STATISTICAL HYPERBOLICITY FOR HARMONIC MEASURE

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ABSTRACT. We consider harmonic measures that arise from random walks on the mapping class group determined by a probability distribution that has finite first moment with respect to the Teichmüller metric, and whose support generates a non-elementary subgroup. We prove that Teichmüller space with the Teichmüller metric is statistically hyperbolic for such a harmonic measure.

1. INTRODUCTION

The notion of *statistical hyperbolicity*, introduced by Duchin-Lelièvre-Mooney [4], encapsulates whether a space is on average hyperbolic at large scales, that is, for any point in the space and spheres centred at that point whether as the radius $r \to \infty$ the average distance between pairs of points on the sphere of radius r is $2r$. To make sense of the average distance, one requires a reasonable measure on spheres.

For many Lebesgue-class measures on Teichmüller space, Dowdall-Duchin-Masur showed that Teichmüller space with the Teichmüller metric is statistically hyperbolic. See [3, Theorems B, C and D]. See also [12]. Here, we consider the same question for harmonic measures that arise from random walks on the mapping class group determined by a probability distribution with finite first moment with respect to the Teichmüller metric, and whose support generates a non-elementary subgroup.

Kaimanovich-Masur showed that a random walk on the mapping class group, whose initial support generates a non-elementary subgroup, converges to the Thurston boundary of Teichmüller space with probability one. This defines a harmonic measure on the Thurston boundary and Kaimanovich-Masur showed that this measure is supported on the set of uniquely ergodic measured foliations. See [10, Theorem 2.2.4] for both statements. Since Teichmüller rays with uniquely ergodic vertical foliations asymptotically converge to this vertical foliation, it is possible to pull back the harmonic measure to the unit cotangent space at a base-point. This allows us to equip spheres in Teichmüller space with a harmonic measure. We can then consider the question of whether Teichmüller space is statistically hyperbolic with respect to these measures.

Our main theorem is:

Theorem 1.1. *Let S be a surface of finite type. Let μ be a probability distribution on the mapping class group $\mathrm{Mod}(S)$ with finite first moment with respect to the Teichmüller metric, and such that the support generates a non-elementary subgroup. Then $\mathcal{T}(S)$ is statistically hyperbolic with respect to the harmonic measure defined by the μ-random walk on $\mathrm{Mod}(S)$.*

When S is a torus or a torus with one marked point or a sphere with four marked points, the Teichmüller space $\mathcal{T}(S)$ with the Teichmüller metric is isometric to $\mathbb{H}$. When μ has finite first moment in the word metric then by a theorem of Guivarch-LeJan [9], the harmonic measure from the μ-random walk is singular with respect to the Lebesgue measure class. See also [1], [2] and [7] for other proofs.

A similar singularity with respect to the class of Lebesgue measures also holds when the complex dimension is greater than one. See [6, Theorem 1.1] for singularity for harmonic measures from finitely supported random walks on $\mathrm{Mod}(S)$ and [8, Theorem 1.4] for singularity for harmonic measures for random walks with finite first moment with respect to a word metric on $\mathrm{Mod}(S)$. Finite first moment with respect to the word metric implies finite first moment with respect to the Teichmüller metric. For this large class of random walks Thereom 1.1 gives a conclusion that is distinct from the main results of Dowdall-Duchin-Masur [3] and, as we outline below, requires different tools.

On the other hand, finite first moment with respect to the Teichmüller metric does not imply the same in the word metric. For the exceptional surfaces above for which the Teichmüller space is $\mathbb{H}$, Furstenberg showed that there is a finite first $d_{\mathbb{H}}$-moment random walk whose harmonic measure on S^1 is absolutely continuous. Thus, for the exceptional surfaces Theorem 1.1 derives statistical hyperbolicity covering both singular and Lebesgue class measures in one statement. For non-exceptional moduli, the Furstenberg problem is still open. Conditional on its solution, Theorem 1.1 then covers singular and also the Lebesgue class measures analysed by Dowdall-Duchin-Masur in one statement.

We will first present the proof of Theorem 1.1 when the complex dimension of $\mathcal{T}(S)$ is greater than one. This is the harder case. For the exceptional surfaces, that is when $\mathcal{T}(S) = \mathbb{H}$, the proof of Theorem 1.1 is obviously easier because the ambient geometry is already hyperbolic. However, as mentioned above, the harmonic measure is singular. So there is something to prove. The argument is straightforward and uses the geodesic separation property for harmonic measure that is already formulated in the proof of the harder case of Theorem 1.1.

In fact, we present the exceptional case as a special case of a more general theorem when the ambient geometry is hyperbolic. We prove:

Theorem 1.2. *Let Γ be a non-uniform lattice in $\mathrm{Isom}(\mathbb{H}^n)$ for $n \geqslant 2$. Let μ be a probability distribution on Γ with finite first $\mathbb{H}^n$-moment such that the support of μ generates a subgroup that contains a pair of loxodromic elements with distinct axes. Then with respect to the harmonic measure defined by the μ-random walk on Γ the space $\mathbb{H}^n$ with the hyperbolic metric is statistically hyperbolic.*

We will present the proof of Theorem 1.2 after the proof of the harder case of Theorem 1.1. This lets us use the geodesic separation property for the harmonic measure formulated in the earlier proof. We note that when $n > 2$, Randecker-Tiozzo proved that a harmonic measure arising from a μ whose support generates Γ and has finite $(n-1)^{\mathrm{th}}$ moment with respect to a word metric, is singular. See [16, Theorem 2]. So Theorem 1.2 has new content.

From now on we assume that the complex dimension of $\mathcal{T}(S)$ is greater than one and present Theorem 1.1 with that assumption.

1.3. **Strategy of the proof.** To derive statistical hyperbolicity, Dowdall-Duchin-Masur set up two properties to check. The first property is called the thickness property. See [3, Definition 5.4]. It states that as the radius of a sphere goes to infinity a typical radial geodesic segment spends a definite proportion of its time in the thick part of Teichmüller space. The second property is called the separation property. See [3, Definition 6.1]. It states that as the radius of a sphere goes to infinity a typical pair of radial geodesic segments exhibit good separation. For Lebesgue-class visual measures, the ergodicity of the Teichmüller geodesic flow is the key tool for the thickness property. For rotationally invariant Lebesgue measures, they verify the separation property by disintegrating the measure along and transverse to Teichmüller discs and then use the hyperbolic geometry of these discs.

For random walks, different tools are needed. The main tool is the ergodicity of the shift map on the space of bi-infinite sample paths. The ergodicity can be leveraged to prove that a typical bi-infinite sample path recurs to a neighbourhood of its tracked geodesic with a

positive asymptotic frequency. Since sample paths lie in a thick part, the recurrence implies that tracked geodesics spend a positive proportion of their time in a thick part. By tweaking the size of the neighbourhood, and hence the thick part, we show that the time spent in the thick part by the tracked geodesic can achieve any positive proportion. While a positive proportion of thickness is suggested directly by the main theorem in [8], the precise quantitative version that we need here requires some work.

For the separation property, we project two fellow travelling radial geodesic segments to the curve complex. By a theorem of Maher-Tiozzo, a typical sample path makes linear progress in the curve complex. Combining this theorem with the recurrence, we show that the projections of fellow travelling radial geodesic segments must nest into a shadow. Also by a proposition in Maher-Tiozzo, the harmonic measure of a shadow tends to zero in the distance of the shadow from the base-point. This then enables us to conclude the required separation property.

1.4. **Acknowledgements:** We thank Maxime Fortier-Bourque for conversations that clarified many points in this work.

2. PRELIMINARIES

2.1. **Statistical hyperbolicity:** Let (X, d) be a metric space. Let $x \in X$. Let $r > 0$ We call the set $S_r(x) = \{x' \in X$ such that $d(x, x') = r\}$ the *sphere* of radius r centred at x. Suppose ν_r is a family of probability measures supported on $S_r(x)$. Provided the limit exists, one defines a numerical index $E(X) := E(X, x, d, \{\nu_r\})$ by

$$E(X) = \lim_{r \to \infty} \frac{1}{r} \int_{S_r(x) \times S_r(x)} d(x', x'') \, d\nu_r(x') d\nu_r(x'')$$

A space is said to be *statistically hyperbolic* if $E(X) = 2$. This is motivated by the fact that $E(\mathbb{H}^n) = 2$ for any dimension equipped with the natural measures on spheres. Moreover, it was demonstrated by Duchin-Lelièvre-Mooney [4, Theorem 4] that $E(G) = 2$ for any non-elementary hyperbolic group G with any choice of generating set. We direct the reader to [4] for further discussion on the sensitivity of E. Indeed, it is not quasi-isometrically invariant, and has dependence on the base-point x and the choice of measures ν_r. Furthermore, δ-hyperbolicity and exponential volume growth are not sufficient to guarantee statistical hyperbolicity.

2.2. **Background on Teichmüller spaces.** Let S be a surface of finite type, that is, S is an oriented surface with finite genus and finitely many marked points. The *Teichmüller space* $\mathcal{T}(S)$ of S is the space of marked conformal structures on S. By the Uniformisation Theorem, if S has negative Euler characteristic then there is a unique hyperbolic metric in each marked conformal class. The *mapping class group* $\mathrm{Mod}(S)$ is the group of orientation preserving diffeomorphisms of S modulo isotopy. The mapping class group $\mathrm{Mod}(S)$ acts on $\mathcal{T}(S)$ by changing the marking. The quotient space $\mathcal{M}(S) = \mathcal{T}(S)/\mathrm{Mod}(S)$ is the *moduli space of Riemann surfaces*. Given $\epsilon > 0$, a marked hyperbolic surface $x \in \mathcal{T}(S)$ is ϵ-*thick* if the hyperbolic length of every closed geodesic on x is at least ϵ. We let $\mathcal{T}_\epsilon(S)$ be the subset of $\mathcal{T}(S)$ consisting of all ϵ-thick marked hyperbolic surfaces. We note that there exists an $\epsilon > 0$ that depends only on the complexity of the surface such that $\mathcal{T}_\epsilon(S)$ is non-empty. See the discussion on the Bers constant in [5, Chapter 12 Section 4.2]. Observe that if x is ϵ-thick then so is gx for any mapping class g. Hence, we deduce that we get a thick-thin decomposition of the moduli space $\mathcal{M}(S)$. The Mumford compactness theorem says that for any $\epsilon > 0$, the thick part $\mathcal{M}_\epsilon(S)$ is compact.

Given a marked conformal surface x, let $\mathcal{Q}(x)$ be the set of meromorphic quadratic differentials on x with simple poles at and only at the marked points. This gives a bundle $\mathcal{Q}$ over $\mathcal{T}(S)$. This bundle is stratified by the orders of the zeroes of the quadratic differentials. By contour integration and a choice of square root, each $q \in \mathcal{Q}(X)$ defines a half-translation structure on S. That is, it defines charts to $\mathbb{C} = \mathbb{R}^2$ with half-translation transition functions of the form $z \to \pm z + c$. It then makes senses to impose the condition that the half-translation surfaces that we consider have unit area. Given a stratum of quadratic differentials, one may fix a basis for the homology of S relative to the marked points and zeroes. One can associate a period to each basis element of the homology by integrating a square root of the quadratic differential over a contour representing it. These periods give local co-ordinates on the stratum and can be used to define the Lebesgue measure class on it. The principal stratum is the stratum of quadratic differentials whose zeroes are all simple. The Lebesgue measure class on the principal stratum can be pushed down to define a Lebesgue measure class on the Teichmüller space $\mathcal{T}(S)$.

The $SL(2, \mathbb{R})$-action on $\mathbb{R}^2$ preserves the area and also the form of the transition functions. Hence, it descends to an action on $\mathcal{Q}$. The compact part $SO(2, \mathbb{R})$ acts by rotations and preserves the conformal structure. The diagonal part of the action given by

$$\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$$

is called the *Teichmüller flow* and we will denote it by ϕ_t. Given a pair x, y of marked hyperbolic surfaces, Teichmüller's theorem states that there is a unit area quadratic differential q on x and a time t such that $\phi_t q$ projects to y. The time t is called the *Teichmüller distance* between x and y.

The Teichmüller distance gives a Finsler metric on $\mathcal{T}(S)$ which we will call the *Teichmüller metric* and denote by d_{Teich}. The mapping class group acts by isometries and hence the Teichmüller distance descends to $\mathcal{M}$. Masur-Wolf [14] showed that Teichmüller space with the Teichmüller metric is not a δ-hyperbolic space. This adds value to the question of whether $\mathcal{T}(S)$ is statistically hyperbolic. Dowdall-Duchin-Masur showed that for measures in the Lebesgue-class, $\mathcal{T}(S)$ is statistically hyperbolic. See [3, Theorems B, C and D].

2.3. **Random walks on the mapping class group.** Let G be a finitely generated group. Let μ be a probability measure on G. A sample path w_n of length n for the μ-random walk on G is a random product $w_n = g_1 g_2 \cdots g_n$ where each g_i is sampled by μ. The n-fold convolution $\mu^{(n)}$ of μ gives the distribution of w_n. If G has an action on a space X, one can use the orbit of a base-point to project the random walk onto X. We are interested in the limiting behaviour of sample paths as $n \to \infty$. For this reason, we will consider the shift on $G^{\mathbb{N}}$. It is convenient to consider both forward and backward random walks. The backward random walk is simply the random walk with respect to the reflected measure $\hat{\mu}$ defined by $\hat{\mu}(g) = \mu(g^{-1})$. We then consider bi-infinite sequences as elements of $G^{\mathbb{Z}}$ with the shift σ acting as a step of the random walk. There is a measure h on $G^{\mathbb{Z}}$ such that the conditional measure for the shift is given by μ. We can separate the forward and backward directions to write h as the product $\nu \times \hat{\nu}$. We call the measure ν the harmonic measure.

By the Nielsen-Thurston classification, a mapping class is finite order, reducible or pseudo-Anosov. A finite order mapping class is an automorphism of some Riemann surface. A reducible mapping class has some power that fixes a multi-curve on the surface. A pseudo-Anosov mapping class f has a Teichmüller axis: an f-invariant bi-infinite Teichmüller geodesic along which the map translates realising the infimum of $d_{\text{Teich}}(x, f(x))$ over $\mathcal{T}(S)$ by this translation. This description of a pseudo-Anosov map implies that the Teichmüller axis is unique and that its vertical and horizontal measured foliations are uniquely ergodic.

A subgroup of $\mathrm{Mod}(S)$ is non-elementary if it contains a pair of pseudo-Anosov mapping classes with distinct Teichmüller axes. Let $x \in \mathcal{T}(S)$ be a base-point. Kaimanovich-Masur showed that if the support of a probability distribution μ on $\mathrm{Mod}(S)$ generates a non-elementary subgroup then for h-almost every sample path $\omega = (w_n)$ the sequence $w_n x$ converges to a projective class of a measured foliation on S. See [10, Theorem 2.2.4]. Thurston showed that there is a natural way in which the space $\mathrm{PMF}(S)$ of projective classes of measured foliations serves as a boundary of $\mathcal{T}(S)$. So the theorem of Kaimanovich-Masur can be rephrased as the convergence to the boundary $\mathrm{PMF}(S)$ for h-almost every sample path. In particular, the measure ν on $\mathrm{Mod}(S)^{\mathbb{N}}$ pushes forward to a measure on $\mathrm{PMF}(S)$. We call this the harmonic measure on $\mathrm{PMF}(S)$.

The distribution μ has *finite first moment* with respect to the Teichmüller metric if

$$\sum_{g \in \mathrm{Mod}(S)} \mu(g) d_{\mathrm{Teich}}(x, gx) < \infty.$$

By [11, Theorem 1.2], finite first moment in the curve complex is sufficient for positive linear drift of typical sample paths when projected to the curve complex using the mapping class group action. Since Teichmüller distance is a coarse upper bound for the curve complex distance, finite first d_{Teich}-moment implies finite first moment in the curve complex. Hence we have positive linear drift of sample paths in the curve complex and consequently in Teichmüller space.

Going back to the work of Kaimanovich-Masur, they showed that if μ has finite entropy and finite first logarithmic moment with respect to the Teichmüller metric then the pushforward measure is measurably isomorphic to ν. See [10, Theorem 2.3.1]. For this reason, and to keep the notation simple, we denote the measure on $\mathrm{PMF}(S)$ by ν even though we do not need the measurable isomorphism.

2.4. **Statistical hyperbolicity for a harmonic measure.** Let $\mathcal{Q}^1(x)$ be the set of unit area quadratic differentials for the marked Riemann surface x. When $q \in \mathcal{Q}^1(x)$ has a uniquely ergodic vertical foliation, the Teichmüller ray $\phi_t q$ converges as $t \to \infty$ to the projective class of the vertical foliation. Since ν is supported on the set of uniquely ergodic foliations, ν can be pulled back to a measure on $\mathcal{Q}^1(x)$. This gives us a measure on every sphere $S_r(x)$. Thus, it makes sense to consider whether $\mathcal{T}(S)$ with the Teichmüller metric is statistically hyperbolic with respect to harmonic measure.

2.5. **Statistical hyperbolicity in Teichmüller space.** Dowdall-Duchin-Masur [3] reduce statistical hyperbolicity of Teichmüller space with the Teichmüller metric for a family of measures $\{\nu_r\}$ to the verification of two properties: the thickness property [3, Definition 5.2] and the separation property [3, Definition 6.1]. We will now state these properties and in Section 4, we will give a quick sketch of how these properties imply statistical hyperbolicity.

For a choice of $\epsilon > 0$ and a geodesic segment $[x, x'] \subset \mathcal{T}(S)$, we denote the proportion of time $[x, x']$ spends in $\mathcal{T}_\epsilon(S)$ by

$$\mathrm{Thick}_\epsilon^\%([x, x']) := \frac{|\{0 \leqslant t \leqslant d_{\mathrm{Teich}}(x, x') : x'_t \in \mathcal{T}_\epsilon(S)\}|}{d_{\mathrm{Teich}}(x, x')},$$

where x'_t is the point at distance t from x along $[x, x']$.

The thickness property is the following.

Definition 2.6 (Thickness property). *A family of measures $\{\nu_r\}$ on spheres in $\mathcal{T}(S)$ has the thickness property if for all $0 < \theta, \eta < 1$ there exists an $\epsilon > 0$ such that*

$$\lim_{r \to \infty} \frac{\nu_r\left(\{x' \in S_r(x) \mid \mathrm{Thick}_\epsilon^\%([x, x'_t]) \geqslant \theta \text{ for all } t \in [\eta r, r]\}\right)}{\nu_r(S_r(x))} = 1,$$

for all $x \in \mathcal{T}(S)$.

The separation property is the following.

Definition 2.7 (Separation property). *A family of measures $\{\nu_r\}$ on spheres in $\mathcal{T}(S)$ has the separation property if for all $M > 0$ and $0 < \eta < 1$, we have*

$$\lim_{r \to \infty} \frac{\nu_r \times \nu_r \left(\{(x', x'') \in S_r(x) \times S_r(x) \mid d_{\text{Teich}}(x'_t, x''_t) \geqslant M \text{ for all } t \in [\eta r, r]\}\right)}{\nu_r \times \nu_r(S_r(x) \times S_r(x))} = 1,$$

for all $x \in \mathcal{T}(S)$.

In the next section, we derive these properties for a harmonic measure that arises from a random walk on the mapping class group determined by a probability distribution with finite first moment with respect to the Teichmüller metric whose support generates a non-elementary subgroup.

3. DERIVATION OF THE THICKNESS AND SEPARATION PROPERTIES

3.1. **Recurrence.** Let x be a base-point in Teichmüller space. Let ω be a bi-infinite sample path. As a convenient notation, we let $x_n = w_n x$ for any $n \in \mathbb{Z}$. In particular, this means that x_0 is the same as the base-point x. For almost every ω, the sequences x_n and x_{-n} as $n \to \infty$ converge projectively to distinct uniquely ergodic measured foliations λ^+ and λ^- respectively. For such sample paths, let γ_ω be the bi-infinite Teichmüller geodesic between λ^+ and λ^-. As convenient notation, let $\gamma = \gamma_\omega$ and let γ_n be a point of γ that is closest to x_n.

Let
$$\Lambda_R = \{\omega \text{ such that } d_{\text{Teich}}(x, \gamma_\omega) < R/3\}$$

By [10, Lemma 1.4.4], the function $\omega \to d_{\text{Teich}}(x, \gamma_\omega)$ is measurable. Recall that h is our notation for the harmonic measure $\nu \times \hat{\nu}$ on bi-infinite sample paths. So if R is large enough then $h(\Lambda_R) > 0$ and $h(\Lambda_R) \to 1$ as $R \to \infty$.
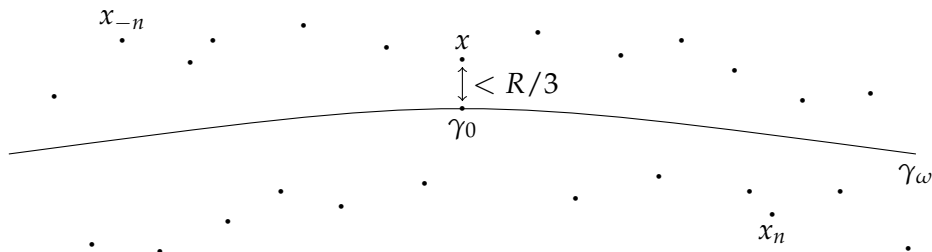

FIGURE 3.2. A sample path ω in Λ_R.

An integer k will be called an *R-recurrence time* for ω if $\sigma^k \omega \in \Lambda_R$. Suppose $j < k$ are R-recurrence times for ω and suppose $d_{\text{Teich}}(x_j, x_k) = 2d$. This distance will be bounded by the sum of the length of each step we do, that is,

$$2d \leqslant \sum_{i=j+1}^{k} d_{\text{Teich}}(x_{i-1}, x_i) = \sum_{i=j+1}^{k} d_{\text{Teich}}(x, g_i x)$$

We let $[\gamma_j, \gamma_k]$ be the segment of γ_ω connecting γ_j and γ_k. We note that

$$\text{Length}[\gamma_j, \gamma_k] \leqslant 2R + 2d.$$

If $2d \leqslant 2R$, then

$$[\gamma_j, \gamma_k] \subset B(x_j, 3R) \cup B(x_k, 3R).$$

So suppose $2d > 2R$. We consider the sub-segments of $[\gamma_j, \gamma_k]$ that might be outside of the union $B(x_j, 3R) \cup B(x_k, 3R)$. We denote the union of these sub-segments by $C_{j,k}$ and let $L(j,k)$ to be the sum of their lengths.

Let $0 < \rho < p < 1$. We choose R large enough such that $h(\Lambda_R) \geqslant p$. Let $n \in \mathbb{N}$ and set

$$E_n^{(1)} = \left\{ \omega \text{ such that } \frac{1}{m} \sum_{0 \leqslant k \leqslant m} \chi_{\Lambda_R}(\sigma^k \omega) < p - \rho \text{ for some } m \geqslant n \right\}.$$

By ergodicity of the shift map σ it follows that $h(E_n^{(1)}) \to 0$ as $n \to \infty$.

Suppose ω is in the complement of $E_n^{(1)}$. Then the number of times $i \in \{0, \cdots, n\}$ such that $\sigma^i \omega \notin \Lambda_R$ is at most $(1 + \rho - p)n$. Let $j_{\min}$ and $j_{\max}$ be the smallest and largest R-recurrence times in $\{0, \ldots, n\}$. Then we note that

$$j_{\min} \leqslant (1 + \rho - p)n \quad \text{and} \quad j_{\max} \geqslant n - (1 + \rho - p)n.$$

By bounding with steps, we get the estimate

$$d_{\text{Teich}}(x, x_{j_{\min}}) \leqslant \sum_{i=1}^{j_{\min}} d_{\text{Teich}}(x, g_i x).$$

We will separate the sum into two sums. The first will contain terms $d_{\text{Teich}}(x, g_i x)$ for which $d_{\text{Teich}}(x, g_i x) \leqslant D$ and the second will contain the rest of the terms. For convenience of notation, we let B be the set of bi-infinite sample paths ω whose first step g_1 satisfies $d_{\text{Teich}}(x, g_1 x) \leqslant D$. The set B depends on the choice of D but we will suppress this from the notation for the moment and point it out when required later. With this notation, the sum above becomes

$$d_{\text{Teich}}(x, x_{j_{\min}}) \leqslant \sum_{i=1}^{j_{\min}} d_{\text{Teich}}(x, g_i x) \chi_B(\sigma^i(\omega)) + \sum_{i=1}^{j_{\min}} d_{\text{Teich}}(x, g_i x) \chi_{\Omega \setminus B}(\sigma^i(\omega)),$$

We can bound the first sum by $\sum_{i=1}^{j_{\min}} D \leqslant (1 + \rho - p)Dn$, and the second one by

$$\sum_{i=1}^{n} d_{\text{Teich}}(x, g_i x) \chi_{\Omega \setminus B}(\sigma^i(\omega)).$$

We denote each of the terms of the sum as b_i^D. They are all independent and identically distributed. Furthermore, we have that $b_i^D \leqslant d_{\text{Teich}}(x, g_i x)$, which, since the measure has finite first d_{Teich}-moment, is integrable. By the strong law of large numbers, the sum divided by n converges almost surely to $\mathbb{E}[b_i^D] = C$. That is, the sets

$$E_n^{(2)} = \left\{ \omega \text{ such that } \frac{1}{m} \sum_{i=0}^{m} b_i^D(\omega) > C + c \text{ for some } m \geqslant n \right\}.$$

satisfy $h(E_n^{(2)}) \to 0$ as $n \to \infty$ for all $c > 0$. As D increases, the value inside the expectation converges point-wise to 0. Hence, by the dominated convergence theorem where we are dominating by $d_{\text{Teich}}(x, gx)$, if D is large enough, C is small.

Assume ω is in the complement of both $E_n^{(1)}$ and $E_n^{(2)}$. Then,

$$d_{\text{Teich}}(x, x_{j_{\min}}) \leqslant ((1 + \rho - p)D + C + c)n,$$

and, by the same reasoning,

$$d_{\text{Teich}}(x_{j_{\max}}, x_n) \leqslant ((1 + \rho - p)D + C + c)n.$$

This implies

$$d_{\text{Teich}}(\gamma_0, \gamma_{j_{\min}}) \leqslant ((1 + \rho - p)D + C + c)n + 2R$$

and
$$d_{\text{Teich}}(\gamma_{j_{\max}}, \gamma_n) \leqslant ((1 + \rho - p)D + C + c)n + 2R.$$

We will now estimate from above the time that the segment $[\gamma_{j_{\min}}, \gamma_{j_{\max}}]$ spends outside the neighbourhood of points along the sample path. A pair $j < k$ of recurrence times is *consecutive* if every J satisfying $j < J < k$ is not a recurrence time.

If $k = j + 1$, the set $C_{j,j+1}$ will be non empty only if $d_{\text{Teich}}(x_j, x_{j+1}) \geqslant 2R$. In that case, if we choose D smaller than R, then $L(j, j+1) \leqslant b_{j+1}^D$.

If $k > j + 1$, we have $k - j - 1$ steps taken outside the R-neighbourhood of the geodesic. We can split these steps depending on whether they are in B or not. Recall that since we are outside the exceptional set $E_n^{(1)}$, we know that the total number of non-recurrence times is bounded above by $(1 + \rho - p)n$. In the estimate for the sum of $L(j, k)$, we note that each non-recurrence time contributes to at most two terms in the sum. We deduce that

$$\sum_{\substack{j < k \\ \text{consecutive}}} L(j, k) \leqslant \sum_{\substack{j+1 = k \\ \text{consecutive}}} b_{j+1}^D + \sum_{\substack{j+1 < k \\ \text{consecutive}}} \sum_{i=j+1}^{k} \left(b_i^D + d_{\text{Teich}}(x, g_i x)\chi_B(\sigma^i(\omega)) \right)$$

$$\leqslant \sum_{i=1}^{n} b_i^D + 2(1 + \rho - p)nD$$

$$\leqslant (2(1 + \rho - p)D + C + c)n.$$

3.3. Linear progress.
The Teichmüller metric is sub-additive along sample paths. By Kingman's sub-additive ergodic theorem, there exists a constant $A \geqslant 0$ such that for almost every sample path ω we have

$$\lim_{n \to \infty} \frac{d_{\text{Teich}}(x_0, x_n)}{n} = A.$$

By [11, Theorem 1.2], $A > 0$.

Let $0 < a < 1$ be a constant smaller than A. Let $n \in \mathbb{N}$. Consider the set of sample paths

$$\Omega_n^{(3)} = \{\omega \text{ such that } (A - a)m < d_{\text{Teich}}(x_0, x_m) < (A + a)m \text{ for all } m \geqslant n\}.$$

Let $E_n^{(3)}$ be the complement $\Omega \setminus \Omega_n^{(3)}$. It follows that $h(E_n^{(3)}) \to 0$ as $n \to \infty$.

3.4. Thickness along tracked geodesics.
Let $0 < \theta' < 1$.

We parameterise the tracked geodesic $\gamma = \gamma_\omega$ by unit speed such that at time zero we are at γ_0, the closest point to $x_0 = x$, and $\gamma(t) \to \lambda^+$ as $t \to \infty$.

Let $\Lambda(r, \theta', \epsilon')$ be the set of sample paths ω such that for all $s > r$ we have

$$\text{Thick}_{\epsilon'}^{\%}[\gamma(0), \gamma(s)] \geqslant \theta'.$$

Proposition 3.5. *Given* $0 < \theta' < 1$ *there exists an* $\epsilon' > 0$ *such that*

$$\lim_{r \to \infty} h(\Lambda(r, \theta', \epsilon')) = 1.$$

Proof. Given $R > 0$ there exists $\epsilon(R) > 0$ such that $B(x_0, 3R) \subset \mathcal{T}_{\epsilon(R)}(S)$. By equivariance, $B(x_n, 3R) \subset \mathcal{T}_{\epsilon(R)}(S)$ for all $n \in \mathbb{Z}$.

Suppose that ω is in the complement $E_n^{(1)} \cup E_n^{(2)}$. We first prove the proposition along the discrete set of times γ_n along γ_ω. By the triangle inequality

$$d_{\text{Teich}}(\gamma_0, \gamma_n) \geqslant d_{\text{Teich}}(x_0, x_n) - d_{\text{Teich}}(x_0, \gamma_0) - d_{\text{Teich}}(x_n, \gamma_n).$$

Since γ_0 is the closest point in γ_ω to x_0

$$d_{\text{Teich}}(x_0, \gamma_0) \leqslant d_{\text{Teich}}(x_0, x_{j_{\min}}) + R \leqslant ((1 + \rho - p)D + C + c)n + R.$$

Similarly

$$d_{\text{Teich}}(x_n, \gamma_n) \leqslant d_{\text{Teich}}(x_n, x_{j_{\max}}) + R \leqslant ((1+\rho-p)D+C+c)n + R.$$

So we get

$$d_{\text{Teich}}(\gamma_0, \gamma_n) \geqslant d_{\text{Teich}}(x_0, x_n) - 2((1+\rho-p)D+C+c)n - 2R.$$

Further assume that ω is also in the complement of $E_n^{(3)}$. We deduce from the above estimates that

$$(3.6) \qquad d_{\text{Teich}}(\gamma_0, \gamma_n) \geqslant (A-a)n - 2((1+\rho-p)D+C+c)n - 2R.$$

We note that the points in the segment $[\gamma_0, \gamma_n]$ that are not in $\mathcal{T}_{\epsilon(R)}(S)$ are in the union of the sets $[\gamma_0, \gamma_{j_{\min}}]$, $[\gamma_{j_{\max}}, \gamma_n]$ and the sets $C_{j,k}$ for all consecutive recurrence pairs $j < k$. The individual upper bounds on the lengths of each set in the union gives us the bound on the thick proportion for a choice of $\epsilon' \leqslant \epsilon(R)$

$$(3.7) \qquad 1 - \text{Thick}_{\epsilon'}^{\%}[\gamma_0, \gamma_n] \leqslant \frac{4((1+\rho-p)D+C+c)}{(A-a) - 2((1+\rho-p)D+C+c) - 2R/n}.$$

Now we make explicit choices as follows.

- Note that as $R \to \infty$, the proportion $p \to 1$ and hence $\min\{(1-p)^{-1/2}, R\} \to \infty$.
- Recall that if we tend $D \to \infty$ then $C \to 0$.
- So if we set $D = \min\{(1-p)^{-1/2}, R\}$ then we can pass to R large enough so that C is small enough.
- Furthermore, note that $D(1-p) \leqslant (1-p)^{1/2}$ which can also be made small with R large enough.
- Recall also that ρ, c and a can be chosen at the outset and set to be small enough.

These choices imply that by choosing R large enough we may arrange that the numerator and the second term in the denominator in 3.7 are small. Once R has been fixed the third term in the denominator goes to zero as $n \to \infty$. In conclusion, by choosing R large enough we may arrange the right hand side of 3.7 to be smaller than $1 - \theta'$.

Now we conclude the proposition as the time $s \to \infty$ along γ_ω. By an argument identical to the derivation of (3.6), we get the upper bound

$$d_{\text{Teich}}(\gamma_0, \gamma_n) \leqslant (A+a)n + 2((1+\rho-p)D+C+c)n + 2R.$$

Given a time $s > 0$, we may choose n to satisfy

$$(A-a)n - 2((1+\rho-p)D+C+c)n - 2R < s < (A+a)n + 2((1+\rho-p)D+C+c)n + 2R.$$

When s is large enough, such a choice always exist. Since we are only interested in the limit as $s \to \infty$, we may make this choice. Further tweaking R and hence p and also tweaking ρ, c and a we may arrange that the ratio of the upper bound to the lower bound in the above inequality is as close to one as we want. This implies that as $s \to \infty$ the thick proportion of $[\gamma_0, \gamma(s)]$ is the same as the thick proportion of $[\gamma_0, \gamma_n]$.

Finally, we note that the set of exceptions is the union $E_n^{(1)} \cup E_n^{(2)} \cup E_n^{(3)}$ whose measure tends to zero as $n \to \infty$. In particular, this implies $h(\Lambda(r, \theta', \epsilon')) \to 1$ as $r \to \infty$, and we are done. $\qquad \square$

As a direct consequence of Proposition 3.5, we get the following conclusion.

Proposition 3.8. *Let $0 < \theta' < 1$. Then there exists an $\epsilon' > 0$ such that for almost every bi-infinite sample path ω there exists t_ω such that for all $t > t_\omega$*

$$\text{Thick}_{\epsilon'}^{\%}[\gamma(0), \gamma(t)] \geqslant \theta'.$$

3.9. **Thickness along rays.** Now let y be some base-point in $\mathcal{T}(S)$. Masur proved that Te-ichmüller rays with the same vertical foliation are asymptotic if the foliation is uniquely ergodic. See [13, Theorem 2]. We now use this result to transfer the thickness estimates from tracked geodesics to corresponding rays from y. Suppose that ω is a typical bi-infinite sample path with the tracked geodesic γ_ω. Let λ_ω^+ be the projective measured foliation that γ_ω converges to in the forward direction. Let ξ_ω be the geodesic ray from y that converges to λ_ω^+. We may parameterise ξ_ω with unit speed so that $\xi_\omega(0) = y$ and $\xi_\omega(t) \to \lambda_\omega^+$ as $t \to \infty$.

Proposition 3.10. *Let $0 < \theta < 1$. Then there exists $\epsilon > 0$ such that for almost every bi-infinite sample path ω there is a time $T_\omega > 0$ such that*

$$\mathrm{Thick}_\epsilon^\%[\xi_\omega(0), \xi_\omega(t)] \geqslant \theta$$

for all $t \geqslant T_\omega$.

Proof. By [10, Theorem 2.2.4], for almost every ω the foliation λ_ω^+ is uniquely ergodic. By Masur's theorem, there is a time $s > 0$ such that for all $t \geqslant s$ we have $d_{\mathrm{Teich}}(\xi_\omega(t), \gamma_\omega) < 1/2$. We may choose $\epsilon \leqslant \epsilon'$ such that the 1-neighbourhood of $\mathcal{T}_{\epsilon'}(S)$ is contained in $\mathcal{T}_\epsilon(S)$. This means after the time s along ξ_ω any ϵ'-thick segment of γ_ω gives an ϵ-thick segment of ξ_ω of at least the same length. We now set $\theta' > \theta$ and use Proposition 3.8. Let $t > s$. The total length of ϵ'-thick segments of γ_ω that are inside a 1-neighbourhood of $[\xi_\omega(s), \xi_\omega(t)]$ is at least $\theta't - s - d_{\mathrm{Teich}}(\gamma(0), y)$. If t is large enough then $\theta't - s - d_{\mathrm{Teich}}(\gamma(0), y) > \theta t$, and we are done. $\square$

As an immediate corollary, we get

Corollary 3.11. *The measures ν_r on spheres arising from the harmonic measure satisfy the thickness property 2.6.*

3.12. **Separation properties.** To prove the separation properties, we project to the complex of curves. Under this projection, Teichmüller geodesics give unparameterised quasi-geodesics and thick segments make coarse linear progress. If a pair of Teichmüller rays with good thickness properties up to a distance r are not well-separated then their projections to the curve complex fellow travel. Hence, the endpoints of their projections lie in some shadow and this shadow is pushed further and further out as $r \to \infty$. So roughly speaking, we may conclude the separation properties from knowing that the harmonic measure of these shadows goes to zero. We now give the details of the argument.

The complex of curves $\mathcal{C}(S)$ is a graph whose vertices are isotopy classes of essential simple closed curves on S. Two vertices α and β have an edge between them if α and β have representatives that are disjoint. The curve complex $\mathcal{C}(S)$ is a locally infinite graph with infinite diameter. Masur-Minsky showed that $\mathcal{C}(S)$ is in fact δ-hyperbolic. See [15, Theorem 1.1]. Recall that the Gromov product on $\mathcal{C}(S)$ is defined as follows: given a base-point x and two points y, z in the space, the Gromov product of y and z based at x is defined as

$$(y, z)_x = \frac{1}{2}\left(d_\mathcal{C}(x, y) + d_\mathcal{C}(x, z) - d_\mathcal{C}(y, z)\right)$$

where $d_\mathcal{C}$ denotes the distance in the curve complex. Let $\partial\mathcal{C}(S)$ be the Gromov boundary of $\mathcal{C}(S)$. Given a number $\tau > 0$, the shadow of y from x with distance parameter $d_\mathcal{C}(x, y) - \tau$ is defined as

$$\mathrm{Shad}_x(y, \tau) = \{z \in \mathcal{C}(S) \cup \partial\mathcal{C}(S) \text{ such that } (y, z)_x \geqslant d_\mathcal{C}(x, y) - \tau\}.$$

There are different definitions of shadows in the literature and the one we use here is from Maher-Tiozzo. See [11, Page 197].

To every marked hyperbolic surface $x \in \mathcal{T}(S)$, one can consider a systole on x, that is, a shortest closed hyperbolic geodesic on x. A systole is always a simple closed curve and hence

can be thought of as a vertex in the curve complex $\mathcal{C}(S)$. This defines a coarse projection $\text{sys} : \mathcal{T}(S) \to \mathcal{C}(S)$ from Teichmüller space to the curve complex. By [15, Theorems 2.3 and 2.6], the projection $\text{sys}(\gamma)$ of a Teichmüller geodesic γ is an unparameterised quasi-geodesic in $\mathcal{C}(S)$.

Let $M > 0$. Suppose ξ_ω and ξ_η are geodesic rays from y chosen with respect to the harmonic measure, where ω and η are the associated bi-infinite sample paths. Let T_ω and T_η be the thresholds given by Proposition 3.10 for ω and η, respectively. Pick T larger than T_ω and T_η. Suppose that $d_{\text{Teich}}(\xi_\omega(T), \xi_\eta(T)) < M$. Since the projection by systoles is coarsely Lipschitz, we may continue to assume $d_\mathcal{C}(\text{sys}(\xi_\omega(T)), \text{sys}(\xi_\eta(T))) < M$.

We consider the projections $\text{sys}([y, \xi_\omega(T)])$ and $\text{sys}([y, \xi_\eta(T)])$. Since ξ_ω and ξ_η spend at least θ proportion of their time in the thick part their projections to $\mathcal{C}(S)$ make linear progress, that is there exists a constant $\kappa > 0$ such that $d_\mathcal{C}(\text{sys}(y), \text{sys}(\xi_\omega(T))) \geqslant \kappa T$ and $d_\mathcal{C}(\text{sys}(y), \text{sys}(\xi_\eta(T))) \geqslant \kappa T$.

Denote by λ_η^+ the limiting point of $\text{sys}(\xi_\eta)$. By hyperbolicity of $\mathcal{C}(S)$ there is a constant $r > 0$ such that the Gromov product between λ_η^+ and $\text{sys}(\xi_\omega(T))$ satisfies

$$(\text{sys}(\xi_\omega(T)), \lambda_\eta^+)_{\text{sys}(y)} \geqslant d_\mathcal{C}(\text{sys}(y), \text{sys}(\xi_\omega(T))) - M - r.$$
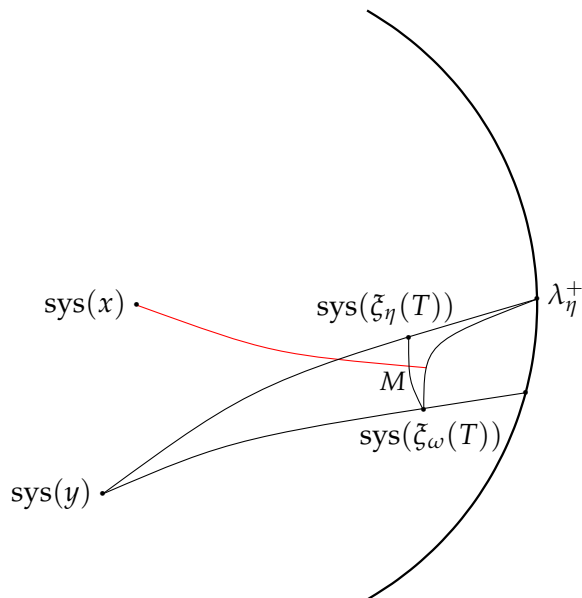

FIGURE 3.13. The Gromov product $(\text{sys}(\xi_\omega(T)), \lambda_\eta^+)_{\text{sys}(x)}$ in $\mathcal{C}(S)$, up to an additive constant.

In order to estimate harmonic measures, we will now pass to $\text{sys}(x)$ as the base-point for Gromov products. By the triangle inequality

$$d_\mathcal{C}(\text{sys}(x), \text{sys}(\xi_\omega(T))) \geqslant d_\mathcal{C}(\text{sys}(y), \text{sys}(\xi_\omega(T))) - d_\mathcal{C}(\text{sys}(x), \text{sys}(y)).$$

Note that $d_\mathcal{C}(\text{sys}(x), \text{sys}(\xi_\omega(T)))$ goes to infinity as $d_\mathcal{C}(\text{sys}(y), \text{sys}(\xi_\omega(T)))$ does. We have $(\text{sys}(\xi_\omega(T)), \lambda_\eta^+)_{\text{sys}(x)} \geqslant d_\mathcal{C}(\text{sys}(x), \text{sys}(\xi_\omega(T))) - r'$, where $r' = M + r + 2d_\mathcal{C}(\text{sys}(x), \text{sys}(y))$. Then λ_η^+ is contained in $\text{Shad}_{\text{sys}(x)}(\text{sys}(\xi_\omega(T)), r')$. By [11, Proposition 5.1], the supremum of the harmonic measure of $\text{Shad}_{\text{sys}(x)}(\text{sys}(\xi_\omega(T)), r')$ tends to zero as $T \to \infty$.

As an immediate corollary we get

Corollary 3.14. *The measures ν_r on spheres arising from the harmonic measure satisfy the separation property 2.7.*

4. STATISTICAL HYPERBOLICITY

For the sake of completeness, we now sketch the spherical version of the argument given by Dowdall-Duchin-Masur [3, Theorem 7.1] of how the thickness and separation properties imply statistical hyperbolicity. The idea is to mimic a proof of the fact that $E(\mathbb{H}^n) = 2$ for the natural measures on spheres which makes use of the δ-hyperbolicity of $\mathbb{H}^n$, and of the speed of separation of geodesics.

However, as discussed above, Teichmüller space is neither δ-hyperbolic, nor negatively-curved in the sense of Busemann. The motivation for the separation property 2.7 is to show instead that most pairs of geodesics after some threshold time become separated by a definite amount. This replaces the use of the negative curvature of $\mathbb{H}^n$. The combination of the following theorem of Dowdall-Duchin-Masur [3, Theorem A] and the thickness property 2.6 then replaces the use of the δ-hyperbolicity of $\mathbb{H}^n$.

Theorem 4.1 ([3], Theorem A)**.** *For any $\epsilon > 0$ and any $0 < \theta' \leqslant 1$, there exist constants C and L such that for any geodesic sub-interval $I \subset [x, x'] \subset \mathcal{T}(S)$ of length at least L and spending at least θ proportion of its time in $\mathcal{T}_\epsilon(S)$, we have*

$$I \cap \mathrm{Nbhd}_C([x, x''] \cup [x', x'']) \neq \varnothing,$$

for all $x'' \in \mathcal{T}(S)$.

We now sketch the proof of statistical hyperbolicity using Theorem 4.1 with the thickness and separation properties, namely 2.6 and 2.7.

Let $x \in \mathcal{T}(S)$. We choose θ large enough, say $\theta = 3/4$, and then for any $0 < p < \eta < 1/3$ we let $\epsilon = \epsilon(\theta, \eta) > 0$ be that guaranteed by the thickness property 2.6. Now choose $\theta' < \theta$, say $\theta' = 1/2$, and let C and L be the constants given by Theorem 4.1 for our choice of θ' and ϵ. The thickness and separation properties then imply that for all r large enough, there exists a subset $P_r \subset S_r(x) \times S_r(x)$ whose complement has $\nu_r \times \nu_r$-measure at most p and is such that for all $(x', x'') \in P_r$ we have $d_{\mathrm{Teich}}(x'_t, x''_t) \geqslant 3C$, and

$$\mathrm{Thick}_\epsilon^{\%}([x, x'_t]), \mathrm{Thick}_\epsilon^{\%}([x, x''_t]) \geqslant \theta,$$

for all $\eta r \leqslant t \leqslant r$. It can be checked that $B(x'_t, C) \cap [x, x''] = \varnothing$ for all such points x'_t.

Choosing r large enough, we can arrange that the interval $I_r = [x'_{\eta r}, x'_{2\eta r}]$ spends at least θ' proportion of its time in $\mathcal{T}_\epsilon(S)$, has length at least L and, since $\eta < 1/3$, is contained in $[x, x']$. By applying Theorem 4.1, we must have that $I_r \cap \mathrm{Nbhd}_C([x, x''] \cup [x', x'']) \neq \varnothing$ and, since we have already noted that $I_r \cap \mathrm{Nbhd}_C([x, x'']) = \varnothing$, it then follows that there exists a point in $[x', x'']$ at distance at most $2\eta r + C$ from x. Hence we have that

$$d_{\mathrm{Teich}}(x', x'') \geqslant (2 - 4\eta)r - 2C,$$

for all $(x', x'') \in P_r$. From which it follows that

$$
\begin{aligned}
E(X) := \lim_{r \to \infty} \frac{1}{r} & \int_{S_r(x) \times S_r(x)} d_{\mathrm{Teich}}(x', x'') \, d\nu_r(x') \, d\nu_r(x'') \\
& \geqslant \liminf_{r \to \infty} \frac{1}{r} \int_{S_r(x) \times S_r(x)} d_{\mathrm{Teich}}(x', x'') \, d\nu_r(x') \, d\nu_r(x'') \\
& \geqslant \liminf_{r \to \infty} \frac{1}{r} (1 - p)((2 - 4\eta)r - 2C) \\
& = (1 - p)(2 - 4\eta).
\end{aligned}
$$

Hence, the result follows as p and η can be taken to be arbitrarily small.

4.2. **Proof of Theorem 1.2.** We now give a quick proof of Theorem 1.2. The ambient geometry in $\mathbb{H}^n$ is already hyperbolic. So we can simply bypass the thickness discussion. All we need is the separation property for the harmonic measure. The argument is similar to the one given for Corollary 3.14 in Section 3.12.

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