd genus it suffices to establish the strong irreducibility of the M-action. For even genus, along with showing the strong irreducibility, we will eliminate all but the symplectic representation by setting up a dimension count. Here, we exploit the additional flexibility that Theorem 5.6 provides.

As we use Dehn twists to build a dimension count, we record the following formula: suppose c, c' are oriented multi-curves. As an element of the homology

$$T(c)(c') = c' + \omega(c', c)c, \tag{9.6}$$

where T is the left Dehn twist and $\omega(*, *)$ is the algebraic intersection number.

The next lemma shows how Dehn twists in M can be used to prove the largeness of a subspace of $H_1(S; \mathbb{R})$ invariant under some finite index subgroup of M.

Lemma 9.7. *Let $V \neq \{0\}$ a subspace of $H_1(S; \mathbb{R})$ on which a finite-index subgroup N of M acts irreducibly. Suppose that $T(v) \in M$ for some $v \in H_1(S; \mathbb{R})$. If there exists $v' \in V$ such that $\omega(v, v') \neq 0$, then $v \in V$.*

Proof. By Equation (9.6) and the hypothesis, the linear combination $T(v)^k(v') - v'$ is then a non-zero multiple of v for every $k \geqslant 1$. Since $|M : N| < \infty$, $T(v)^k \in N$ for some $k \geqslant 1$. The lemma follows. □

We now state and prove a strong irreducibility criterion that we will use throughout the discussion that follows.

Lemma 9.8. *Let B be a finite set of cycles in $H_1(S; \mathbb{R})$ such that*

- *the span of B is $H_1(S; \mathbb{R})$;*
- *for any pair $u \neq u' \in B$, there exists a* chain $u = u_0, \ldots, u_k = u'$ *such that $\omega(u_j, u_{j+1}) \neq 0$ for all $0 \leqslant j \leqslant k-1$; and*
- *$T(u)$ is in M for all $u \in B$.*

Then, M acts strongly irreducibly on $H_1(S; \mathbb{R})$.

Proof. Let $V \neq \{0\}$ be a subspace on which a finite-index subgroup N of M acts irreducibly. By hypothesis, it suffices to show that B is contained in V.

Since B spans $H_1(S; \mathbb{R})$, there exists $v \in V$, $u \in B$ such that $\omega(v, u) \neq 0$. Then, by Lemma 9.7, $u \in V$. Let $u' \neq u$ be an element of B. By hypothesis, there is a chain $u = u_0, \ldots, u_k = u'$ such that $\omega(u_j, u_{j+1}) \neq 0$ for all $0 \leqslant j \leqslant k-1$. By applying Lemma 9.7 inductively, it follows that for all j the cycles u_j are contained in V. We conclude that B is contained in V, so the lemma follows. □

Remark 9.9. *Explicit sets B to which we will apply the previous lemma, will not always be a basis of $H_1(S;\mathbb{R})$; extra cycles may be needed to satisfy the chain hypothesis.*

9.10. **Minimal strata.** For $g \geqslant 3$, let $d = 2g$ and

$$\pi_d = \begin{pmatrix} 1 & 2 & 1 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & d-1 \\ 2 & 4 & 3 & 5 & 4 & 8 & 7 & \cdots & d & d-1 & d \end{pmatrix}.$$

The rooted differentials in $\mathcal{C}_{\pi_d}$ belong to $\mathcal{Q}(4g-4)$.

As the first step, we will prove that the action of the monodromy group on $H_1(S;\mathbb{R})$ is strongly irreducible. To do so, we will choose an explicit set of cycles spanning the homology. The parity of g will dictate a choice of slightly different sets of cycles. See Figure 9.11a for a flat surface in $\mathcal{Q}(16)$ for $g=5$, and Figure 9.11b and Figure 9.11c for a flat surface in $\mathcal{Q}(12)$ for $g=4$. Note, however, that the stratum $\mathcal{Q}(12)$ has two components, and that the figure depicts a quadratic differential in $\mathcal{Q}(12)^{\mathrm{reg}}$. We will complete the proof for $g=4$ in Appendix B, but for now we will only show the strong irreducibility for $\mathcal{Q}(12)^{\mathrm{reg}}$. The pattern for the cycles in even genus greater than four remains the same as in Figure 9.11b and Figure 9.11c, so we don't need a separate figure.

In the second step, we will prove Zariski density of the monodromy group for even genus (odd genera being directly covered by Filip's result).

In our proofs, we will use a subset of the cycles (and their combinations) that are core curves of cylinders on the flat surface. Dehn twists in such cycles are in the monodromy group, so we may use them in our computations.

Let M be the monodromy group and let $\mathfrak{m}$ be the Lie algebra of its Zariski closure. We set $\varepsilon = (-1)^g$.

We consider the collection of the following homology classes:

- c_1 and c_d are the homology classes of the dashed curves;
- c_2 and c_{d-1} are the homology classes of the dash-dotted curves; and
- $c_3, \ldots, c_{d-2}$ are the homology classes of the solid curves.

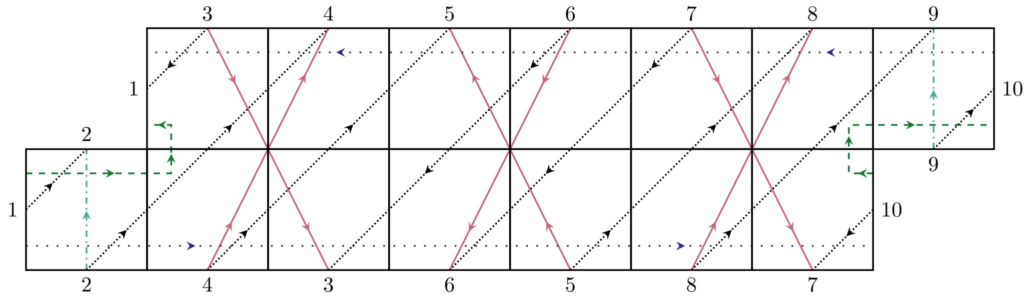
These curves form a basis for $H_1(S;\mathbb{R})$ (which is symplectic if ordered appropriately). The densely dotted slope-1 curve is $b = \sum_{i=1}^{d} c_i$ and the loosely dotted horizontal curve is $p = c_1 + \varepsilon c_d$. In even genus, we also need the curve $b' = \sum_{i=1}^{d-4} c_i - c_{d-1} - c_d$ obtained by modifying b. Note that the set $\{c_2, \ldots, c_{d-1}, b, p\}$ is a basis for $H_1(S;\mathbb{R})$ when g is odd and the set $\{c_2, \ldots, c_{d-1}, b, b', p\}$ is a spanning set for $H_1(S;\mathbb{R})$ when g is even.

Lemma 9.12. *The action of M on $H_1(S;\mathbb{R})$ is strongly irreducible.*

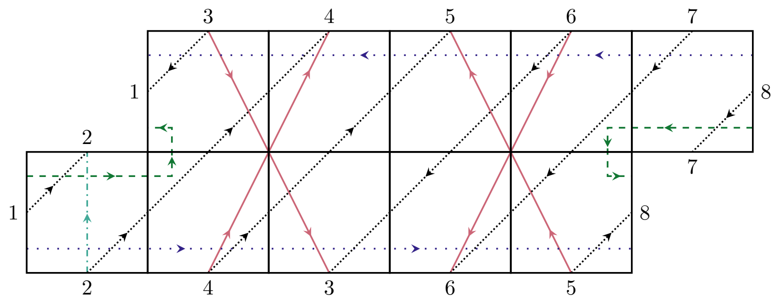
Proof. We set

$$B = \begin{cases} \{c_2, \ldots, c_{d-1}, b, p\} & \text{if } g \text{ is odd} \\ \{c_2, \ldots, c_{d-1}, b, b', p\} & \text{if } g \text{ is even} \end{cases}$$

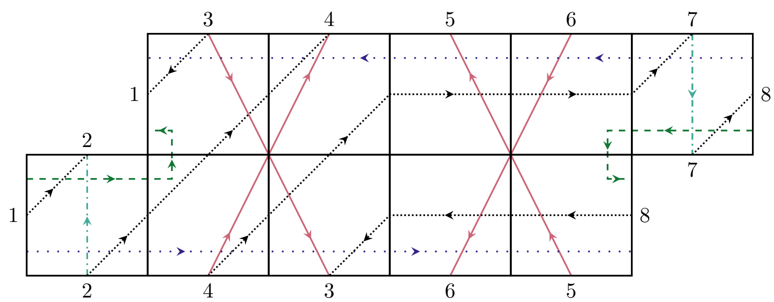
As directly seen from Figure 9.11a and Figure 9.11b, the cycles in the set $\{c_2, \ldots, c_{d-1}, b\}$ are given by core curves of cylinders on the corresponding flat surfaces. Now consider the cylinder with core curve $c_{d-3} - c_{d-2}$. A left-handed shear applied inside of this cylinder makes the four horizontal saddle connections have slope one. Equivalently, it performs a one-quarter Dehn twist. This straightens the modified slope-one curve b'; see Figure 9.11c. Thus b' is the core curve of a cylinder on the deformed surface. We deduce that for any cycle u in B the Dehn twist $T(u)$ is in M.



(A) Odd g.



(B) Even g with the slope-1 curve b.



(C) Even g with the modified slope-1 curve b'.

FIGURE 9.11. Basis and useful curves.

We also note that for any pair $u \neq u'$ in B there exists a chain $u = u_0, \ldots, u_k = u'$ such that $\omega(u_j, u_{j+1}) \neq 0$ for all $0 \leqslant j \leqslant k-1$.

It follows that the set B satisfies the hypothesis of Lemma 9.8 and thus the action of M on $H_1(S;\mathbb{R})$ is strongly irreducible. $\square$

The Dehn twist $T(c)$ along a homology cycle c is also a symplectic transvection. For the following calculation we will think of it as such. If $T(c)^k \in M$, then $T(c)^k - \mathrm{Id} \in \mathfrak{m}$. As notation, let $D(c) = T(c) - \mathrm{Id}$, $E(c) = T(c)^2 - \mathrm{Id}$.

The cycles $c_2, \ldots, c_{d-1}$ and p are realised as core curves of cylinders on the surface. Hence, the Dehn twists $T(c_i)$ for $i = 2, \ldots, d-1$ and $T(p)$ are in M. Observe that:

$$\begin{aligned}
D(c_2)(c_1) &= -c_2 & D(c_2)(c_i) &= 0 \text{ for } i \neq 1 \\
D(c_{2i+1})(c_{2i+2}) &= c_{2i+1} & D(c_{2i+1})(c_j) &= 0 \text{ for } j \neq 2i+2 \\
D(c_{2i+2})(c_{2i+1}) &= -c_{2i+2} & D(c_{2i+2})(c_j) &= 0 \text{ for } j \neq 2i+1 \\
D(c_{d-1})(c_d) &= -c_{d-1} & D(c_{d-1})(c_j) &= 0 \text{ for } j \neq d \\
D(p)(c_2) = p,\ D(p)(c_{d-1}) &= \varepsilon p & D(p)(c_j) &= 0 \text{ for } j \notin \{2, d-1\}
\end{aligned}$$

All elements above belong to $\mathfrak{m}$.

Moreover, for each $1 \leqslant i, j \leqslant g-2$ we have that the elements $T(c_{2i+1}+c_{2j+1})^2$, $T(c_{2i+1}+c_{2j+2})^2$ and $T(c_{2i+2}+c_{2j+2})^2$ all belong to M. Indeed, if $i = j$ then $|\omega(c_{2i+1}, c_{2j+2})| = 1$, so $T(c_{2i+1}+c_{2j+2}) \in M$. Otherwise, this follows from a result by the fourth author using b as the auxiliary vector [Gut19, Corollary 2.8]. Observe that

$$\begin{aligned}
E(c_{2i+1}+c_{2j+1})(c_{2i+2}) &= 2(c_{2i+1}+c_{2j+1}) \\
E(c_{2i+1}+c_{2j+1})(c_{2j+2}) &= 2(c_{2i+1}+c_{2j+1}) \\
E(c_{2i+1}+c_{2j+2})(c_{2i+2}) &= 2(c_{2i+1}+c_{2j+2}) \\
E(c_{2i+1}+c_{2j+2})(c_{2j+1}) &= -2(c_{2i+1}+c_{2j+2}) \\
E(c_{2i+2}+c_{2j+2})(c_{2i+1}) &= -2(c_{2i+2}+c_{2j+2}) \\
E(c_{2i+2}+c_{2j+2})(c_{2j+1}) &= -2(c_{2i+2}+c_{2j+2}).
\end{aligned}$$

The number of elements of the form $D(*)$ is $d-1 = 2g-1$. The number of elements of the form $E(*)$ are $\binom{2g-4}{2}$.

With basis $\{c_1, \dots, c_d\}$, we identify the vector space of linear transformations of $H_1(S;\mathbb{R})$ with $\mathrm{Mat}_{d\times d}(\mathbb{R})$. We may then associate a matrix to the elements $D(*)$ and $E(*)$ considered above. Let $M_{i,j}$ be the matrix with the (i,j)-entry one and all other entries zero. We may then write the matrices for $D(*)$ and $E(*)$ as linear combinations of $M_{i,j}$.

From the calculations above, we note that

- The matrix for $D(c_{2i+1})$ is $M_{(2i+1),(2i+2)}$.
- The matrix for $D(c_{2i+2})$ is $-M_{(2i+2),(2i+1)}$.
- The matrix $M_{(2i+2),(2j+1)}$ features only in the linear combination of the matrix for $E(c_{2i+1}+c_{2j+1})$.
- The matrix $M_{(2i+2),(2j+2)}$ features only in the linear combination of the matrix for $E(c_{2i+1}+c_{2j+2})$.
- The matrix $M_{(2i+1),(2j+2)}$ features only in the linear combination of the matrix for $E(c_{2i+2}+c_{2j+2})$.

It follows that all the elements considered above are linearly independent in $\mathrm{Mat}_{d\times d}(\mathbb{R})$. Thus,

$$\dim_{\mathbb{R}} \mathfrak{m} \geqslant \binom{2g-4}{2} + 2g - 1 = 2g^2 - 7g + 9.$$

We conclude that if $g \geqslant 6$, we have that

$$\begin{aligned}
&\dim_{\mathbb{R}} \mathfrak{m} > \dim_{\mathbb{R}} \mathfrak{su}(g/2, g/2) = g^2 - 1 \\
&\dim_{\mathbb{R}} \mathfrak{m} > \dim_{\mathbb{R}} \mathfrak{so}(2n-1, 2) = \log_2(2g)\log_2(8g^2) \text{ for } n = \log_2(2g) \\
&\dim_{\mathbb{R}} \mathfrak{m} > \dim_{\mathbb{R}} \mathfrak{so}(2n-2, 2) = \log_2(4g)\log_2(8g^2) \text{ for } n = \log_2(4g) \\
&\dim_{\mathbb{R}} \mathfrak{m} > \dim_{\mathbb{R}} \mathfrak{so}^*(g/2) = g(g-1)/2.
\end{aligned}$$

Hence, in these cases, $\mathfrak{m} = \mathfrak{sp}(2g, \mathbb{R})$ and that $M = \mathrm{Sp}(2g, \mathbb{R})$. Recall directly from the list that $\mathfrak{m}$ is $\mathfrak{sp}(2g, \mathbb{R})$ when g is odd, as the action of M on $H_1(S;\mathbb{R})$ is strongly irreducible.

Since minimal strata only occur for $g \geqslant 3$, the only remaining case is $g = 4$. We treat this case in Appendix B.

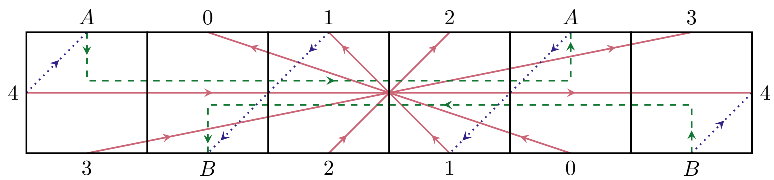


FIGURE 9.14. Curves for the hyperelliptic case.

9.13. **Hyperelliptic components with two singularities.** Let $r, s \geqslant 1$ be odd integers. Consider the permutation $\pi_{r,s}$ defined by

$$\begin{pmatrix} A & 0 & 1 & 2 & \cdots & r-1 & A & r & r+1 & \cdots & r+s \\ r+s & r+s-1 & \cdots & r & B & r-1 & r-2 & r-3 & \cdots & 0 & B \end{pmatrix}.$$

The embodying component for this permutation is $\mathcal{Q}(2r, 2s)^{\text{hyp}}$ and we can assume that $r \geqslant s$. See Figure 9.14 for a flat surface in $\mathcal{Q}(6, 2)^{\text{hyp}}$. The genus of the underlying surface is $g = (r + s + 2)/2$. Let:

- $c_0, \ldots, c_{r+s}$ be the homology classes of the solid curves;
- c_A and c_B be the homology classes of the dashed curves; and
- c_{AB} be the homology class of the dotted curve.

These cycles form a spanning set of the relative homology $H_1(S, Z; \mathbb{Z})$. As a relative cycle, $c_{AB} = -c_s + c_{r+s} - c_A + c_B$.

In absolute homology, $c_A + c_B = 0$. Excising c_B (or c_A) we obtain exactly $2g = r + s + 2$ curves. So $\{c_0, \ldots, c_{r+s}\} \cup \{c_A\}$ is a basis of $H_1(S; \mathbb{Z})$. Note that

- all cycles in $\{c_0, \ldots, c_{r+s}\} \cup \{c_{AB}\}$ are represented by core curves of cylinders and hence all $T(c_j)$, for $0 \leqslant j \leqslant r + s$, and $T(c_A)$ are in M; and
- any pair of cycles in $\{c_0, \ldots, c_{r+s}\}$ intersect. Moreover, c_{AB} intersects c_0.

Thus, the basis satisfies the hypothesis of Lemma 9.8 which proves that the action of M on $H_1(S; \mathbb{R})$ is strongly irreducible.

We again consider the Dehn twist $T(c)$ in a cycle c as a symplectic transvection. If $T(c)^k \in M$, then $T(c)^k - \mathrm{Id} \in \mathfrak{m}$. As notation, let $D(c) = T(c) - \mathrm{Id}$.

The cycles c_i, for $0 \leqslant i \leqslant r + s$, and $c_i + c_j$, for $0 \leqslant i < j \leqslant r + s$, are core curves of cylinders on the flat surface. Hence, $T(c_i), T(c_i + c_j) \in M$ for each $0 \leqslant i < j \leqslant r + s$ and, thus, $D(c_i), D(c_i + c_j) \in \mathfrak{m}$. As before, we use the basis $\{c_0, c_1, \ldots, c_{r+s}, c_A\}$ to identify linear transformations of $H_1(S; \mathbb{R})$ with $\mathrm{Mat}_{2g\times 2g}(\mathbb{R})$. Again as before, for $0 \leqslant i, j \leqslant r + s$ let $M_{i,j}$ be the matrix with (i, j)-entry one and all remaining entries zero. Similarly, we have the definitions for $M_{i,A}$ and $M_{A,A}$.

We use the following notation. If $P(i, j)$ is a logical proposition on i and j, we define

$$[\![P(i,j)]\!] = \begin{cases} 1 & \text{if } P(i,j) \text{ is true} \\ 0 & \text{if } P(i,j) \text{ is false.} \end{cases}$$

We then note that

$$D(c_i) = -\sum_{k<i} M_{i,k} + \sum_{k>i} M_{i,k} + [\![i < r]\!] M_{i,A}.$$

Similarly, with $i < j$ note that

$$D(c_i + c_j) = -\,(M_{i,i} + M_{j,i}) + (M_{i,j} + M_{j,j})$$

$$\begin{aligned}
&- 2\sum_{k<i}(M_{i,k} + M_{j,k}) + 2\sum_{j<k}(M_{i,k} + M_{j,k}) \\
&+ 2[\![i < j < r]\!](M_{i,A} + M_{j,A}) \\
&+ [\![i < r \leqslant j]\!](M_{i,A} + M_{j,A}).
\end{aligned}$$

Lemma 9.15. *The collection of matrices* $\{D(c_i)\} \cup \{D(c_i + c_j)\}_{i<j}$ *are linearly independent.*

Proof. Any linear combination can be regrouped as $\sum_{i=0}^{r+s} S_i$ where

$$S_i = a_i D(c_i) + \sum_{j=i+1}^{r+s} a_{i,j} D(c_i + c_j).$$

In particular, when $i = r + s$ the summation is empty and $S_{r+s} = a_{r+s} D(c_{r+s})$.

Note that the matrices in S_0 are the only ones in the whole linear combination whose first row does not vanish. Inductively, the matrices in S_{i+1} are the only ones outside $S_0, \dots, S_i$ whose $(i+1)$-th row is empty. Hence, it suffices to argue that each such collection $\{D(c_i), D(c_i+c_{i+1}), \dots, D(c_i+c_{r+s})\}$ is linearly independent.

Let $0 \leqslant i \leqslant r+s$ and let W be the i-th row of $D(c_i)$. Moreover, for $i+1 \leqslant j \leqslant r+s$, let V_j be the i-th row of $D(c_i + c_j)$.

Claim 9.16. *The row vectors* $W, V_{i+1}, \dots, V_{r+s}$ *are linearly independent.*

Proof (Claim). We index the standard basis in $\mathbb{R}^{2g}$ as $\{e_0, \dots, e_{r+s}, e_A\}$. From the formulae above, we deduce

$$W = -\sum_{k<i} e_k + \sum_{k>i} e_k + [\![i < r]\!] e_A.$$

and for $i + 1 \leqslant j \leqslant r + s$

$$\begin{aligned}
V_j = &- 2\sum_{k<i} e_k - e_i + e_j + 2\sum_{k>j} e_k \\
&+ (2[\![i < j < r]\!] + [\![i < r \leqslant j]\!]) e_A
\end{aligned}$$

Consider $V_{r+s} = -2\sum_{k<i} e_k - e_i + e_{r+s} + [\![i < r]\!] e_A$. Observe that the linear combination of canonical vectors realising $V_j - 2V_{r+s}$ does not feature e_{r+s} for $i+1 \leqslant j < r+s$. Thus, we redefine V_j to be $V_j - 2V_{r+s}$ for such j and continue inductively. That is, the next step redefine V_j to be $V_j - 2V_{r+s-1}$ for every $i+1 \leqslant j < r+s-1$.

We obtain that

$$\begin{aligned}
V_j = 2(-1)^{j+1} &\sum_{k<i} e_k + (-1)^{j+1} e_i + e_j \\
&+ 2(-1)^j [\![i < j < r]\!] e_A \\
&+ (-1)^j [\![i < r \leqslant j]\!] e_A.
\end{aligned}$$

We also apply a similar process to W, redefining it to be $W - V_{r+s} - V_{r+s-1} - \cdots - V_{i+1}$. Then, we obtain that

$$W = \begin{cases} -\sum_{k<i} e_k + [\![i < r]\!] e_A & \text{if } i \text{ is even} \\ +\sum_{k\leqslant i} e_k - [\![i < r]\!] e_A & \text{if } i \text{ is odd.} \end{cases}$$

We observe

- for $i+1 \leqslant j \leqslant r+s$, the vector e_j is featured only in the linear combination for V_j, and

- the vector W is in the kernel of the projection $\mathbb{R}^{2g} \to \mathbb{R}^{r+s-i}$ given by $(u_0, \ldots, u_{r+s}) \mapsto (u_{i+1}, \ldots, u_{r+s})$.

We thus conclude the claim. □

By the claim, the collection $\{D(c_i)\} \cup \{D(c_i + c_j)\}_{i<j}$ for each fixed i is linearly independent. The lemma then follows from our grouping. □

By Lemma 9.15,

$$\dim_{\mathbb{R}} \mathfrak{m} \geqslant \binom{2g-1}{2} + 2g - 1 = g(2g-1).$$

We conclude that if $g \geqslant 5$, we have that

$$\begin{aligned}
&\dim_{\mathbb{R}} \mathfrak{m} > \dim_{\mathbb{R}} \mathfrak{su}(g/2, g/2) = g^2 - 1 \\
&\dim_{\mathbb{R}} \mathfrak{m} > \dim_{\mathbb{R}} \mathfrak{so}(2n-1, 2) = \log_2(2g) \log_2(8g^2) \text{ for } n = \log_2(2g) \\
&\dim_{\mathbb{R}} \mathfrak{m} > \dim_{\mathbb{R}} \mathfrak{so}(2n-2, 2) = \log_2(4g) \log_2(8g^2) \text{ for } n = \log_2(4g) \\
&\dim_{\mathbb{R}} \mathfrak{m} > \dim_{\mathbb{R}} \mathfrak{so}^*(g/2, g/2) = g(g-1)/2.
\end{aligned}$$

Thus, we obtain that $\mathfrak{m} = \mathfrak{sp}(2g, \mathbb{R})$ and that $M = \mathrm{Sp}(2g, \mathbb{R})$ for $g \geqslant 5$.

To extend to $g = 4$, we include $D(c_{AB})$ in the collection of matrices and prove linear independence. Note that

$$\begin{aligned}
D(C_{AB}) = 3&(M_{0,0} - M_{s,0} - M_{A,0}) \\
&+ 4 \sum_{k<s} (M_{0,k} - M_{s,k} - M_{A,k}) \\
&+ 2 \sum_{k=s+1}^{r-1} (M_{0,k} - M_{s,k} - M_{A,k}).
\end{aligned}$$

We expand the collection S_0 in the proof of Lemma 9.15 to include $D(C_{AB})$. Let $\{W, V_1, \ldots, V_{r+s}$ be the vectors that arise from S_0 in Claim 9.16. It suffices to show that the first row of $D(C_{AB})$ cannot be written as a linear combination of $\{W, V_1, \ldots, V_{r-1}\}$. This is readily seen after the vectors $W, V_1, \ldots, V_{r-1}$ are redefined as per the Gaussian elimination process described in the proof of Claim 9.16.

For $g = 3$, we have directly from the list that $\mathfrak{m} = \mathfrak{sp}(2g, \mathbb{R})$ since the action of M on $H_1(S; \mathbb{R})$ is strongly irreducible.

The only remaining case is $g = 2$. Here, we have that

$$\begin{aligned}
\dim_{\mathbb{R}} \mathfrak{m} \geqslant 6 > 3 = \dim_{\mathbb{R}} \mathfrak{su}(1,1) \\
\mathfrak{sp}(4, \mathbb{R}) \cong \mathfrak{so}(3,2) \\
\dim_{\mathbb{R}} \mathfrak{sp}(4, \mathbb{R}) = 10 < 15 = \dim_{\mathbb{R}} \mathfrak{so}(4,2),
\end{aligned}$$

so the only possibility for $\mathfrak{m}$ is $\mathfrak{sp}(4, \mathbb{R}) \cong \mathfrak{so}(3,2)$.

9.17. **Exceptional non-minimal strata.** In Table 1, we exhibit an explicit simple extension from a component of a minimal stratum to each non-hyperelliptic component of exceptional non-minimal strata that was not already treated by the fourth author [Gut17, Table 1], except for $\mathcal{Q}(9, -1)^{\mathrm{irr}}$. Indeed, there does not exist a simple extension from $\mathcal{Q}(8)$ to this last component as noted by Lanneau [Lan08], so we treat it separately in Appendix B. These computations were performed by using the `surface_dynamics` package for SageMath [Ste+20].

Source	Target	Permutation
$\mathcal{Q}(8)$	$\mathcal{Q}(9,-1)^{\mathrm{reg}}$	$\begin{pmatrix} 1 & 2 & 1 & 3 & 4 & 3 & 5 & 2 \\ & 6 & 5 & 4 & A & A & 6 & \end{pmatrix}$
$\mathcal{Q}(12)^{\mathrm{reg}}$	$\mathcal{Q}(6,6)^{\mathrm{reg}}$	$\begin{pmatrix} 1 & 2 & A & 3 & 4 & 5 & A & 6 & 7 & 5 \\ & 2 & 4 & 6 & 8 & 7 & 8 & 3 & 1 & \end{pmatrix}$
$\mathcal{Q}(12)^{\mathrm{irr}}$	$\mathcal{Q}(6,6)^{\mathrm{irr}}$	$\begin{pmatrix} & 1 & 2 & 3 & 4 & 3 & 5 & 6 & 7 & \\ 8 & 1 & 6 & 8 & A & 4 & 2 & 7 & A & 5 \end{pmatrix}$
$\mathcal{Q}(12)^{\mathrm{reg}}$	$\mathcal{Q}(9,3)^{\mathrm{reg}}$	$\begin{pmatrix} 1 & 2 & 3 & A & 4 & 3 & 5 & 6 & 7 \\ 2 & 4 & 6 & 8 & 7 & 8 & 5 & A & 1 \end{pmatrix}$
$\mathcal{Q}(12)^{\mathrm{irr}}$	$\mathcal{Q}(9,3)^{\mathrm{irr}}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & A & 4 & A & 6 & 7 \\ & 2 & 6 & 8 & 5 & 3 & 7 & 8 & 1 & \end{pmatrix}$

TABLE 1. Explicit extensions into non-hyperelliptic components of exceptional strata with less than three singularities. The permutation in the third column represents the component in the second column. Erasing letters A and B produces a permutation representing the component of a minimal stratum in the first column.

10. Simplicity

We now prove the Kontsevich–Zorich conjecture.

Theorem 10.1. *The Kontsevich–Zorich cocycle has a simple spectrum for all components of all strata of abelian differentials. The plus and minus Kontsevich–Zorich cocycles also have a simple spectrum for all components of all strata of quadratic differentials.*

Proof. Let $\mathcal{C}$ be any component of a stratum of abelian or quadratic differentials. We have the following facts.

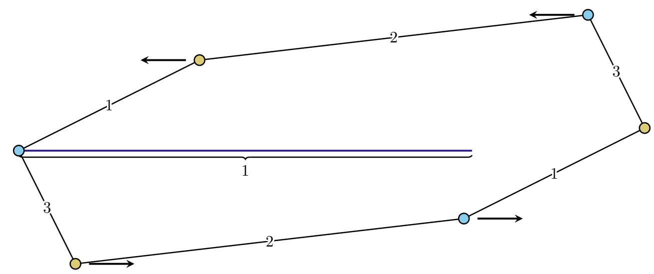
- The Teichmüller flow on a finite cover of $\mathcal{C}$ admits a coding as a countable shift with approximate product structure.
- We have established that the plus and minus cocycles lifted to this cover are locally constant and integrable.
- We have also established that associated monoids for the plus and minus cocycles (lifted to this cover) are Zariski dense in the corresponding symplectic groups.

By Criterion 6.9, it follows that the Lyapunov spectra for the plus and minus cocycles are simple. $\square$

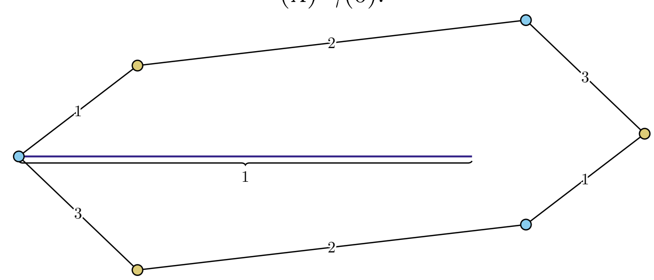
Appendix A. Examples

A.1. **The decomposition of $\mathcal{C}^{\mathrm{root}}$ is not polytopal.** In this section, we present an explicit example showing that the decomposition of $\mathcal{C}^{\mathrm{root}}$ into the union of the $\overline{\mathcal{C}_\pi}$, where $\pi \in \mathcal{R}$, is "not polytopal". Indeed, a compact arc in $\mathcal{C}^{\mathrm{root}}$ may intersect $\mathcal{S} = \mathcal{C}^{\mathrm{root}} - \bigcup_{\pi \in \mathcal{R}} \mathcal{C}_\pi$ infinitely many times even if it is transverse to $\mathcal{V}$.

Let $\mathcal{C} = \mathcal{H}(0,0)$ and consider the curve shown in Figure A.2. The rooted differential $\gamma(1)$ contains a "wide" vertical cylinder, that is, a vertical cylinder such that an arc of length one emanating from the root ends before it crosses the cylinder entirely. On the other hand, we assume that that the rooted differential $\gamma(0)$ is doubly non-vanishing, and that it admits a normalised zippered rectangles construction for the underlying permutation $\pi = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right)$.



(A) $\gamma(0)$.



(B) $\gamma(1)$.

FIGURE A.2. A curve $\gamma\colon [0,1] \to \mathcal{H}(0,0)^{\mathrm{root}}$ that ends at $\mathcal{V}$. The real periods are deformed following the arrows, while the imaginary periods remain constant.

Starting from $s = 0$ and as s increases, the normalised base-arc shrinks until its length is exactly equal to $1 + \min\{x_1, x_3\}$ at $s = s_1 > 0$. Thus, $\gamma(s)$ admits a zippered rectangles construction for every $0 \leqslant s < s_1$, but $\gamma(s_1)$ does not. Indeed, $\gamma(s_1)$ belongs to a flow face. As γ passes through this flow face, a forward Rauzy move must be performed to again obtain a normalised zippered rectangles construction. The winning letter is 3, so the resulting permutation after the Rauzy move is again π.

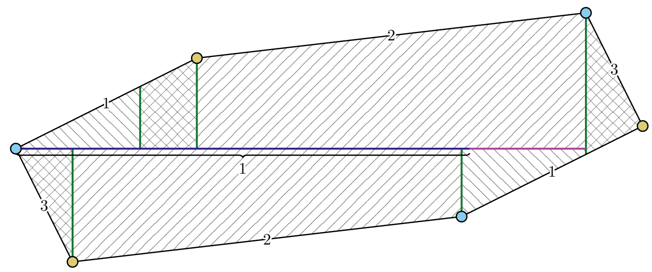
This process continues inductively. Indeed, starting from $s = s_k$, for any integer $k \geqslant 1$, and as s increases, the normalised base-arc continues to shrink until the curve hits the flow face again at $s = s_{k+1} > s_k$. Thus, $\gamma(s)$ admits a normalised zippered rectangles construction for every $s_k < s < s_{k+1}$. A Rauzy move must be performed when the curve crosses the flow face; the winning letter continues to be 3. Hence, the resulting permutation is again π.

In summary, there exists a countable collection $0 < s_1 < \cdots < s_k < s_{k+1} < \cdots$ such that:

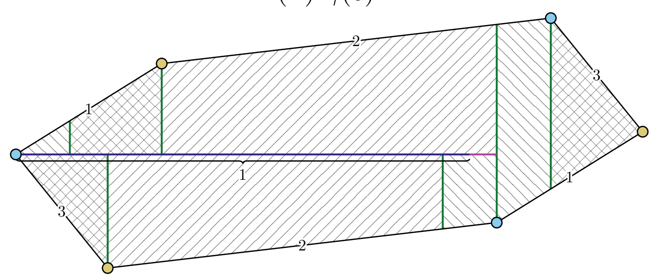
(1) $\gamma(s)$ admits a normalised zippered rectangles construction for $0 \leqslant s < s_1$ and every $s_k < s < s_{k+1}$ for $k \geqslant 1$;
(2) $\gamma(s_k)$ belongs to a flow face for every $k \geqslant 1$.

Thus, γ intersects $\mathcal{S}$ infinitely many times and there is no finite Rauzy–Veech sequence shadowing γ. Moreover, as this process unfolds, the width of the rectangle R_1 goes to zero, while its height diverges grows indefinitely. Hence, the normalised zippered rectangles constructions along γ become more and more degenerate and do not converge to a well-defined element of P_π. See Figure A.3 for an illustration of this phenomenon.

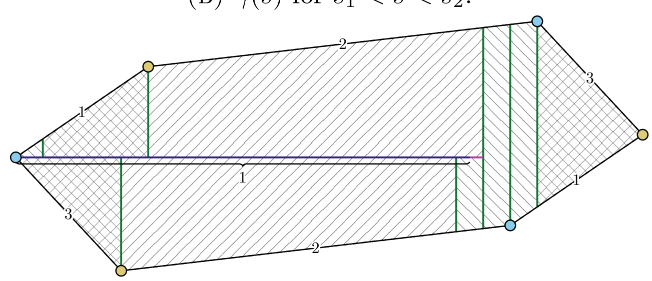
Similar examples exist also for *toppling faces*, that is, when either some width x_α or some zipper height goes to zero. It is possible to find a compact arc in $\mathcal{C}^{\mathrm{root}}$ transverse to $\mathcal{V}$ that intersects infinitely many toppling faces. Indeed, a simple way



(A) $\gamma(0)$.



(B) $\gamma(s)$ for $s_1 < s < s_2$.



(C) $\gamma(s)$ for $s_2 < s < s_3$.

FIGURE A.3. Normalised zippered rectangles construction of $\gamma(s)$.

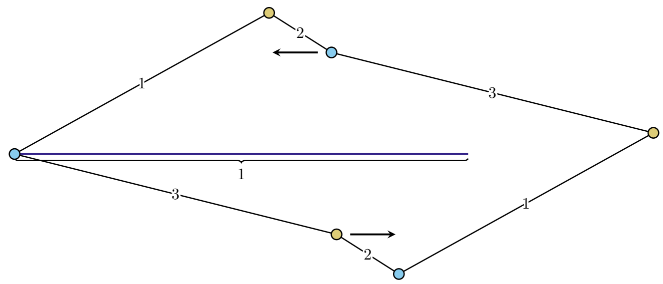
to obtain such an example is to consider a horizontal slit J that does not meet the normalised base-arc I, and such that some vertical segments emanating from I meet J before their first return. Then, the slit can be rotated until it becomes vertical (in a way that it still does not meet the base-arc). This forces the widths of some rectangles to hit zero infinitely many times before the slit becomes vertical.

A.4. **Crossing a toppling face.** In this section, we present concretely the construction used in the based loop theorem, namely Theorem 4.23, in which any loop in $\pi_1(\mathcal{C}^{\mathrm{root}}, q_0)$ is written as a finite concatenation of paths that are forward (or backward) Teichmüller segments or are contained inside a polytope. The path in $\mathcal{C}^{\mathrm{root}}$ that we present is not closed, but it still illustrates the key point. For more complicate paths or loops, this procedure has to be done several times.

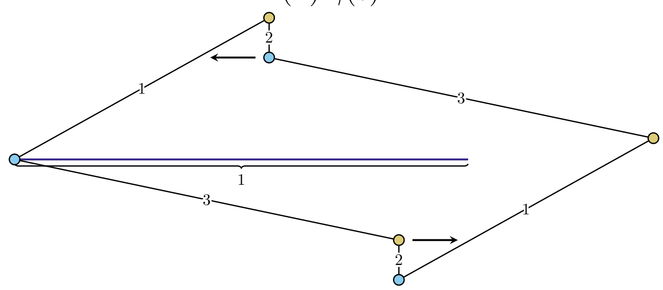
Let $\mathcal{C} = \mathcal{H}(0,0)$. Consider the path $\gamma\colon [0,1] \to \mathcal{C}^{\mathrm{root}}$ illustrated in Figure A.5. Assume that $\gamma(0)$ and $\gamma(1)$ are doubly non-vanishing, while $\gamma(1/2)$ has a vertical saddle connection and, thus, belongs to $\mathcal{V}$.

Since $\gamma(0)$ is doubly non-vanishing, it admits a normalised base-arc. The resulting zippered rectangles construction, with underlying permutation $\pi = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right)$, is shown in Figure A.6.

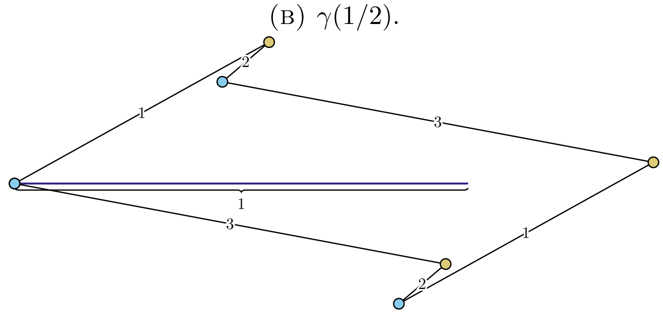
If $0 \leqslant s < 1/2$, a parameter $(x_s, y_s) \in P_\pi$ of this zippered rectangles construction satisfies $q_\pi(x_s, y_s) = \gamma(s)$. As s increases towards $1/2$, these parameters



(A) $\gamma(0)$.



(B) $\gamma(1/2)$.



(C) $\gamma(1)$.

Figure A.5. A curve $\gamma\colon [0,1] \to \mathcal{H}(0,0)^{\mathrm{root}}$ that passes through $\mathcal{V}$. The real periods are deformed following the arrows, while the imaginary periods remain constant.

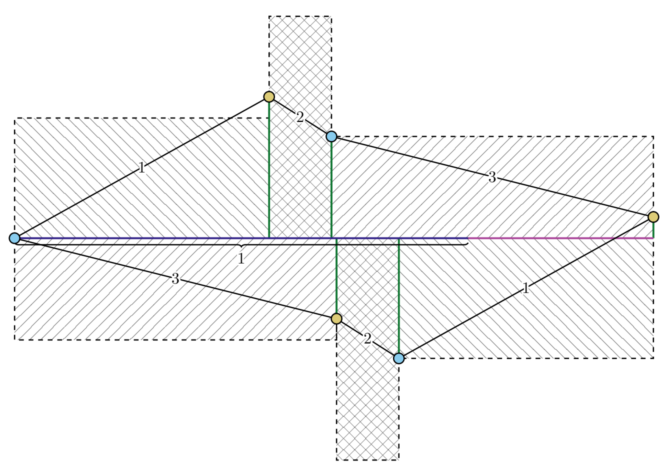
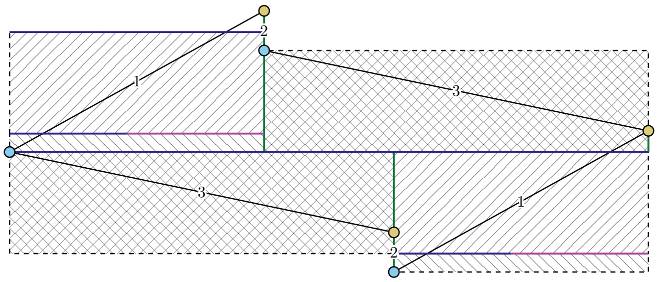
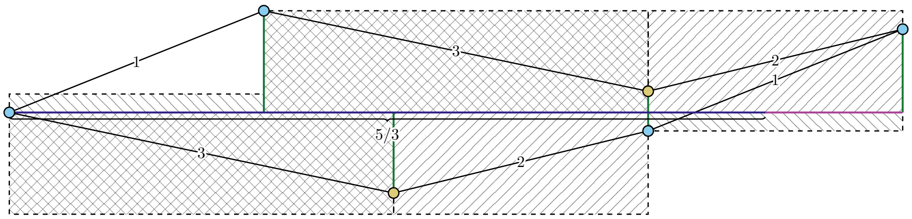


Figure A.6. Normalised zippered rectangles construction of $\gamma(0)$.

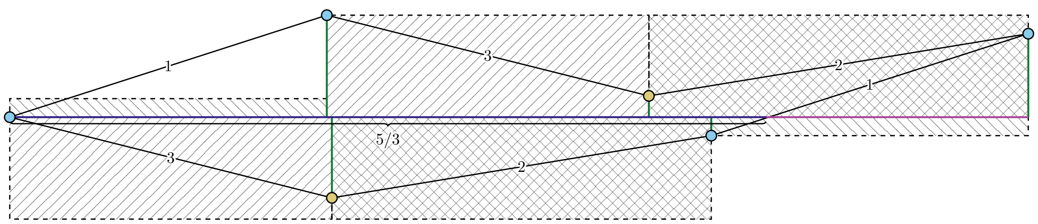


(A) Before cutting and pasting.

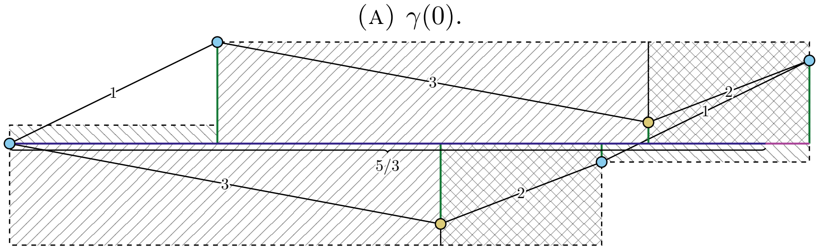


(B) After cutting and pasting by two backward Rauzy moves.

FIGURE A.7. Zippered rectangles construction of $\gamma(1/2)$ with a base-arc of length at least $5/3$.



(A) $\gamma(0)$.

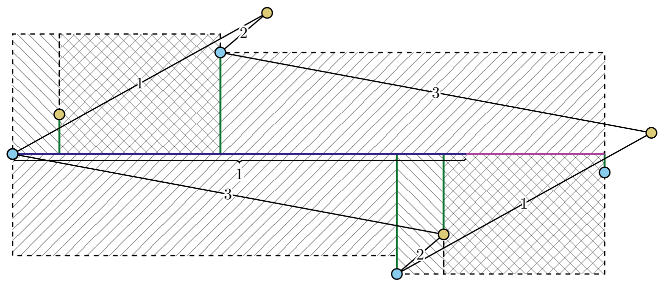


(B) $\gamma(1)$.

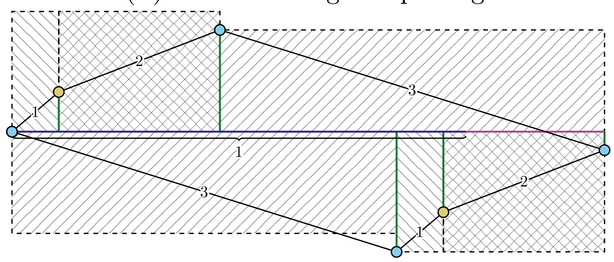
FIGURE A.8. Zippered rectangles constructions with a base-arc of length at least $5/3$.

approach the boundary of P_π and $\gamma(1/2) \notin \mathcal{C}_\pi$. In particular, as s increases to $1/2$, the width x_2 tends to zero while all other parameters stay bounded away from zero and infinity, and thus $\gamma(1/2)$ can be said to lie on a toppling face.

On the other hand, $\gamma(1/2)$ is not doubly vanishing. It thus admits a base-arc. We take a base-arc of length at least $5/3$, since the interior of any horizontal segment with length at least $5/3$ meets every leaf of the vertical foliation. The resulting (unnormalised) zippered rectangles construction, with underlying permutation $\sigma = \left(\begin{smallmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{smallmatrix}\right)$, is shown in Figure A.7. As Teichmüller flow by $T = -\log(5/3)$ normalises the base-arc, we have that $g_T(\gamma(1/2)) \in \mathcal{C}_\sigma$. Observe that σ is obtained from π by two backward Rauzy moves.



(A) Before cutting and pasting.



(B) After cutting and pasting by two backward Rauzy moves followed by two forward Rauzy moves.

FIGURE A.9. Normalised zippered rectangles construction of $\gamma(1)$.

For “large” deformations of parameters, the zippered rectangles construction with a base-arc of length at least $5/3$ is contained in $\mathcal{C}_\sigma$ after Teichmüller flow by $T = -\log(5/3)$. In particular, we have that $g_T(\gamma(s)) \in \mathcal{C}_\sigma$ for every $s \in [0,1]$. Let $(x'_s, y'_s) \in P_\sigma$ such that $q_\sigma(x'_s, y'_s) = g_T(\gamma(s))$. Figure A.8 shows these zippered rectangles constructions for $\gamma(0)$ and $\gamma(1)$.

Finally, $\gamma(1)$ is also doubly non-vanishing. Thus it admits a normalised base-arc. The resulting zippered rectangles construction, with underlying permutation $\tau = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right)$, is shown in Figure A.9. Observe that τ is obtained from π by two backward Rauzy moves followed by two forward Rauzy moves.

If $1/2 < s \leqslant 1$, a parameter (x_s, y_s) of this zippered rectangles construction satisfies $q_\tau(x_s, y_s) = \gamma(s)$. As s decreases towards $1/2$, these parameters approach the boundary of P_τ and $\gamma(1/2) \notin \mathcal{C}_\tau$.

Putting everything together, we obtain three open sets $U_0, U_{1/2}, U_1 \subseteq \mathcal{C}^{\mathrm{root}}$ satisfying:

- $U_0 = q_\pi(W_0)$, where $W_0 \subseteq P_\pi$ is an open set containing (x_0, y_0) whose closure is contained in P_π;
- $U_{1/2} = g_{-T}(q_\sigma(W_{1/2}))$, where $W_{1/2} \subseteq P_\sigma$ is an open set containing (x'_s, y'_s) for every $s \in [0,1]$ whose closure is contained in P_σ; and
- $U_1 = q_\tau(W_1)$, where $W_1 \subseteq P_\tau$ is an open set containing (x_1, y_1) whose closure is contained in P_τ.

Then, γ is homotopic, relative to its endpoints, to the concatenation of the paths:

- $g_t\gamma(0)$ for $t \in [0,T]$;
- $g_T\gamma(s)$ for $s \in [0,1]$; and
- $g_{-t}\gamma(1)$ for $t \in [0,T]$.

Therefore, the combinatorial description of this concatenation is

$$\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix} \xrightarrow{\mathrm{b}^{-1}} \begin{pmatrix}1 & 3 & 2\\ 3 & 2 & 1\end{pmatrix} \xrightarrow{\mathrm{t}^{-1}} \begin{pmatrix}1 & 3 & 2\\ 3 & 2 & 1\end{pmatrix} \xrightarrow{\mathrm{b}} \begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix} \xrightarrow{\mathrm{t}} \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$$

which is the (undirected) Rauzy–Veech sequence shadowing γ.

Appendix B. Zariski density of the remaining cases

In this section, we explicitly check the Zariski density for the plus piece of the four remaining components, namely $\mathcal{Q}(5,-1)$, $\mathcal{Q}(9,-1)^{\mathrm{irr}}$, $\mathcal{Q}(12)^{\mathrm{reg}}$ and $\mathcal{Q}(12)^{\mathrm{irr}}$. We do this by using the following sufficient criterion.

Criterion B.1 ([PR14, Theorem 9.10])**.** *Let G be a subgroup of $\mathrm{Sp}(2g,\mathbb{Z})$. We have that G is Zariski dense in $\mathrm{Sp}(2g,\mathbb{R})$ provided the Zariski closure of G is not a power of $\mathrm{SL}(2,\mathbb{R})$, and there exist elements $A, B \in G$ satisfying:*

1. *A is Galois-pinching in the sense of Matheus–Möller–Yoccoz [MMY15]. That is, all of its eigenvalues are real and have distinct moduli, and the Galois group of its characteristic polynomial is maximal; and*
2. *B has infinite order and does not commute with A.*

Since A is symplectic, its characteristic polynomial P is reciprocal. Thus, the Galois group of P is contained inside an appropriate hyperoctahedral group. Hence, this group is maximal if and only if it has order $2^g g!$. Moreover, if a monodromy group is a power of $\mathrm{SL}(2,\mathbb{R})$, then it has more than one compact factor, which is forbidden for strongly irreducible pieces [Fil17, Theorem 1.2; EFW18, Theorem 1.1]. Thus, if we can establish Criterion B.1 together with Lemma 9.8, we obtain the Zariski density of G inside $\mathrm{Sp}(2,\mathbb{R})$.

For all of the remaining components, we will follow the same strategy to show that the hypotheses of Criterion B.1 hold. We will start with a specific permutation π. We will then exhibit two cycles δ_1 and δ_2 based at π in the reduced Rauzy diagram such that their induced matrices A_1 and A_2 in a preferred basis (for the action on absolute homology) can be combined to produce the matrices A and B that the criterion requires. Specifically, in all cases, B can be taken to be A_1 and A can be taken to be A_1A_2.

These cycles δ_1 and δ_2 were found by a randomised computer search on the reduced Rauzy diagrams. We tried to choose cycles that are relatively short to avoid the resulting matrices having entries that are greater than 100.

B.2. Zariski density of $\mathcal{Q}(5,-1)$.

Let

$$\pi = \begin{pmatrix}1 & 2 & 3 & 2 & 4\\ 4 & 5 & 5 & 3 & 1\end{pmatrix}$$

and

$$\delta_1 = \mathrm{b}^3\mathrm{t}^2\mathrm{b}^3\mathrm{t}\mathrm{b}^2\mathrm{t}\mathrm{b}^3\mathrm{t}^2\mathrm{b}^3$$
$$\delta_2 = \mathrm{t}^2\mathrm{btbtbt}^3\mathrm{btbtb}^2\mathrm{t}^2$$

Consider the four curves $c_1, \ldots, c_4$ depicted in Figure B.3 as solid or dashed lines. These cycles form a basis for the absolute homology as their intersection matrix is

$$\Omega = \begin{pmatrix}0 & 0 & -1 & -1\\ 0 & 0 & 1 & 0\\ 1 & -1 & 0 & -1\\ 1 & 0 & 1 & 0\end{pmatrix}$$

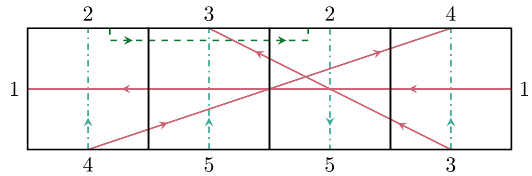


FIGURE B.3. Representative of $\mathcal{Q}(5,-1)$.

that has determinant 1. On the other hand, the cycle $v \in H_1(S;\mathbb{R})$ depicted in Figure B.3 as dash-dotted vertical lines can be written as $v = -c_2 + c_3 + c_4$. Thus, the set $B = \{c_1, c_3, c_4, v\}$ readily satisfies the hypotheses of Lemma 9.8, so the M-action is strongly irreducible.

In the chosen basis, the matrices induced by δ_1^2 and δ_2^2 are

$$A_1 = \begin{pmatrix} 1 & -2 & -2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 2 & -1 \\ 1 & 2 & -1 & 2 \\ -2 & 1 & -2 & 0 \end{pmatrix}$$

Then, $A = A_1 A_2$ has the form

$$A = \begin{pmatrix} -1 & 0 & 1 & -2 \\ 2 & 2 & -4 & 3 \\ 2 & 6 & -7 & 0 \\ 0 & 5 & -4 & -4 \end{pmatrix}$$

The characteristic polynomial P of A is $P(t) = t^4 + 10t^3 + 22t^2 + 10t + 1$. We verified in Magma [BCP97] that A is Galois pinching, that is, it satisfies condition (1) of Criterion B.1. Setting $B = A_1$, we similarly check that B satisfies condition (2) of the criterion. Thus, the plus piece of $\mathcal{Q}(5,-1)$ is Zariski dense inside $\mathrm{Sp}(4,\mathbb{R})$.

B.4. **Zariski density of $\mathcal{Q}(9,-1)^{\mathrm{irr}}$.** Let

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 3 \\ 7 & 7 & 6 & 5 & 4 & 2 & 1 \end{pmatrix}$$

and

$$\delta_1 = \mathrm{b}^4\mathrm{t}^5\mathrm{b}^3\mathrm{t}\mathrm{b}^5\mathrm{t}\mathrm{b}^6\mathrm{t}^2$$
$$\delta_2 = \mathrm{b}^4\mathrm{t}^2\mathrm{b}^3\mathrm{t}^3\mathrm{b}^7\mathrm{t}^3\mathrm{b}\mathrm{t}^3\mathrm{b}\mathrm{t}^2\mathrm{b}^2\mathrm{t}^3\mathrm{b}^2\mathrm{t}^2\mathrm{b}^2\mathrm{t}^2\mathrm{b}^2$$

Consider the six curves $c_1, \ldots, c_6$ depicted in Figure B.5 as solid or dashed lines. These cycles form a basis for the absolute homology, as their intersection matrix is

$$\Omega = \begin{pmatrix} 0 & -1 & 0 & -1 & -1 & -1 \\ 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 & -1 & -1 \\ 1 & 1 & -1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 1 & 1 & 0 \end{pmatrix}$$

that has determinant 1. On the other hand, the cycle $v \in H_1(S;\mathbb{R})$ depicted in Figure B.5 as dash-dotted vertical lines can be written as $v = c_2 + c_3$. Thus, the set $B = \{c_1, c_2, c_4, c_5, c_6, v\}$ readily satisfies the hypotheses of Lemma 9.8, so the M-action is strongly irreducible.

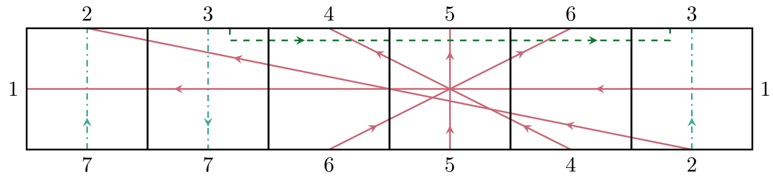


FIGURE B.5. Representative of $\mathcal{Q}(9,-1)^{\mathrm{irr}}$.

In the chosen basis, the matrices induced by δ_1^2 and δ_2^2 are

$$A_1 = \begin{pmatrix} 0 & 2 & -1 & -1 & -1 & -2 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -2 & -2 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 3 & -1 & 2 & 2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -3 & -10 & -2 & -4 & -6 & -4 \\ 1 & 3 & 0 & 2 & 2 & 0 \\ 2 & 3 & -2 & 0 & -1 & 0 \\ 0 & -2 & -2 & -1 & -2 & -2 \\ -3 & -7 & 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Then, $A = A_1 A_2$ has the form

$$A = \begin{pmatrix} -2 & 0 & 2 & 0 & -1 & -1 \\ -6 & -1 & 3 & -2 & 0 & -3 \\ 7 & -1 & -2 & 2 & 3 & 1 \\ 3 & -3 & 2 & 1 & 3 & -2 \\ 5 & -3 & 3 & 2 & 3 & -2 \\ 8 & -2 & -2 & 2 & 5 & 0 \end{pmatrix}$$

The characteristic polynomial P of A is $P(t) = t^6 + t^5 - 22t^4 - 52t^3 - 22t^2 + t + 1$. Again, we use Magma to check that A is Galois pinching. Setting $B = A_1$, we can readily check that B satisfies condition (2) of the criterion. Thus, the plus piece of $\mathcal{Q}(9,-1)^{\mathrm{irr}}$ is Zariski dense inside $\mathrm{Sp}(6,\mathbb{R})$.

B.6. **Zariski density of $\mathcal{Q}(12)^{\mathrm{reg}}$.** Let

$$\pi = \begin{pmatrix} 1 & 2 & 1 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 3 & 6 & 5 & 8 & 7 & 8 \end{pmatrix}$$

and

$$\delta_1 = \mathrm{b}^4\mathrm{t}^2\mathrm{b}\mathrm{t}^6\mathrm{b}\mathrm{t}^4\mathrm{b}^2\mathrm{t}\mathrm{b}^4\mathrm{t}^2\mathrm{b}^7\mathrm{t}^2\mathrm{b}^2\mathrm{t}\mathrm{b}^5\mathrm{t}^4\mathrm{btbtb}$$
$$\delta_2 = \mathrm{btbt}^3\mathrm{bt}^6\mathrm{b}^2\mathrm{t}^4\mathrm{bt}^3\mathrm{b}^2\mathrm{t}^3\mathrm{b}^2\mathrm{tb}^4\mathrm{t}^2\mathrm{bt}^3\mathrm{b}^2\mathrm{t}^2\mathrm{b}^3\mathrm{tb}^5$$

Let M be the monodromy group of $\mathcal{Q}(12)^{\mathrm{reg}}$. We have that the M-action is strongly irreducible by Lemma 9.12.

Consider the six curves $c_1, \ldots, c_6$ shown in Figure B.7. Ordered appropriately, these curves form a symplectic basis. In this basis, the matrices induced by δ_1^2 and δ_2^2 are

$$A_1 = \begin{pmatrix} 2 & 5 & -1 & 7 & 6 & 2 & 1 & 0 \\ -1 & 0 & 1 & -1 & -2 & 1 & 3 & 0 \\ -1 & 2 & -1 & 2 & 1 & 1 & -2 & 0 \\ -1 & -4 & 1 & -6 & -5 & -2 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 2 & 5 & 0 \\ -1 & -3 & 1 & -4 & -4 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -5 & 7 & -11 & -10 & 2 & 11 & -1 \end{pmatrix}$$

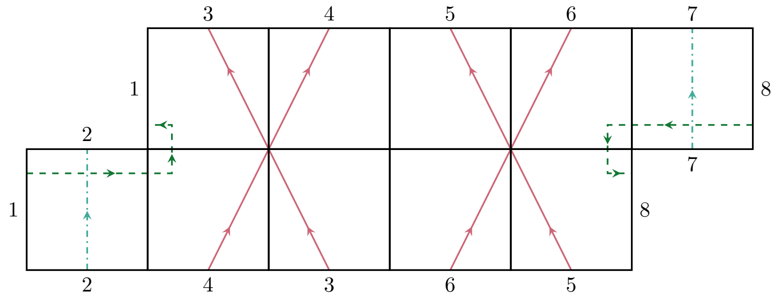


FIGURE B.7. Representative of $\mathcal{Q}(12)^{\mathrm{reg}}$.

$$A_2 = \begin{pmatrix} 1 & 3 & -5 & 1 & 6 & -7 & -3 & -1 \\ 0 & -2 & 2 & 0 & -3 & 3 & 1 & 1 \\ 4 & -1 & -1 & 2 & 3 & -4 & 1 & 0 \\ -7 & 5 & -3 & -2 & -1 & 3 & -5 & -1 \\ 5 & -5 & 3 & 2 & -2 & 1 & 5 & 2 \\ 2 & 0 & -1 & 0 & 2 & -3 & 0 & -1 \\ 8 & -1 & -4 & 3 & 7 & -10 & 0 & -1 \\ 2 & 3 & -5 & 1 & 6 & -7 & -3 & -2 \end{pmatrix}$$

Then, $A = A_1 A_2$ has the form

$$A = \begin{pmatrix} 17 & -7 & 15 & -17 & 5 & 11 & 36 & 20 \\ 23 & -13 & 25 & -38 & 8 & 24 & 62 & 33 \\ 6 & 0 & 0 & 7 & 0 & -7 & -5 & -2 \\ 33 & -20 & 34 & -50 & 6 & 37 & 89 & 51 \\ 28 & -16 & 31 & -47 & 9 & 33 & 81 & 44 \\ 15 & -7 & 12 & -15 & 3 & 8 & 27 & 15 \\ 21 & -6 & 5 & 12 & -6 & -6 & 7 & 11 \\ 1 & -1 & 0 & 1 & -2 & 1 & 1 & 2 \end{pmatrix}$$

The characteristic polynomial P of A is $P(t) = t^8 + 20t^7 - 1686t^6 - 24t^5 + 36258t^4 - 24t^3 - 1686t^2 + 20t + 1$. Using Magma, we can check that A is Galois pinching. Setting $B = A_1$, we can also similarly check that B satisfies condition (2) of the criterion. Thus, the plus piece of $\mathcal{Q}(12)^{\mathrm{reg}}$ is Zariski dense inside $\mathrm{Sp}(8, \mathbb{R})$.

B.8. **Zariski density of $\mathcal{Q}(12)^{\mathrm{irr}}$.** Let

$$\pi = \begin{pmatrix} 1 & 2 & 1 & 3 & 4 & 5 & 6 & 7 \\ 2 & 6 & 5 & 4 & 3 & 8 & 7 & 8 \end{pmatrix}$$

and

$$\delta_1 = \mathrm{b}^3\mathrm{t}\mathrm{b}^2\mathrm{t}^5\mathrm{b}\mathrm{t}^5\mathrm{b}^2\mathrm{t}^3\mathrm{b}\mathrm{t}^2\mathrm{b}^2\mathrm{t}^6\mathrm{b}\mathrm{t}^2\mathrm{b}^2\mathrm{t}\mathrm{b}\mathrm{t}^3\mathrm{b}^2\mathrm{t}^6\mathrm{b}^4$$
$$\delta_2 = \mathrm{b}^5\mathrm{t}^5\mathrm{b}\mathrm{t}\mathrm{b}\mathrm{t}^4\mathrm{b}^3\mathrm{t}^7\mathrm{b}^2\mathrm{t}\mathrm{b}\mathrm{t}^4\mathrm{b}^4$$

Consider the six curves $c_1, \ldots, c_6$ depicted in Figure B.9 as solid, dashed or dash-dotted lines. These cycles form a basis for the absolute homology as their

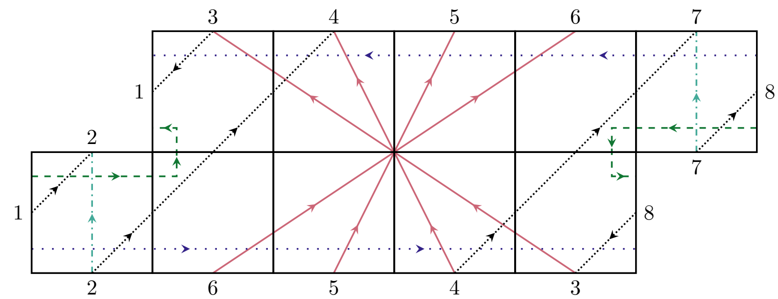


FIGURE B.9. Representative of $\mathcal{Q}(12)^{\mathrm{irr}}$.

intersection matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

and has determinant 1. On the other hand, the cycle b, depicted as the slope-1 densely dotted lines, can be written as $b = c_1 + c_2 - c_3 + c_4 + c_7 - c_8$, and the cycle p depicted as loosely dotted horizontal lines can be written as $p = c_1 + c_8$. Thus, the set $B = \{c_2, c_3, c_4, c_5, c_6, c_7, b, p\}$ readily satisfies the hypotheses of Lemma 9.8, so the M-action is strongly irreducible.

In the chosen basis, the matrices induced by δ_1^2 and δ_2^2 are

$$A_1 = \begin{pmatrix} -1 & 6 & -2 & 6 & 11 & 8 & 6 & -4 \\ 2 & -5 & 0 & -6 & -9 & -6 & -4 & 6 \\ -1 & -3 & -2 & -6 & -8 & -5 & -3 & 4 \\ -1 & -5 & -1 & -9 & -11 & -7 & -5 & 6 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 6 & 8 & 4 & 3 & -4 \\ -2 & -4 & 2 & -4 & -8 & -6 & -5 & 0 \\ 0 & -6 & 2 & -6 & -11 & -8 & -6 & 3 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -2 & 1 & 1 & 1 & -1 & -2 & -3 & 0 \\ 3 & -5 & 3 & 3 & 3 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & -2 & -1 & 2 & 2 & 0 \\ -3 & 4 & -3 & -3 & -3 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -7 & 13 & -9 & -9 & -7 & 2 & 5 & -1 \end{pmatrix}$$

Then, $A = A_1 A_2$ has the form

$$A = \begin{pmatrix} 6 & -15 & 1 & 1 & 6 & 12 & 2 & 43 \\ -19 & 27 & 3 & 5 & 4 & -25 & 4 & -43 \\ -7 & -19 & 2 & 1 & 12 & 18 & -2 & 51 \\ -33 & 11 & 6 & 9 & 22 & -11 & 4 & 13 \\ -41 & 34 & 8 & 11 & 21 & -33 & 8 & -29 \\ -24 & 30 & 5 & 7 & 8 & -28 & 6 & -40 \\ -15 & 24 & 3 & 5 & 4 & -23 & 5 & -35 \\ 32 & -12 & -4 & -6 & -16 & 10 & 0 & 5 \end{pmatrix}$$

The characteristic polynomial P of A is $P(t) = t^8 - 47t^7 - 794t^6 + 11691t^5 - 22022t^4 + 11691t^3 - 794t^2 - 47t + 1$. By using Magma again, we can explicitly check that A is Galois pinching. Setting $B = A_1$, we can similarly check that B satisfies condition (2) of the criterion. Thus, the plus piece of $\mathcal{Q}(12)^{\mathrm{irr}}$ is Zariski dense inside $\mathrm{Sp}(8, \mathbb{R})$.

References

[Aar97] J. Aaronson. "An introduction to infinite ergodic theory". Vol. 50. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997, pp. xii+284. ISBN: 0-8218-0494-4. DOI: 10.1090/surv/050.

[AGY06] A. Avila, S. Gouëzel, and J.-C. Yoccoz. "Exponential mixing for the Teichmüller flow". In: *Publ. Math. Inst. Hautes Études Sci.* 104 (2006), pp. 143–211. DOI: 10.1007/s10240-006-0001-5.

[AMY18] A. Avila, C. Matheus, and J.-C. Yoccoz. "Zorich conjecture for hyperelliptic Rauzy-Veech groups". In: *Math. Ann.* 370.1-2 (2018), pp. 785–809. DOI: 10.1007/s00208-017-1568-5.

[AR12] A. Avila and M. J. Resende. "Exponential mixing for the Teichmüller flow in the space of quadratic differentials". In: *Comment. Math. Helv.* 87.3 (2012), pp. 589–638. DOI: 10.4171/CMH/263.

[AV07a] A. Avila and M. Viana. "Simplicity of Lyapunov spectra: a sufficient criterion". In: *Port. Math. (N.S.)* 64.3 (2007), pp. 311–376. DOI: 10.4171/PM/1789.

[AV07b] A. Avila and M. Viana. "Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture". In: *Acta Math.* 198.1 (2007), pp. 1–56. DOI: 10.1007/s11511-007-0012-1.

[BCP97] W. Bosma, J. Cannon, and C. Playoust. "The Magma algebra system. I. The user language". In: *J. Symbolic Comput.* 24.3-4 (1997). Computational algebra and number theory (London, 1993), pp. 235–265. DOI: 10.1006/jsco.1996.0125.

[Bel+19] M. Bell, V. Delecroix, V. Gadre, R. Gutiérrez-Romo, and S. Schleimer. "Coding Teichmüller flow using veering triangulations". Preprint (2019). arXiv: 1909.00890 [math.DS].

[Ben97] Y. Benoist. "Propriétés asymptotiques des groupes linéaires". In: *Geom. Funct. Anal.* 7.1 (1997), pp. 1–47. DOI: 10.1007/PL00001613.

[BL09] C. Boissy and E. Lanneau. "Dynamics and geometry of the Rauzy-Veech induction for quadratic differentials". In: *Ergodic Theory Dynam. Systems* 29.3 (2009), pp. 767–816. DOI: 10.1017/S0143385708080565.

[BMP03] M. Boileau, S. Maillot, and J. Porti. "Three-dimensional orbifolds and their geometric structures". Vol. 15. Panoramas et Synthèses [Panoramas and Syntheses]. Société Mathématique de France, Paris, 2003, pp. viii+167. ISBN: 2-85629-152-X.

[Boi20] C. Boissy. "Moduli space of meromorphic differentials with marked horizontal separatrices". In: *Algebr. Geom. Topol.* 20.5 (2020), pp. 2373–2412. DOI: 10.2140/agt.2020.20.2373.

[Cal20] A. Calderon. “Connected components of strata of Abelian differentials over Teichmüller space”. In: *Comment. Math. Helv.* 95.2 (2020), pp. 361–420. DOI: 10.4171/CMH/491.

[CM14] D. Chen and M. Möller. “Quadratic differentials in low genus: exceptional and non-varying strata”. In: *Ann. Sci. Éc. Norm. Supér. (4)* 47.2 (2014), pp. 309–369. DOI: 10.24033/asens.2216.

[CS19a] A. Calderon and N. Salter. “Higher spin mapping class groups and strata of Abelian differentials over Teichmüller space”. Preprint (2019). arXiv: 1906.03515 [math.GT].

[CS19b] A. Calderon and N. Salter. “Relative homological representations of framed mapping class groups”. In: *Bull. Lond. Math. Soc.* (2019). DOI: 10.1112/blms.12412.

[CS20] A. Calderon and N. Salter. “Framed mapping class groups and the monodromy of strata of Abelian differentials”. Preprint (2020). arXiv: 2002.02472 [math.GT].

[DN88] C. Danthony and A. Nogueira. “Involutions linéaires et feuilletages mesurés”. In: *C. R. Acad. Sci. Paris Sér. I Math.* 307.8 (1988), pp. 409–412.

[DN90] C. Danthony and A. Nogueira. “Measured foliations on nonorientable surfaces”. In: *Ann. Sci. École Norm. Sup. (4)* 23.3 (1990), pp. 469–494.

[EFW18] A. Eskin, S. Filip, and A. Wright. “The algebraic hull of the Kontsevich-Zorich cocycle”. In: *Ann. of Math. (2)* 188.1 (2018), pp. 281–313. DOI: 10.4007/annals.2018.188.1.5.

[EKZ14] A. Eskin, M. Kontsevich, and A. Zorich. “Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow”. In: *Publ. Math. Inst. Hautes Études Sci.* 120 (2014), pp. 207–333. DOI: 10.1007/s10240-013-0060-3.

[EMM15] A. Eskin, M. Mirzakhani, and A. Mohammadi. “Isolation, equidistribution, and orbit closures for the $\mathrm{SL}(2, \mathbb{R})$ action on moduli space”. In: *Ann. of Math. (2)* 182.2 (2015), pp. 673–721. DOI: 10.4007/annals.2015.182.2.7.

[Fil16] S. Filip. “Splitting mixed Hodge structures over affine invariant manifolds”. In: *Ann. of Math. (2)* 183.2 (2016), pp. 681–713. DOI: 10.4007/annals.2016.183.2.5.

[Fil17] S. Filip. “Zero Lyapunov exponents and monodromy of the Kontsevich-Zorich cocycle”. In: *Duke Math. J.* 166.4 (2017), pp. 657–706. DOI: 10.1215/00127094-3715806.

[Gut17] R. Gutiérrez-Romo. “Simplicity of the Lyapunov spectra of certain quadratic differentials”. Preprint (2017). arXiv: 1711.02006 [math.DS].

[Gut19] R. Gutiérrez-Romo. “Classification of Rauzy-Veech groups: proof of the Zorich conjecture”. In: *Invent. Math.* 215.3 (2019), pp. 741–778. DOI: 10.1007/s00222-018-0836-7.

[Ham18] U. Hamenstädt. “Typical properties of periodic Teichmüller geodesics: stretch factors”. Preprint (2018). URL: http://www.math.uni-bonn.de/people/ursula/tracefield.pdf.

[Hir94] M. W. Hirsch. “Differential topology”. Vol. 33. Graduate Texts in Mathematics. Corrected reprint of the 1976 original. Springer-Verlag, New York, 1994, pp. x+222. ISBN: 0-387-90148-5.

[Joh80] D. Johnson. “Spin structures and quadratic forms on surfaces”. In: *J. London Math. Soc. (2)* 22.2 (1980), pp. 365–373. DOI: 10.1112/jlms/s2-22.2.365.

[KMS86] S. Kerckhoff, H. Masur, and J. Smillie. “Ergodicity of billiard flows and quadratic differentials”. In: *Ann. of Math. (2)* 124.2 (1986), pp. 293–311. DOI: 10.2307/1971280.

[Kon97] M. Kontsevich. “Lyapunov exponents and Hodge theory”. In: *The mathematical beauty of physics (Saclay, 1996)*. Vol. 24. Adv. Ser. Math. Phys. World Sci. Publ., River Edge, NJ, 1997, pp. 318–332.

[KZ03] M. Kontsevich and A. Zorich. "Connected components of the moduli spaces of Abelian differentials with prescribed singularities". In: *Invent. Math.* 153.3 (2003), pp. 631–678. DOI: 10.1007/s00222-003-0303-x.

[KZ97] M. Kontsevich and A. Zorich. "Lyapunov exponents and Hodge theory". Preprint (1997). arXiv: hep-th/9701164 [hep-th].

[Lan04] E. Lanneau. "Hyperelliptic components of the moduli spaces of quadratic differentials with prescribed singularities". In: *Comment. Math. Helv.* 79.3 (2004), pp. 471–501. DOI: 10.1007/s00014-004-0806-0.

[Lan08] E. Lanneau. "Connected components of the strata of the moduli spaces of quadratic differentials". In: *Ann. Sci. Éc. Norm. Supér. (4)* 41.1 (2008), pp. 1–56. DOI: 10.24033/asens.2062.

[Mas82] H. Masur. "Interval exchange transformations and measured foliations". In: *Ann. of Math. (2)* 115.1 (1982), pp. 169–200. DOI: 10.2307/1971341.

[Mat21] C. Matheus. Personal communication (2021).

[MMY15] C. Matheus, M. Möller, and J.-C. Yoccoz. "A criterion for the simplicity of the Lyapunov spectrum of square-tiled surfaces". In: *Invent. Math.* 202.1 (2015), pp. 333–425. DOI: 10.1007/s00222-014-0565-5.

[PR14] G. Prasad and A. S. Rapinchuk. "Generic elements in Zariski-dense subgroups and isospectral locally symmetric spaces". In: *Thin groups and superstrong approximation.* Vol. 61. Math. Sci. Res. Inst. Publ. Cambridge Univ. Press, Cambridge, 2014, pp. 211–252.

[Rau79] G. Rauzy. "Échanges d'intervalles et transformations induites". In: *Acta Arith.* 34.4 (1979), pp. 315–328. DOI: 10.4064/aa-34-4-315-328.

[Ste+20] W. Stein et al. "Sage Mathematics Software (Version 9.1)". http://www.sagemath.org. The Sage Development Team. 2020.

[Tre13] R. Treviño. "On the non-uniform hyperbolicity of the Kontsevich-Zorich cocycle for quadratic differentials". In: *Geom. Dedicata* 163 (2013), pp. 311–338. DOI: 10.1007/s10711-012-9751-z.

[Vee82] W. A. Veech. "Gauss measures for transformations on the space of interval exchange maps". In: *Ann. of Math. (2)* 115.1 (1982), pp. 201–242. DOI: 10.2307/1971391.

[Vee86] W. A. Veech. "The Teichmüller geodesic flow". In: *Ann. of Math. (2)* 124.3 (1986), pp. 441–530. DOI: 10.2307/2007091.

[Yoc10] J.-C. Yoccoz. "Interval exchange maps and translation surfaces". In: *Homogeneous flows, moduli spaces and arithmetic.* Vol. 10. Clay Math. Proc. Amer. Math. Soc., Providence, RI, 2010, pp. 1–69.

[Zor08] A. Zorich. "Explicit Jenkins-Strebel representatives of all strata of abelian and quadratic differentials". In: *J. Mod. Dyn.* 2.1 (2008), pp. 139–185. DOI: 10.3934/jmd.2008.2.139.

[Zor18] A. Zorich. Personal communication (2018).

[Zor97] A. Zorich. "Deviation for interval exchange transformations". In: *Ergodic Theory Dynam. Systems* 17.6 (1997), pp. 1477–1499. DOI: 10.1017/S0143385797086215.

[Zor99] A. Zorich. "How do the leaves of a closed 1-form wind around a surface?" In: *Pseudoperiodic topology.* Vol. 197. Amer. Math. Soc. Transl. Ser. 2. Amer. Math. Soc., Providence, RI, 1999, pp. 135–178. DOI: 10.1090/trans2/197/05.

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